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TEMPERLEY-LIEB-JONES ALGEBRA APPROACH TO HEISENBERG SPIN MODELS

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Temperley-Lieb-Jones Algebra Approach to Heisenberg Spin Models [1]

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Abstract

We present a method in dealing with the Heisenberg spin$\frac{1}{2}$ models in one and higher dimensions by Temperley-Lieb-Jones algebraic structures. It is shown that the model in higher dimensions can always be reduced into the one in one dimension with some non-nearest interactions and Temperley-Lieb-Jones algebraic structures for the isotropic case. Some examples are given with detailed calculations.

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1 Heisenberg Spin Models and Bi-algebras

1.1 Lie Bi-algebra $su(2)$

For completeness and facilitating the forthcoming discussions, we choose to start our discussion from the very beginning of Lie bi-algebra $su(2)$.

Lie algebra $su(2)$ is spanned by $\{J^{\pm}, J^3\}$ and defined by the following commutation relations

$$[J^+, J^-] = 2J^3 , \quad [J^3, J^{\pm}] = \pm J^{\pm} , \tag{1}$$

and for the convenience of the following description, we supplement it with unity 1. The center of this algebra is therefore spanned by 1 and the Casimir element $C = J^+J^- + J^3(J^3 - 1)$.

The algebra structure is built with two maps, the unit map u and the map of

multiplication m,

$$\begin{aligned} m: \quad & su(2)\otimes su(2) \longrightarrow su(2) \\ u: \quad & \mathbf{C} \longrightarrow su(2) \end{aligned} \tag{2}$$

and m is associative.

A co-algebra structure can also be defined based on the co-unit map ϵ and the map of co-multiplication Δ, which are algebra homomorphisms, i.e., $\Delta(ab) = \Delta(a)\Delta(b)$, $\epsilon(ab) = \epsilon(a)\epsilon(b)$, $\forall a, b \in su(2)$. The maps turn the algebra into its tensor representation as shown below

$$\begin{aligned} \Delta: \quad & su(2) \longrightarrow su(2)\otimes su(2) \\ \epsilon: \quad & su(2) \longrightarrow \mathbf{C} \end{aligned} \tag{3}$$

and Δ is associative, i.e., $(\Delta\otimes id)\Delta(a) = (id\otimes\Delta)\Delta(a)$, $\forall a \in SU(2)$. The actions of Δ and ϵ on generators $J^{\pm,3}$ are explicitly

$$\begin{aligned} \Delta(J^{\pm,3}) &= J^{\pm,3}\otimes \mathbf{1} + \mathbf{1}\otimes J^{\pm,3}\,, \\ \epsilon(J^{\pm,3}) &= 1\,. \end{aligned} \tag{4}$$

As algebra $su(2)$ has the co-algebra structure, it is a Lie bi-algebra.

The Casimir element of this Lie bi-algebra is the diagonalization of C, often called the (1-st rank) co-Casimir, which reads,

$$\begin{aligned} \Delta(C) &= J^+J^-\otimes\mathbf{1} + \mathbf{1}\otimes J^+J^- + J^+\otimes J^- + J^-\otimes J^+ \\ &+\mathbf{1}\otimes(J^3)^2 + (J^3)^2\otimes\mathbf{1} + 2J^3\otimes J^3 - \mathbf{1}\otimes J^3 - J^3\otimes\mathbf{1} \end{aligned} \tag{5}$$

1.2 Permutation Algebra and Isotropic Chain

For the case of spin-$\frac{1}{2}$ representation, formula (5) gives rise to

$$\Delta(C) = \sigma^+\otimes\sigma^- + \sigma^-\otimes\sigma^+ + \frac{1}{2}\sigma^3\otimes\sigma^3 \tag{6}$$

where $\sigma^{\pm,3}$ are Pauli matrices. Therefore the operator

$$E = \frac{1}{2}\cdot\mathbf{1}\otimes\mathbf{1} - \Delta(C) = \begin{pmatrix} 0 & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & 0 \end{pmatrix} \tag{7}$$

is in the center of the Lie bi-algebra $su(2)\otimes su(2)$. Hence it is true that

$$e_i = \mathbf{1}_1\otimes\mathbf{1}_2\otimes\cdots\otimes\mathbf{1}_{i-1}\otimes(E)_{i,i+1}\otimes\mathbf{1}_{i+2}\otimes\cdots\otimes\mathbf{1}_{N+1} \tag{8}$$

(where $i = 1, 2, \cdots N$) is $su(2)$-invariant, i.e.,

$$[e_i, \bullet] = 0\,, \quad \forall\ \bullet\in\ su(2)^{\otimes N}. \tag{9}$$

It is interesting that e_i are elements of the N-dimensional permutation group algebra P_N satisfying

$$\begin{aligned} & e_i^2 = 2e_i \\ & e_ie_{i\pm1}e_i = e_i \\ & [e_i, e_j] = 0\,, \quad |i-j|\geq 2 \end{aligned} \tag{10}$$

which is a special case of Temperley-Lie-Jones (T-L-J) algebra[1]. Therefore the Hamiltonian for spin-$\frac{1}{2}$ Heisenberg chain

$$\begin{aligned} H_{XXX} &= G\sum_{i=1}^{N} e_i = G\sum_{i=1}^{N}\left(\Delta\left(\frac{1}{2}-C\right)\right)^{(i,i+1)}\,, \\ &= G\left\{-\sum_{i=1}^{N}\left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{1}{2}\sigma_i^3\sigma_{i+1}^3\right) + \frac{1}{2}N\right\}\,. \end{aligned} \tag{11}$$

is invariant under the action of Lie bi-algebra $su(2)$, therefore $su(2)$ is the *hidden symmetry* of this system. When $G > 0$ the system is ferromagnetic and when $G < 0$ it is antiferromagnetic.

