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R-MATRICES FOR SOME INHOMOGENEOUS QUANTUM GROUPS

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R-matrices for some inhomogeneous quantum groups

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Abstract

In this paper, some new generalized inhomogeneous quantum groups corresponding to the homogeneous multiparameter quantum groups $GL_{X,q_{ij}}(N)$ are constructed, furthermore, the R-matrices for these inhomogeneous quantum groups are found out.

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1. Introduction

The quantum groups are now widely used in the 2-dim conoformal field theory, 2-D gravity and 2-dim intergrable model so that they are being paid more and more attention. The concept of quantum groups originated from the quantum inverse scattering method(see ref.[1]-[2]) and in middle 1980s Drinfel'd found out the algebraic constructions associated with it are closely related to Hopf algebra. According to the viwpoints of Reshetikhin et.al[3], the quantum groups are the noncommutative function rings over the classical groups, i.e. they are either noncommutative or noncocommutative Hopf algbras while their realization spaces also have noncommutative geometric structure.

In group thoery it has been known how to construct a matrix representation of the usual undeformed inhomogeneous groups acting on the affine space with translation and rotation parts. If we take x^i as the coordinate functions of the translation part, and T^i_j as the matrix of homogeneous part acting on the translation vectors x^i, then the matrix representation of the corresponding inhomogeneous group is

$$T^I = \begin{pmatrix} T^i_j & x^i \\ 0 & 1 \end{pmatrix}. \tag{1.1}$$

In the undeformed case all these matrix entries commute with one another and generate the function algebra over the inhomogeneous group. But how about the quantum (deformed) groups?

Inhomogeneous quantum groups corresponding to homogeneous quantum groups

$U_q(N), SO_q(N)$ have been constructed in ref.[4], however, the relevant R-matrix has not been given out. In ref.[5] the $\mathcal{R}-$ matrices for the inhomogeneous quantum groups $IGL_q(N)$ has been discovered, at the same time the beautiful structure associated with these inhomogeneous quantum groups has been displayed by Castellani. In his approach the quantum group and its quantum plane are combined together, according to the relation

$$\mathcal{R}^{AB}{}_{EF}T^E_C T^F_D = T^B_F T^A_E \mathcal{R}^{EF}{}_{CD}, \tag{1.2}$$

where T^A_B is given in (1.1) and $\mathcal{R}$ is

$$\mathcal{R}^{AB}{}_{CD} = \begin{pmatrix} R^{ab}{}_{cd} & 0 & 0 & 0 \\ 0 & q^{-1} & q-q^{-1} & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{1.3}$$

Obviously, a new structure appears: the quantum plane is no longer the left or right module but the bimodule structure. In the quantum group case, the matrix entries which do not commute with one another generate the q-deformed Hopf algebra of functions over the inhomogeneous groups. Especially, in order to reflect the covariance of the generators under the homogeneous group the new Hopf algebra must be a left covariant bimodule of the original Hopf algebra which corresponds to the homogeneous quantum group (see ref.[4]). This nice inhomogeneous quantum group structure should have got important application, for example, one of them has been used to discuss the quantum Poincare group[6]. However the full meaning of the inhomogeneous quantum groups is not so clear for us. It is worthwhile to search for its deeper understanding.

In this paper we turn our attention to the multiparameter quantum groups $GL_{X,q_{ij}}(N)$. At first we construct new inhomogeneous quantum groups $IGL_{X,q_{ij}}(N)$ following the formalism in ref.[4] and [5]. Secondly, we construct a new type of inhomogeneous quantum groups $SIGL_{X,q_{ij}}(N)$, which is a little different from the first one. Thirdly, we introduced another type of inhomogeneous quantum groups $I^2GL_{X,q_{ij}}(N)$. All the $\mathcal{R}$-matrices for the inhomogeneous quantum groups above are given. We will find that the Hecke relation play an important role in our constructions. The new $\mathcal{R}$-matrices for these inhomogeneous quantum groups should satisfy the same Hecke relation as the original one.

In Sec.2, we briefly review the quantum groups $GL_{X,q_{ij}}(N)$ which has been discussed in ref.[7] and [8] in order to explain our notation. In Sec.3, three types of inhomogenous quantum groups are investigated. The corresponding $\mathcal{R}$-matrices are examined through the $RTT = TTR$ relation and the Yang- Baxter equation.

