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GENERAL SOLUTION OF REFLECTION EQUATION FOR EIGHT VERTEX MODEL

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General solution of reflection equation for eight vertex model [1]

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Abstract

We discuss the general solutions of reflection equations for eight vertex model. And the hamiltonian with four free parameters are given. This model is integrable and some properties is discussed.

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It is well-known that integrability is a consequence of Yang-Baxter equation (YBE) [1,2] in two dimensional statistical models and conformal field theory [2,3]. The YBE takes the form

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{1}$$

where R matrix can be considered as the factorized S matrix in two timespace dimensions by

$$S^{ab}_{cd}(u) = R^{ab}_{\bar{c}d}(u). \tag{2}$$

It can also be thought as the statistical boltzmann weight for a vertex model in two dimensional statistics and the Lax-Pair of a one-dimensional quantum spin chain system. In order to diagonalize such system, one has usually to impose the periodic condition on it [1,2,4]. With YBE and periodic condition, Baxter has set up the relation between eight vertex and H_{xyz} model [2]. One between six vertex and H_{xxz} model can be obtained from the former by taking trigonometrical limit. Although the periodic condition has not explicitly given in most of close systems, it almost appears in all systems solved by using Bethe Ansatz Method and Quantum Inverse Scattering Method.

Recently, much more attention has been paid on the integrable systems with nontrivial boundary condition, which was initiated by Cherednik and Sklyanin [5,6] to study finite size systems. In this case, YBE is not sufficient condition of integrability of open model. The reflection equations have played an important role, which was first proposed by Cherednik and Sklyanin. They take the forms

$$R_{12}(u-v)\overset{1}{K}_-(u)R_{21}(u+v)\overset{2}{K}_-(v) = \overset{2}{K}_-(v)R_{12}(u+v)\overset{1}{K}_-(u)R_{21}(u-v) \tag{3}$$

and

$$R_{12}(-u+v)\overset{1}{K}{}^{t_1}_+(u)R_{21}(-u-v-2\eta)\overset{2}{K}{}^{t_2}_+(v) = \overset{2}{K}{}^{t_2}_+(v)R_{12}(-u-v-2\eta)\overset{1}{K}{}^{t_1}_+(u)R_{21}(-u+v) \tag{4}$$

where R matrix must have PT and crossing symmetries [6]. For other kinds of R matrices, the generalized reflection equations appear in references [7] and

[8]. In two dimensional field theory, K_+ and K_- describe the scattering of the particles by the left and right walls respectively. For statistical model, these two K matrices determine the boundary condition and contribute nontrivial terms to the hamiltonian. Mathematically, YBE can related to some algebras such as quantum group and Sklyanin algebra, while the reflection equations are an effective tool to show that open systems with proper boundary term enjoy quantum symmetry [9,10,11].

In this note we present the general solution of reflection equations for eight vertex model. And the nontrivial boundary terms competable with integrability was obtained, which has four free parameters. Under some special choices of the parameters, it degenerates to known results in references [6,12,13].

model: The boltzmann weight of 8-vertex model was first found by Baxter [2]. He has also set up the relation between 8-vertex and H_{xyz} model. In the notation of Faddeev and Takhatajan [4], the R-matrix can be written as

$$R_{12}(u) = \begin{pmatrix} a(u) & 0 & 0 & d(u) \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ d(u) & 0 & 0 & a(u) \end{pmatrix} \tag{5}$$

here

$$\begin{aligned} a(u) &= sn(u+2\eta), & b(u) &= sn(u) \\ c(u) &= sn(2\eta), & d(u) &= ksn(u+2\eta)sn(u)sn(2\eta) \end{aligned} \tag{6}$$

This R matrix defined by equation 5 satisfies the assumption of Sklyanin [6], i.e.

