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August 1993

GENERALIZED BOGLIUBOV TRANSFORMATION FOR q-FERMIONS

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ASITP-93-43

Generalized Bogliubov Transformation for q-Fermions [1]

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Abstract

The q-analog of the Bogliubov transformation for two q-fermions is found. And the Clebsh-Gordon coefficients related to this transformation is constructed in term of a q-polynomials.

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Bogliubov transformation (BT) which is the linear isomorphism of the algebra of fermian and bosonian operators has played an important role in quantum mechanics, statistics and solid state physics. The generator of BT can be written as a unitary operator. Recently, much more attention have been paid in the deformation of boson, fermion and paraparticle which can be used to realize the deformation of lie-algebra and super-lie-algebra [1-6]. One obvious question is how to construct the the BT for q-boson, q-fermion and q-paraparticles. Zhedanov has solved this problem for q-boson [3]. The generalized BT and the quantum Clebsh-Gordon coefficients of q-boson have been given. The operator of the transformation is unitary. The purpose of this paper is to discuss the generalized BT and the Clebsh-Gordon coefficients for q-fermions.

Let us consider two q-fermions which obey the following commutating relations

$$\begin{aligned}
\tilde{A}_-\tilde{A}_+ + q^{1/2}\tilde{A}_+\tilde{A}_- &= q^{-\tilde{A}_0/2} & [\tilde{A}_0, \tilde{A}_\pm] &= \pm\tilde{A}_\pm \\
\tilde{B}_-\tilde{B}_+ + q^{1/2}\tilde{B}_+\tilde{B}_- &= q^{-\tilde{B}_0/2} & [\tilde{B}_0, \tilde{B}_\pm] &= \pm\tilde{B}_\pm \\
\tilde{A}_-\tilde{A}_+ &= (\tilde{A}_0+1)_q & \tilde{A}_+\tilde{A}_- &= (\tilde{A}_0)_q \\
\tilde{B}_-\tilde{B}_+ &= (\tilde{B}_0+1)_q & \tilde{B}_+\tilde{B}_- &= (\tilde{B}_0)_q
\end{aligned} \tag{1}$$

$$[A_0, B_i] = [B_0, A_i] = [A_0, B_0] = \{A_i, B_j\} = 0, \qquad (i, j = \pm) \tag{2}$$

where

$$(x)_q = \frac{q^{1/2}(q^{-x/2} - (-1)^x q^{x/2})}{1+q}$$

For the convenience, we introduce a transformation

$$\begin{aligned}
A_\pm = q^{(A_0+1)/4} &= \tilde{A}_\pm & A_0 &= \tilde{A}_0 \\
B_\pm = q^{(B_0+1)/4} &= \tilde{B}_\pm & B_0 &= \tilde{B}_0
\end{aligned} \tag{3}$$

and rewrite the algebra as

$$\begin{aligned}
A_-A_+ + qA_+A_- &= q^{1/2} & [A_0, A_\pm] &= \pm A_\pm \\
A_-A_+ &= [A_0+1] & A_+A_- &= [A_0]
\end{aligned} \tag{4}$$

where

$$[x] = \frac{q^{1/2}(1-(-q)^x)}{1+q}$$

and algebra B_i has similar relation as A_i. As in the q-boson case, one can choose the canonical bases of q-fermion

$$A_0|n_A>=n_A|n_A> \qquad A_+|n_a>=\sqrt{[n_A+1]}|n_A+1> \qquad A_-|n_a>=\sqrt{[n_A]}|n_A-1> \tag{5}$$

This bases can be realized in term of operator A_+ acting on the vacuum state $|0>$

$$|n_A> \;=\; \frac{1}{\sqrt{[n_A]!}}(A_+)^{n_A}|0> \qquad A_-|0> \;=\; 0 \tag{6}$$

where $[n]! = [n][n-1]\cdots[2][1], [0] = 0, [0]! = 1$.(For q-fermion B, there are same canonical bases $|n_B>$.) One can show that the state $|n_A>$ satisfies orthogonal condition

$$<n'_A|n_A>=0$$

In the proof of it, a useful formula $A_-(A_+)^n - (-1)^n(A_+)^n A_- = [n]$ is taken.