2. Multiparameter quantum groups $GL_{X,q_{ij}}$ and quantum plane

Let

$$(R_{X,q_{ij}})^{ij}{}_{kl} = \delta^i_k \delta^j_l (\delta^{ij} + \Theta^{ji}\frac{1}{q_{ij}} + \Theta^{ij}\frac{q_{ji}}{X}) + \delta^i_l \delta^j_k (1 - X^{-1}), \tag{2.1}$$

here

$$\Theta^{ij} = \begin{cases} 1, & if \quad i > j, \\ 0, & otherwise. \end{cases} \tag{1}$$

One can easily prove that this $R_{X,q_{ij}}$ satisfies the Yang- Baxer equation

$$R^{a_1b_1}{}_{a_2b_2}R^{a_2c_1}{}_{a_3c_2}R^{b_2c_2}{}_{b_3c_3} = R^{b_1c_1}{}_{b_2c_2}R^{a_1c_2}{}_{a_2c_3}R^{a_2b_2}{}_{a_3b_3}. \tag{2.2}$$

and if we take

$$q_{ij} = q, \quad (i < j), \quad X = q^2,$$

in (2.1), we will get the R-matrix for quantum groups $GL_q(N)$. Here, the Hecke relation for $\hat{R} = PR$ is to be

$$\hat{R}^2 = (1 - X^{-1})\hat{R} + X^{-1}I. \tag{2.3}$$

Associated with $R_{X,q_{ij}}$, we get the multiparameter quatum group $GL_{X,q_{ij}}(N)$. From relation

$$R^{ab}{}_{ef}t^e_c t^f_d = t^b_f t^a_e R^{ef}{}_{cd}, \tag{2.4}$$

for $i, j, k, l = 1, \cdots, n$, with $i < j, k < l$, we have

$$\begin{aligned} t^i_k t^i_l &= p_{kl} t^i_l t^i_k \\ t^i_k t^j_k &= q_{ij} t^j_k t^i_k \\ t^i_l t^j_k &= \frac{q_{ij}}{p_{kl}} t^j_k t^i_l \\ t^i_k t^j_l &= \frac{p_{kl}}{q_{ij}} t^j_l t^i_k + (p_{kl} - q_{kl}{}^{-1}) t^i_l t^j_k \end{aligned} \tag{2.5}$$

where $p_{ij} = \dfrac{X}{q_{ij}}$.

The projecting operators corresponding to quantum group $GL_{X,q_{ij}}(N)$ is the same as the ones for $GL_q(N)$, i.e.

$$\mathcal{A} = \frac{I - \hat{R}}{1 + X^{-1}}, \quad \mathcal{T} = \frac{\hat{R} + X^{-1}I}{1 + X - 1}, \tag{2.6}$$

and satisfy

$$\mathcal{A}^2 = \mathcal{A}, \quad \mathcal{T}^2 = \mathcal{T}, \tag{2.7}$$

$$\mathcal{A} + \mathcal{T} = 1, \quad \mathcal{A}\mathcal{T} = 0. \tag{2.8}$$

Quantum vector space and exterior quantum vector space that correspond to $\hat{R}_{X,q_{ij}}$ can be calculated directly, the result is

$$\mathcal{A}^{ij}{}_{kl}x^k x^l = 0 \Rightarrow x^i x^j = q_{ij} x^j x^i, \quad i < j, \tag{2.9}$$

$$\mathcal{T}^{ij}{}_{kl}\xi^i \xi^j = 0 \Rightarrow \begin{cases} \xi^i \xi^j = -p_{ij}{}^{-1}\xi^j \xi^i, & i < j \\ \xi^{i^2} = 0. \end{cases} \tag{2.10}$$

The determinant of the quantum group $GL_{X,q_{ij}}$ can be given

$$Det^{GL}(T) = \sum_{\sigma \in S_n} \left[\sum_{\substack{i<k \\ \sigma(i) > \sigma(k)}} (-p_{\sigma(k)\sigma(i)}) \right] t^1_{\sigma(1)} t^2_{\sigma(2)} \cdots t^n_{\sigma(n)}. \tag{2.11}$$