$$P_{12}R_{12}(u)P_{12} = R_{12}(u), \tag{7}$$

$$R_{12}^{t_1}(u) = R_{12}^{t_2}(u), \tag{8}$$

$$R_{12}(u)R_{12}(-u) = g(u)id, \tag{9}$$

$$R_{12}^{t_1}(u)R_{12}^{t_2}(u-2\eta) = \tilde{g}(u)id, \tag{10}$$

and has the property

$$R(0) = sn(2\eta)P_{12} \tag{11}$$

where P_{12} is a permutation operator, $g(u)$ and $\tilde{g}(u)$ are two scale functions. We do not need their explicit form here. For eight vertex model, the $L(u)$ operator is

$$L_j(u) = \sum_{i=0}^{3} w_i(u)\sigma_0^i\sigma_j^i \tag{12}$$

where

$$\begin{aligned} w_0(u) + w_3(u) &= a(u), & w_0(u) - w_3(u) &= b(u) \\ w_1(u) + w_2(u) &= c(u), & w_1(u) - w_2(u) &= d(u) \end{aligned} \tag{13}$$

which satisfies the Yang-Baxter relation

$$R_{12}(u-v)\overset{1}{L}_j(u)\overset{2}{L}_j(v) = \overset{2}{L}_j(v)\overset{1}{L}_j(u)R_{12}(u-v) \tag{14}$$

In order to set up the relation between eight vertex and H_{xyz}, one can define monodromy matrix as

$$\tau(u) = trL_N(u)L_{N-1}(u)\cdots L_1(u) \tag{15}$$

It is easy to show that it consists a commutative family. Differentiating $log\tau(u)$ respect u at $u=0$, one can get hamiltonian H_{xyz} [2].

Before solving the reflection equations, we review some useful formulas about Jacobi elliptic function.

$$sn^2(u) + cn^2(u) = 1, \qquad dn^2(u) + k^2sn^2(u) = 1 \tag{16}$$

$$sn(u+v) = \frac{sn(u)cn(v)dn(v) + sn(v)cn(u)dn(u)}{1 - k^2sn^2(u)sn^2(v)} \tag{17}$$

$$cn(u+v) = \frac{cn(u)cn(v) - sn(u)sn(v)dn(u)dn(u)}{1 - k^2sn^2(u)sn^2(v)} \tag{18}$$

$$dn(u+v) = \frac{dn(u)dn(v) - sn(u)cn(u)sn(v)cn(v)}{1 - k^2sn^2(u)sn^2(v)} \tag{19}$$

here k is the modulus of Jacobi elliptic function, $sn(u)$ is odd function, while $cn(u)$ and $dn(u)$ are even functions.

general solution: Now, we seek the general solution of reflection equation 3. Let the general form be

$$K_-(u) = \begin{pmatrix} \alpha(u) & \beta(u) \\ \gamma(u) & \delta(u) \end{pmatrix} \tag{20}$$

Substituting this K matrix into equation 3, we get sixteen equations about four arbitrary functions. But the number of independent equations is ten. The simplest two equations are

$$a(u+v)d(u-v)[\alpha(u)\alpha(v)-\delta(u)\delta(v)] = a(u-v)d(u+v)[\alpha(u)\delta(v)-\alpha(v)\delta(u)] \tag{21}$$

and

$$a(u-v)c(u+v)[\beta(u)\gamma(v)-\beta(v)\gamma(u)] = d(u-v)b(u+v)[\gamma(u)\gamma(v)-\beta(u)\beta(v)] \tag{22}$$

Taking use of formula 17, one can find from equation 21

$$\alpha(u) = f(u,\xi)sn(\xi+u), \qquad \delta(u) = f(u,\xi)sn(\xi-u) \tag{23}$$

where $f(u,\xi)$ is a arbitrary function and ξ is arbitrary complex variable. Equation 22 gives the ratio of two function

$$\frac{\beta(u)}{\gamma(u)} = -ksn^2(u) \tag{24}$$

It is clear that there is only one arbitrary function due to the constrain 24, while there exist eight equations to be satisfied. Generally, such question has not solution. Fortunately, in our case, these equations are coincident each other. In the following, we explicitly give a equation which corresponds to the first row and second column element of reflection equation 3.

$$\begin{aligned} &a(u-v)a(u+v)\alpha(u)\beta(v) + b(u+v)d(u-v)\gamma(u)\delta(v) \\ &+ a(u-v)c(u+v)\beta(u)\delta(v) + d(u+v)d(u-v)\delta(u)\beta(v) = \\ &a(u+v)c(u-v)\alpha(v)\beta(u) + b(u+v)b(u-v)\beta(v)\alpha(u) \\ &+ c(u+v)c(u-v)\beta(v)\delta(u) + d(u+v)b(u-v)\alpha(v)\gamma(u) \end{aligned} \tag{25}$$