The BT of q-fermions is an automorphism which preserves the algebra equation 4. Under the transformation, two q-fermions A and B change into two new q-fermions C and D. For C q-fermion, we introduce a transformation

$$\begin{aligned} C_- &= \sqrt{1-sin^2\theta(-q)^{A_0}}A_- + sin\theta q^{A_0}B_- \\ C_+ &= A_+\sqrt{1-sin^2\theta(-q)^{A_0}} + B_+ sin\theta q^{A_0} \end{aligned} \tag{7}$$

It is easy to show that C_- and C_+ satisfy

$$C_-C_+ + qC_+C_- = q^{1/2}$$

by taking care of the relation

$$F(A_0)A_\pm = A_\pm F(A_0 \pm 1) \tag{8}$$

The operator C_0 can be defined through

$$C_0 = C_+C_-$$

With the help of the above relations, one can show

$$C_-C_+ = [C_0+1] \qquad [C_0, C_\pm] = \pm C_\pm$$

From equation and equation we can express C_0 as

$$\begin{aligned} (-q)^{C_0} \;=\; & (sin^2\theta - q)(-q)^{A_0-1} - (1-q)sin^\theta(-q)^{2A_0-1} + sin^\theta(-q)^{2A_0+B_0} \\ & + \sqrt{1-sin^2\theta(-q)^{A_0}}sin\theta q^{A_0}(q^{1/2}+q^{-1/2})A_-B_+ \\ & - \sqrt{1-sin^2\theta(-q)^{A_0-12}}sin\theta q^{A_0-1}(q+1)q^{1/2})A_+B_- \end{aligned} \tag{9}$$

Generally, the inverse of $(-q)^{C_0}$ is not simple as $(-q)^{A_0}$. This is due to the appearance of $A_i, B_i, (i=\pm)$ in $(-q)^{C_0}$. Hence, it must contain $(A_-B_+)^n$ and $(A_+B_-)^n$ for arbitrary n. Now, we return to consider the unity of the transformation. If q be real, then $(C_-)^\dagger = C_+$. The the transformation can be written as

$$C_\pm = U_q(\theta)A_\pm U_q^\dagger(\theta) \tag{10}$$

This unitary matrix $U_q(\theta)$ is unique existent as the transformation keeps the algebra structure. But, it is very difficult to find the explicit expression. We do not discuss it here. Next, we consider the limit of $C_\pm$ under q approaches into unit. It is obvious that

$$C_\pm = cos(\theta)A_\pm + sin(\theta)B_\pm \tag{11}$$

In taking limit, we have used the following formula

$$F(A_0)A_- = A_-F(0) \qquad F(A_0)A_+ = A_+F(1) \tag{12}$$

The above formula can be get from the property of fermion. The limit of $U_q(\theta)$ is coincident with one $U(\theta)$ in fermian case which is known as $exp\{a_2a_1^\dagger - a_1a_2^\dagger\}$.

In q-fermian case, $A_0 + B_0$ is a invariant which commutes with A_0 and C_0. Thus, under the BT, $A_0 + B_0 \to C_0 + D_0$ is still a invariant, i.e.

$$C_0 + D_0 = U_q(\theta)(A_0+B_0)U_q^\dagger(\theta) = A_0 + B_0 \tag{13}$$

This can be considered as the definition of D_0 which is a polynomial about A_0, B_0, A_-B_+ and A_+B_-. On the other hand, $D_\pm$ must satisfy the same algebra relation as equation 4 and

$$D_\pm = U_q(\theta)B_\pm U_q^\dagger(\theta)$$

hence, $D_\pm$ is not simple as $C_\pm$ but a complicated function of A_i and B_i. Therefore, we get a nonsymmetric generalized BT for q-fermions which does like the symmetric one for fermions. Fortunately, although we have only half part of BT, it is enough to consider the Clebsh-Gordon (C-G) coefficients of q-fermions.