The relation between the matrix element and the determinant is no longer commutative and has been given out in ref.[7]:

$$t^i_k \mathcal{D} = \frac{\prod_{\alpha=1}^{k-1} q_{\alpha k}}{\prod_{\beta=1}^{i-1} q_{\beta i}} \frac{\prod_{\gamma=k+1}^{n} p_{k\gamma}}{\prod_{\delta=i+1}^{n} p_{i\delta}} \mathcal{D} t^i_k. \tag{2.12}$$

here $\mathcal{D}$=$Det^{GL}(T)$. In general, q_{ij} cannot be equal to $p_{ij}(\forall i, j)$ that makes the $\mathcal{D}$ be not the center of $GL_{X,q_{ij}}(N)$.

Definition 2.1:

$$GL_{X,q_{ij}}(N) := \frac{C[t^i_j, \mathcal{D}^{-1}]}{RT_1T_2 - T_1T_2R, \mathcal{D}\mathcal{D}^{-1} - 1, \mathcal{D}^{-1}\mathcal{D} - 1}$$

Theorem 2.2:

$GL_{X,q_{ij}}(N)$ is a Hopf algebra with coproduct Δ

$$\Delta(1) = 1 \otimes 1, \quad \Delta(t^i_j) = t^i_k \otimes t^k_j$$

counit ϵ

$$\epsilon(1) = 1, \quad \epsilon(t^i_j) = \delta^i_j$$

antipode S

$$S(t^i_j) = (T^{-1})^i_j$$

multiplication m on $GL_{X,q_{ij}}(N)$

$$m(t^i_j \otimes t^k_l) = t^i_j t^k_l$$

3. Inhomogeneous quantum groups

The concept of inhomogeneous quantum groups was first defined in ref.[4], and following the work in [4], the $\mathcal{R}$-matrix corresponding to $IGL_q(N)$ was given out obviously in ref.[5]. Our work is to construct the $\mathcal{R}$-matrix for multiparameter inhomogeneous quantum group $IGL_{X,q_{ij}}(N)$ at first, and then we introduce two new types of inhomogeneous quantum groups who are different from the first one.

3.1 The first type of inhomogeneous quantum groups

With the method used in [4] and [5], we can define the inhomogeneous quantum groups $IGL_{X,q_{ij}}(N)$ as follows.

Definition 3.1.1

The multiparamer inhomogeneous quantum group $IGL_{X,q_{ij}}(N)$ is the associative algebra A generated by

(1)the non-commuting matrix entries $T^A_B = (t^a_b, t^a_\cdot = x^a, t_b = 0, t_\cdot = 1)$:

$$T^A_B \equiv \begin{pmatrix} t^a_b & x^a \\ 0 & I \end{pmatrix}, \tag{3.1.1}$$

satisfying the following relations

$$R^{ab}{}_{ef} t^e_c t^f_d = t^b_f t^a_e R^{ef}{}_{cd}, \tag{3.1.2}$$

$$x^a t^b_c = X R^{ba}{}_{ef} t^e_c x^f, \tag{3.1.3}$$

$$\mathcal{A}^{ab}{}_{cd} x^c x^d = 0, \tag{3.1.4}$$

$\mathcal{A}$ being the projecting operator defined in (2.6), $R^{ab}{}_{ef}$ being the R-matrix for $GL_{X,q_{ij}}(N)$(see (2.1)).

(2) the inverse $\mathcal{D}^{-1}$ of the determinant $\mathcal{D} = Det^{GL}(T)$ defined by

$$\mathcal{D}^{-1}\mathcal{D} = \mathcal{D}\mathcal{D}^{-1} = I, \tag{3.1.5}$$

the $Det^{GL}(T)$ and the relation between $\mathcal{D}$ and t^i_j have been given out in eq.(2.11),(2.12).