From the above equation, we can find

$$\frac{\alpha(v)\beta(u)}{\alpha(u)\beta(v)} = \frac{sn(v+\xi)[1-k^2sn^2(v)sn^2(\xi)\ sn(2u)sn^2(u)}{sn(u+\xi)[1-k^2sn^2(u)sn^2(\xi)\ sn(2v)sn^2(v)} \tag{26}$$

Comparing equation 26 with equation 23, one get

$$\begin{aligned} f(u,\xi) &= \nu[1-k^2sn^2(u)sn^2(\xi)] \\ \beta(u) &= \mu sn(2u)sn^2(u) \end{aligned} \tag{27}$$

here μ, ν are arbitrary constants. So we obtain the general form of $K(u)$

$$\begin{aligned} \alpha(u) &= \nu[1-k^2sn^2(u)sn^2(\xi)]sn(\xi+u), \qquad \beta(u) = \mu sn(2u)sn^2(u) \\ \delta(u) &= \nu[1-k^2sn^2(u)sn^2(\xi)]sn(\xi-u), \qquad \gamma(u) = -\mu k^{-1}sn(2u) \end{aligned} \tag{28}$$

When the constant μ is zero, one can find that the K matrix is diagonal, which appears in references [5,6,13]. Because of the scale invariance of K matrix, we can normalize it such that $\nu = 1$.

Next, we study the trigonometrical limit of our result. The trigonometrical limit means that the variable τ in theta function approaches into $i\infty$. In this limit, the jacobi elliptic functions and modulu change into

$$\begin{aligned} &sn(u) \longrightarrow sin(u) \\ &cn(u) \longrightarrow con(u) \\ &dn(u) \longrightarrow 1, \qquad k \longrightarrow 0 \end{aligned} \tag{29}$$

It is easy to find

$$\alpha(u) = sn(\xi+u), \qquad \delta(u) = sn(\xi-u). \tag{30}$$

But one must be careful to deal with β and γ due to $k=0$. In fact the simple limit only gives a special solution. In order to get general one, we return to study equation 22. It is not difficult to find

$$\frac{\beta(u)}{\gamma(u)} = \frac{\beta(v)}{\gamma(v)} = const. \tag{31}$$

substituting this into equation 4, we get

$$K_-(u) = \begin{pmatrix} sin(\xi+u) & \mu sin(2u) \\ \nu sin(2u) & sin(\xi-u) \end{pmatrix} \tag{32}$$

This is the result of de Vega et al [12]. In the similar way, one can find the solution of equation 4. However, there exists an automorphism which maps K_- into K_+

$$\phi : K_-(u) \longmapsto K_+(u) = K_-^t(-u-\eta) \tag{33}$$

The proof is direct calculation. One can take the transport of equation 3 and make substitution $u \to -u-\eta$ and $v \to -v-\eta$. Therefore, we get

$$K_+(u) = \begin{pmatrix} p(u)sn(\zeta-u-\eta) & -\lambda k^{-1}sn(-2u-2\eta) \\ \rho sn(-2u-2\eta)sn^2(-u-\eta) & p(u)sn(\zeta+u+\eta) \end{pmatrix} \tag{34}$$

here ζ, λ, ρ are arbitrary complex variables and $p(u) = [1-k^2sn^2(-u-\eta)sn^2(\zeta)]$.

Besides, the K matrix has the property

$$K_-(0) = \nu sn(\xi) \tag{35}$$

and

$$K_+(0) = \begin{pmatrix} p(0)sn(\zeta-\eta) & -\lambda k^{-1}sn(-2\eta) \\ \rho sn(-2\eta)sn^2(-\eta) & p(0)sn(\zeta+\eta) \end{pmatrix} \tag{36}$$

hamiltonian: We now consider the hamiltonian of open chain system. With the help of Yang-Baxter relation and reflection equations, $\mathcal{T}(u)$ satisfies the reflection equation

$$R_{12}(u-v)\overset{1}{\mathcal{T}}(u)R_{21}(u+v)\overset{2}{\mathcal{T}}(v) = \overset{2}{\mathcal{T}}(v)R_{12}(u+v)\overset{1}{\mathcal{T}}(u)R_{12}(u-v) \tag{37}$$

here $\mathcal{T}(u)$ is defined by

$$\begin{aligned} \mathcal{T} &= L_N(u)\cdots L_2(u)L_1(u)K_-(u)L_1^{-1}(u)L_2^{-1}(u)\cdots L_N^{-1}(u) \\ L_j^{-1}(u) &= \sigma^2 L_j^t(-u-2\eta)\sigma^2 \end{aligned} \tag{38}$$