Indeed, the formula 7 is a addition rule for two independent q-fermions yielding third one. From equation 13, we know the transformation does not violate the invariant $A_0 + B_0$. Let it be $2j$ for a given representation, one can get a $2j+1$ dimensional space on which the mutually commutating operators A_0 and C_0 act. The overlap function W_{nm} is defined

$$W_{nm} \;=\; {}_C< m;j|n;j>_A \;=\; {}_A< m;j|U_q(\theta)|n;j>_A \tag{14}$$

On the other hand, these overlop functions are the C-G coefficients which map a vector of space $\mathcal{H}_A$ onto vectors of space $\mathcal{H}_C$

$$|n;j>_A = \sum_{m=0}^{2j} W_{nm}|m;j>_C \tag{15}$$

The operator $(-q)^{C_0}$ acting on the vector $|n;j>_C$ gives

$$(-q)^{C_0}|n;j> = \beta_n|n;j>_C - \alpha_{n+1}|n+1;j>_C + \alpha_n|n;j>_C \tag{16}$$

where

$$\begin{aligned} \alpha_n &= (1 + sin^2\theta((-q)^{2j} + (-1/q))(-q)^n - sin^2\theta(1-1/q)(-q)^{2n} \\ \alpha_n^2 &= (1 - sin^2\theta(-q)^{n-1})(-q)^{2n-2}(1-(-q)^n)(1-(-q)^{2j+1-n}) \end{aligned} \tag{17}$$

Substituting equation 9 into equation 16, one can get the recurrent relation of W_{nm} which involves three terms

$$\alpha_{n-1}W_{n-1m} - \alpha_{n+1}W_{n+1m} + \beta_n W_{nm} = (-q)^m W_{nm} \tag{18}$$

and

$$-\alpha_1 W_{1m} + \beta_0 W_{0m} = (-q)^m W_{nm} \tag{19}$$

Let

$$W_{nm} = W_{0m}P_n(m) \tag{20}$$

we get

$$\alpha_n P_{n-1}(m) - \alpha_{n+1}P_{n+1}(m) + \beta_n W_n(m) = (-q)^m P_n(m) \tag{21}$$

and

$$P_0(m) = 1 \qquad P_1(m) = \beta_0 - \frac{(-q)^m}{\alpha_1} \tag{22}$$

The polynomial P_n, governed by the above equations, is unique. The orthogonal property of $P_n(m)$ is

$$\sum_{m=0}^{2j} w_n P_n(m)P_{n'}(m) = \delta_{nn'} \qquad w_n = |W_{0n}|^2 \tag{23}$$

In the above equation, the quantity w_{0n} is undetermined. We must point that w_{0n} depends upon j, but we have omitted it. In order to find w_n, we calculate

$$\begin{aligned} {}_C< m|C^-|0;j> &= \sqrt{[m+1]}W_{om+1}(j) \\ &= {}_C< m|\{\sqrt{1-(-q)^{A_0}sin^2\theta}A_- + sin\theta q^{A-0}B_-\}|0;j> \end{aligned} \tag{24}$$

So, we have a recurrent relation

$$\sqrt{[m+1]}W_{0m+1}(j) = sin\theta\sqrt{[2j]}W_{0m}(j-\frac{1}{2}) \qquad (0 \le m \le 2j) \tag{25}$$

From this relation, we find

$$W_{0m}(j) = \frac{\sqrt{[2j]!}sin^m\theta}{\sqrt{[m]![2j-m]!}}W_{00}(j-\frac{m}{2}) \tag{26}$$

Now, our question changes into finding $W_{00}(l)$, which equals to determine the vacuum of algebra C. We assume that

$$|0>_C = \sum_{m,n=0}^{\infty} f_{m,n}|m>_A \otimes |n>_B \tag{27}$$

Operator C_- acts on the vacuum, giving

$$f_{m,n}\sqrt{[m+1](1-sin^2\theta(-q)^{m-1})} = -q^{m-1}sin\theta\sqrt{[n+1]}f_{m-1,n+1} \tag{28}$$

It is not difficult to find

$$f_{m,n} = \frac{(-1)^n q^{m(m-1)}}{\sqrt{[m]![n]!\prod_{l=0}^{m}(1-sin^2\theta(-q)^l)}sin^n\theta} \tag{29}$$

and

$$c < 0|0;l >= f_{0,2l} = \frac{(-1)^{2l}}{sin^{2l}\theta cos\theta\sqrt{[2l]!}} \tag{30}$$

Substituting this expression into equation 26, we find the results

$$W_{0m}(j) = \frac{\sqrt{[2j]!}}{cos\theta sin^{2(j-m)}\theta[2j-m]!\sqrt{[m]!}} \tag{31}$$

Acknowledgement

The author is thankful to CCAST (World Laboratory) for hospitality and support.

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