The relations between new element x^a and $\mathcal{D}, \mathcal{D}^{-1}$ is

$$x^j \mathcal{D} = Y \mathcal{D} x^j, Y = X \cdot (\prod_{i=1}^{j-1} \frac{X}{q_{ij}}) \cdot (\prod_{i=j+1}^{n} q_{ji}), \tag{3.1.6}$$

$$x^j\mathcal{D}^{-1} = Y^{-1}\mathcal{D}^{-1}x^j, \tag{3.1.7}$$

which can be readily checked using the eq.(3.1.3). If we let $q_{ij} = X^{\frac{1}{2}}$, we find

$$x^j\mathcal{D} = X^{\frac{n+1}{2}}\mathcal{D}x^j, \tag{3.1.8}$$

Theorem 3.1.2

The algebra $\mathcal{A}$ is a Hopf algebra with coproduct Δ

$$\Delta(1) = 1 \otimes 1, \quad \Delta(T^A_B) = T^A_C \otimes T^C_B, \tag{3.1.9}$$

$$\Delta(\mathcal{D}) = \mathcal{D} \otimes \mathcal{D}, \quad \Delta(\mathcal{D}^{-1}) = \mathcal{D}^{-1} \otimes \mathcal{D}^{-1}, \tag{3.1.10}$$

counit ϵ

$$\epsilon(1) = 1, \quad \epsilon(T^A_B) = 1, \quad \epsilon(\mathcal{D}) = \epsilon(\mathcal{D}^{-1}) = 1, \tag{3.1.11}$$

and antipode S

$$S(T^A_B) = (T^{-1})^A_B, \quad S(\mathcal{D}) = \mathcal{D}^{-1}, \quad S(\mathcal{D}^{-1}) = \mathcal{D}, \tag{3.1.12}$$

where

$$(T^{-1})^A_B \equiv \begin{pmatrix} S(t^a_b) & -S(t^a_b)x^b \\ 0 & I \end{pmatrix}. \tag{3.1.13}$$

From (3.1.9), let $B = \cdot$, we have

$$\Delta(x^a) = t^a_b \otimes x^b + x^a \otimes I, \tag{3.1.14}$$

From (3.1.12),(3.1.13), we get

$$S(x^a) = -S(t^a_b)x^b, \tag{3.1.15}$$

From these relations, we can easily check that

$$\Delta(x^a\mathcal{D} - Y\mathcal{D}x^a) = 0, \quad \Delta(x^a\mathcal{D}^{-1} - Y^{-1}\mathcal{D}^{-1}x^a) = 0.$$

The $\mathcal{R}$ matrix for $IGL_{X,q_{ij}}(N)$ can be constructed

$$\mathcal{R}^{AB}{}_{CD} = \begin{pmatrix} R^{ab}{}_{cd} & 0 & 0 & 0 \\ 0 & X^{-1} & 1-X^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.1.16}$$

so that it satisfy all the relations for a R-matrix:

(1) $RTT = TTR$ relation

$$\mathcal{R}^{AB}{}_{EF}T^E_C T^F_D = T^B_F T^A_E \mathcal{R}^{EF}{}_{CD}, \tag{3.1.17}$$

can reproduce the relations (3.1.2),(3.1.3) and (3.1.4). If let $A = a, B = b, C = \cdot, D = d$, we have

$$R^{ab}{}_{ef}x^e t^f_d = (1-X^{-1})x^b t^a_d + t^b_d x^a, \tag{3.1.18}$$

which is equivalent to (3.1.3) and the Hecke relation satified by $R^{ab}{}_{cd}$.

(2) Hecke relation

The Hecke relation for the R-matrix which corresponds to $GL_{X,q_{ij}}(N)$ has been given out in (2.3), the same relation must be satisfied by $\mathcal{R}$

$$\hat{\mathcal{R}}^2 = (1-X^{-1})\hat{\mathcal{R}} + X^{-1}I, \tag{3.1.19}$$

or in another form

$$\hat{\mathcal{R}} = (1-X^{-1})I + \hat{\mathcal{R}}^{-1}, \tag{3.1.20}$$

(3) Yang-Baxer equation

$$\mathcal{R}^{A_1B_1}{}_{A_2B_2}\mathcal{R}^{A_2C_1}{}_{A_3C_2}\mathcal{R}^{B_2C_2}{}_{B_3C_2} = \mathcal{R}^{B_1C_1}{}_{B_2C_2}\mathcal{R}^{A_1C_2}{}_{A_2C_3}\mathcal{R}^{A_2B_2}{}_{A_3B_3}. \tag{3.1.21}$$

It is easy to verify that $\mathcal{R}$ is a solution of above equation.