From this reflection equation and YBE, one can show that

$$t(u) = trK_+(u)\mathcal{T}(u) \tag{39}$$

consists a commutative family. The proof is similar with one given by Sklyanin [6]. Differentiating $t(u)$ with respect to u at $u=0$, one can find the Hamiltionian

of the open chain

$$H = \sum_{j=1}^{N-1} H_{j,j+1} + \frac{1}{2}\overset{1}{\mathcal{K}'}_- + \frac{Tr_0 \overset{0}{\mathcal{K}}_+(0)H_{0,N}}{Tr_0 \overset{0}{\mathcal{K}}_+(0)} \tag{40}$$

where

$$H_{j,j+1} = P_{jj+1}R'_{jj+1}(u)|_{u=0} \tag{41}$$

is the bulk term of eight vertex model. For the convenience, we take the ν in $K_-(u)$ be $sn^{-1}(\xi)$. Substituting K_-, K_+ and R matrix into equation 40, we get

$$\begin{aligned} H = \ & \textstyle\sum_{j=1}^{N-1} J_1\sigma_j^1\sigma_{j+1}^1 + J_2\sigma_j^2\sigma_{j+1}^2 + J_3\sigma_j^3\sigma_{j+1}^3 \\ & + a_1\sigma_1^1 + a_2\sigma_1^2 + a_3\sigma_1^3 + b_1\sigma_N^1 + b_2\sigma_N^2 + b_3\sigma_N^3 \end{aligned} \tag{42}$$

where

$$\begin{aligned} a_1 &= \frac{-\mu}{2ksn(\xi)}, \quad J_1 = 1+k^2sn^2(2\eta) \\ a_2 &= \frac{i\mu}{2ksn(\xi)}, \quad J_2 = 1-k^2sn^2(2\eta) \\ a_3 &= \frac{cn(\xi)dn(\xi)}{2sn(\xi)}, \quad J_3 = cn(\eta)dn(\eta) \\ b_1 &= \frac{J_1sn(4\eta)[\lambda k^{-1} - \rho sn^2(2\eta)]}{2sn(\zeta)cn(2\eta)dn(2\eta)} \\ b_2 &= \frac{iJ_2sn(4\eta)[\lambda k^{-1} + \rho sn^2(2\eta)]}{2sn(\zeta)cn(2\eta)dn(2\eta)} \\ b_3 &= \frac{-sn(2\eta)cn(\zeta)dn(\zeta)}{sn(\zeta)} \end{aligned} \tag{43}$$

conclusion: We have solved the general solution of reflection equations for eight vertex model, and obtained the hamiltonian with four parameters which describe the boundary conditions. This is different from one of six vertex model. This new model is integrable due to the existence of commutative family $t(u)$. However, how to diagonalize this model is still an open question. This is very complicated. One way may be generalized QISM proposed by Sklyanin. All commutative relation defined by equation 38 will be useful.

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References

[1] C.N. Yang, Phys. Rev. Lett. **19**(1967)1312-1314.

[2] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, New York, 1982).

[3] H.J.de Vega, Int. Mod. Phys. **A4**(1983)2371.

[4] L.D. Faddeev and L.A. Takhtajan, Russ. Math. Surv. **34**(1979)11.

[5] I.V.Cherednik, Theor.Math.Phys.**61**(1984)911; **17**(1983)77.

[6] E.K.Sklyanin, J. of Phys. **A21**(1988)2375.

[7] Manchenski and Nepomenchie J. of Phys. **A24**(1991)L17.

[8] R.H.Yue and Y.X.Chen, J. of Phys. **A25**(1993)(in press).

[9] V.Pasquier and H.Saleur, Nucl. Phys. **B330**(1989)523.

[10] L.Mezincescu and R.I.Nepomechie, Mod. Phys. Lett. **A6**(1991)2497.

[11] B.Y.Hou, K.J.Shi, Z.X.Yang and R.H.Yue J. Phys. **A24**(1991)3284.

[12] de Vega and A.Gonzalez Ruiz, LPTHE-PAR-92-45.

[13] R.H.Yue, AS-ITP-93-42, CCAST-93-16.