Using a concrete formula to define $IGL_{X,q_{ij}}(N)$, we get

$$IGL_{X,q_i}(N) := \frac{C[T^A_B, \mathcal{D}^{-1}]}{\mathcal{R}T_1T_2 - T_2T_1\mathcal{R}, \mathcal{D}\mathcal{D}^{-1} - 1, \mathcal{D}^{-1}\mathcal{D} - 1} \tag{3.1.22}$$

3.2 The second type of inhomogeneous quantum groups

In subsection 3.1, we have constructed a kind of inhomogeneous quantum groups using the method in ref.[4]. Where we let x^a be the coordinate functions of the translation part, t^a_b be the matrix of the homogeneous part acting on the translation vectors so that we get the matrix representation of the inhomogeneous quantum group:

$$T^A_B = \begin{pmatrix} t^a_b & x^a \\ 0 & 1 \end{pmatrix}. \tag{3.2.1}$$

Here we let the exterior quatum vecter ξ^a be the coodinate functions of the translation part taking place of x^a, we get the matrix representation of the new type of inhomogeneous quantum group:

$$\dot{T}^A_B = \begin{pmatrix} t^a_b & \xi^a \\ 0 & 1 \end{pmatrix}, \tag{3.2.2}$$

i.e.

$$\dot{T}^A_B = (t^a_b, \dot{T}^a_{\cdot} = \xi^a, \dot{T}^{\cdot}_b = 0, \dot{T}^{\cdot}_{\cdot} = 1).$$

We demand that the $\dot{\mathcal{R}}$ corresponding to this new inhomogeneous quantum group $SIGL_{X,q_{ij}}(N)$ should satisfy the Hecke relation (3.1.19), YBE (3.1.20). And from its

$RTT = TTR$ relation we must get

$$\mathcal{T}^{ab}{}_{cd}\xi^c\xi^d = 0, \tag{3.2.3}$$

where $\mathcal{T}$ is defined in (2.6).

The $\dot{\mathcal{R}}$-matrix can be constructed as follows

$$(\dot{\mathcal{R}}_{X,q_{ij}})^{AB}{}_{CD} = \begin{pmatrix} (R_{X,q_{ij}})^{ab}{}_{cd} & 0 & 0 & 0 \\ 0 & -1 & 1-X^{-1} & 0 \\ 0 & 0 & -X^{-1} & 0 \\ 0 & 0 & 0 & -X^{-1} \end{pmatrix}, \tag{3.2.4}$$

From $RTT = TTR$ relation, we have

$$\xi^e t^f_d = -(R_{x,q_{ij}})^{fe}{}_{ba} t^b_d \xi^a, \tag{3.2.5}$$

such that we can calculate the commuting relation between ξ^a and $Det^{SIGL}(T)$

$$\xi^j\dot{\mathcal{D}} = Z\dot{\mathcal{D}}\xi^j, \tag{3.2.6}$$

here we let $\dot{\mathcal{D}} = Det^{SIGL}(T) = Det^{GL}(T)$ and

$$Z = (-1)^n \cdot (\prod_{i=1}^{j-1}\frac{1}{q_{ij}}) \cdot (\prod_{i=j+1}^{n}\frac{q_{ji}}{X}), \tag{3.2.7}$$

the inverse $\dot{\mathcal{D}}^{-1}$ of determinant $\dot{\mathcal{D}}$ can be defined as in (3.1.5), then we get

$$x^j\dot{\mathcal{D}}^{-1} = Z^{-1}\dot{\mathcal{D}}^{-1}x^j, \tag{3.2.8}$$

Definition 3.2.1:

The inhomogeneous quantum groups $SIGL_{X,q_{ij}}(N)$ is defined as

$$SIGL_{X,q_{ij}}(T) := \frac{C[\dot{T}^A_B, \dot{\mathcal{D}}^{-1}]}{\dot{\mathcal{R}}\dot{T}_1\dot{T}_2 - \dot{T}_2\dot{T}_1\dot{\mathcal{R}}, \dot{\mathcal{D}}\dot{\mathcal{D}}^{-1} - 1, \dot{\mathcal{D}}^{-1}\dot{\mathcal{D}} - 1}. \tag{3.2.9}$$

Theorem 3.2.2:

The inhomogeneous quantum group $SIGL_{X,q_{ij}}(T)$ is a Hopf algebra with coproduct Δ

$$\Delta(1) = 1 \otimes 1, \quad \Delta(\dot{T}^A_B) = \dot{T}^A_C \otimes \dot{T}^C_B, \tag{3.2.10}$$

counit ϵ

$$\epsilon(\dot{T}^A_B) = \delta^A_B, \tag{3.2.11}$$

antipode S

$$S(\dot{T}^A_B) = (\dot{T}^{-1})^A_B = \begin{pmatrix} S(t^a_b) & -S(t^a_b)\xi^b \\ 0 & 1 \end{pmatrix}, \tag{3.2.12}$$

here $A = (a, \cdot), B = (b, \cdot)$.

The proof is obvious.

3.3 The third type of inhomogenous quantum groups

We have discussed two kinds of inhomogeneous quantum groups $IGL_{X,q_{ij}}(N)$ and $SIGL_{X,q_{ij}}(N)$ in above sections, and now we begin to investigate another kind of inhomogeneous quantum groups. We can generalize the idea we used above: we add another translation part on (3.1.1), then we get the matrix representation of new inhomogeneous quantum group:

$$\bar{T}^A_B = \begin{pmatrix} t^a_b & x^a & y^a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.3.1}$$

here $\bar{T}^A_B = (t^a_b, t^a_\cdot = x^a, t^a_\cdot = y^a, t^\cdot_a = t^\cdot_\cdot = 0, t^\cdot_\cdot = t^\cdot_\cdot = 1, t^\cdot_\cdot = t^\cdot_\cdot = 0)$

The relations between the matrix elements are

$$R^{ab}{}_{ef} t^e_c t^f_d = t^b_f t^a_e R^{ef}{}_{cd}, \tag{3.3.2}$$

$$x^a t^b_c = X R^{ba}{}_{ef} t^e_c x^f, \tag{3.3.3}$$

$$y^a t^b_c = X R^{ba}{}_{ef} t^e_c y^f, \tag{3.3.4}$$

$$\mathcal{A}^{ab}{}_{cd} x^c x^d = 0, \tag{3.3.5}$$

$$\mathcal{A}^{ab}{}_{cd} y^c y^d = 0, \tag{3.3.6}$$

$$R^{ab}{}_{ef} x^e y^f = X^{-1} y^b x^a + (1 - X^{-1}) x^b y^a, \tag{3.3.7}$$

$$R^{ab}{}_{ef} y^e x^f = x^b y^a, \tag{3.3.8}$$

here R being the R-matrix for quantum group $GL_{X,q_{ij}}(N)$(see (2.1)) and $\mathcal{A}$ is the projecting operator of $GL_{X,q_{ij}}(N)$

$$\mathcal{A} = \frac{I - \hat{R}}{1 + X^{-1}}, \tag{3.3.9}$$

We can construct the $\bar{\mathcal{R}}$-matrix corresponding to this new inhomogeneous quan-

tum groups $I^2GL_{X,q_{ij}}(N)$ as follows

$$\bar{\mathcal{R}}^{AB}{}_{CD} = \begin{pmatrix} R^{ab}{}_{cd} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X^{-1} & 1-X^{-1} & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-X^{-1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X^{-1} & 1-X^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.3.10)$$

It is not hard to check that the $\bar{\mathcal{R}}$ satisfy the $RTT = TTR$ relation, Hecke relation, and the Yang-Baxer equation. For example, the $RTT = TTR$ relation is nothing but the formulae (3.3.2-8). One can find similar relation in ref.[3] and [9] in which it is used to discuss the quantum coset space and quantum Minkowski space. We will give the exact relation between quantum coset space, quantum Minkowski space, and our construction in the forthcoming paper.

It is easy to find out that the determinant and the relation between the determinant and the group elements are not changed, one can see them in (2.11), (2.12),(3.1.6),(3.1.7).

Definition 3.3.1:

The inhomogeneous quantum groups $I^2GL_{X,q_{ij}}(N)$ is defined as

$$I^2GL_{X,q_{ij}}(N) := \frac{C[\bar{T}^A_B, \mathcal{D}^{-1}]}{\bar{\mathcal{R}}\bar{T}_1\bar{T}_2 - \bar{T}_2\bar{T}_1\bar{\mathcal{R}}, \mathcal{D}\mathcal{D}^{-1}-1, \mathcal{D}^{-1}\mathcal{D}-1}. \quad (3.3.11)$$

Theorem 3.3.2

The inhomogeneous quantum group $I^2GL_{X,q_{ij}}(N)$ is a Hopf algebra with coproduct Δ

$$\Delta(\bar{T}^A_B) = \bar{T}^A_C \otimes \bar{T}^C_B, \quad (3.3.12)$$

counit ϵ

$$\epsilon(\bar{T}^A_B) = \delta^A_B, \quad (3.3.13)$$

antipode S

$$S(\bar{T}^A_B) = (\bar{T}^{-1})^A_B, \quad (3.3.14)$$

where

$$(\bar{T}^{-1})^A_B = \begin{pmatrix} S(t^a_b) & -S(t^a_b)x^b & -S(t^a_b)y^b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.3.15)$$

We can also consider the matrix representation of a new inhomogeneous quantum group just like

$$\mathrm{T}^A_B = \begin{pmatrix} t^a_b & \xi^a & \eta^a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.3.16)$$

The relations among the matrix entries change into

$$R^{ab}{}_{ef}t^e_c t^f_d = t^b_f t^a_e R^{ef}{}_{cd}, \quad (3.3.17)$$

$$\xi^e t^f_d = -R^{fe}{}_{ba} t^b_d \xi^a, \tag{3.3.18}$$

$$\eta^e t^f_d = -R^{fe}{}_{ba} t^b_d \eta^a, \tag{3.3.19}$$

$$T^{ab}{}_{cd} \xi^c \xi^d = 0, \tag{3.3.20}$$

$$T^{ab}{}_{cd} \eta^c \eta^d = 0, \tag{3.3.21}$$

$$R^{ab}{}_{ef} \xi^e \eta^f = X^{-1} \eta^b \xi^a + (1 - X^{-1}) \xi^b \eta^a, \tag{3.3.22}$$

$$R^{ab}{}_{ef} \eta^e \xi^f = \xi^b \eta^a, \tag{3.3.23}$$

here R is still the R-matrix for $GL_{X,q_{ij}}(N)$, and T is the projector we defined in (2.6)

$$T = \frac{\hat{R} + X^{-1} I}{1 + X^{-1}}, \tag{3.3.24}$$

Then we can give out the R-matrix for this new inhomogeneous quantum group $\tilde{I}^2 GL_{X,q_{ij}}(N)$:

$$\mathbb{R}^{AB}{}_{CD} = \begin{pmatrix} R^{ab}{}_{cd} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1-X^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -X^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -X^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-X^{-1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1-X^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -X^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -X^{-1} \end{pmatrix}, \tag{3.3.25}$$

The discussions about determinant, $RTT = TTR$ relation and the Yang-Baxer equation in this case is slmilar to the previous ones we have given.

We have discussed three types of inhomogeneous quantum groups corresponding to homogeneous quantum group $GL_{X,q_{ij}}(N)$ and give out the relevent R-matrices. What about the physical meaning of these results, we will discuss later.

References

[1] E.K.Sklyanin, L.A.Takhtadzhyan, and L.D.Faddeev, Teoret.Mat.Fiz.40(1979) 194.

[2] L.D.Faddeev, *Intergrable models in 1+1-dimensional quantum field theory* (Les Houches Lectures 1982), Elsevier, Amsterdam.1984.

[3] N.Yu. Reshetikhin, L.A. Takhtajan and L.D. Faddeev, Leningrad Math. J. 1(1990)193.

[4] M.Schlieker et.al, Z.Phys.C53(1992)79.

[5] L.Castellani, Phys.Lett B 298(1993)335.

[6] L.Castellani, On the quantum Poincare group.Torino preprint DFTT-57/92.

[7] A.Schirrmacher, Z.Phys.C50(1991)321.

[8] A.Schirrmacher, J.Phys.A 24(1991)L1249.

[9] S.A.Merkulov, Z.Phys.C52(1991)583.