1.3 Q-deformed Bi-algebra $su_q(2)$

The q-deformed algebra $su_q(2)$ is generated by $\{J^{\pm,3}, \mathbf{1}\}$ [2]:

$$[J^+, J^-] = [2J^3]_q\,, \quad [J^3, J^\pm] = \pm J^\pm\,, \tag{12}$$

where $[x]_q = \dfrac{q^x - q^{-x}}{q - q^{-1}}$, and the center is spanned by $\{1,\ C_q\}$ and C_q is the Casimir element

$$C_q = J^+J^- + \left[J^3 - \frac{1}{2}\right]_q^2 - \left[\frac{1}{2}\right]_q^2\,, \tag{13}$$

satisfying

$$[C_q\,,\ \bullet] = 0\,, \quad \forall\ \bullet\in\ su_q(2)\,. \tag{14}$$

Recall that the co-multiplication map of the Hopf $su_q(2)$,

$$\Delta: su_q(2) \to su_q(2)\otimes su_q(2) \tag{15}$$

is defined explicitly by

$$\begin{aligned} \Delta(J^3) &= J^3\otimes\mathbf{1} + \mathbf{1}\otimes J^3\,, \\ \Delta(J^\pm) &= J^\pm\otimes q^{J^3} + q^{-J^3}\otimes J^\pm\,, \\ \Delta(1) &= \mathbf{1}\otimes\mathbf{1}\,, \end{aligned} \tag{16}$$

which is algebra homomorphism, i.e., $\Delta(ab) = \Delta(a)\Delta(b)$, $\forall a,\ b\in su_q(2)$. When $q\to 1$, algebra $su_q(2)$ becomes $su(2)$ and the Casimir $C_q \to C = J^+J^- + \left(J^3 - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2$.

It is a direct result that the comultiplication of Casimir element is the center of the tensorized algebra $su_q(2)\otimes su_q(2)$, i.e.,

$$\left[\Delta(C_q)\ ,\ \Delta(J^3)\right] = \left[\Delta(C_q)\ ,\ \Delta(J^\pm)\right] = 0\ . \tag{17}$$

As we are interested in the spin-$\frac{1}{2}$ representation, we take

$$J^\pm = \sigma^\pm\ ,\quad J^3 = \frac{1}{2}\sigma^3\ , \tag{18}$$

where $\sigma^{\pm,3}$ are Pauli matrices (note that the spin-$\frac{1}{2}$ representation should be taken after the comultiplication operation, or one will get incorrect results). A simple calculation leads to

$$\begin{aligned}\Delta\left(C_q'\right) &= \frac{q+q^{-1}}{4}1\otimes 1 - \left[\sigma^+\otimes\sigma^- + \sigma^-\otimes\sigma^+ + \frac{q+q^{-1}}{4}\sigma^3\otimes\sigma^3\right.\\ &\quad \left. -\frac{q-q^{-1}}{4}\left(\sigma^3\otimes 1 - 1\otimes\sigma^3\right)\right]\\ &= \begin{pmatrix} 0 & & & \\ & q & -1 & \\ & -1 & q^{-1} & \\ & & & 0\end{pmatrix}\ ,\end{aligned} \tag{19}$$

where $C_q' = \left(q+q^{-1}\right)1 - C_q$ is a shifted Casimir.

1.4 T-L-J Algebra and Anisotropic Chain

The comultiplication of C_q' (co-Casimir) given in (19) is just the e matrix in [2] and the E matrix in [4] in constructing the T-L-J algebra elements e_i (with $i = 1, 2, \cdots, N$) , i.e,

$$e_i = 1^{(1)}\otimes 1^{(2)}\otimes\cdots\otimes 1^{(i-1)}\otimes\Delta\left(C_q'\right)^{(i,i+1)}\otimes 1^{(i+2)}\otimes\cdots\otimes 1^{(N+1)}\ , \tag{20}$$

and will be denoted simply

$$e_i = \Delta\left(C_q'\right)^{(i,i+1)}\ . \tag{21}$$

These elements act on the tensorized space

$$V = V_{1/2}^{(1)}\otimes V_{1/2}^{(2)}\otimes\cdots\otimes V_{1/2}^{(i)}\otimes\cdots V_{1/2}^{(N+1)}\ , \tag{22}$$

where spaces $V_{1/2}$ are spin-$\frac{1}{2}$ representation space of $su_q(2)$, the indices 1, 2, $\cdots$, i, $\cdots$, $N+1$ denote their positions. It is easy to check that e_i satisfy the TLJ algebraic relations

$$\begin{aligned} &e_i^2 = [2]_q e_i\ ,\\ &e_i e_{i\pm 1} e_i = e_i\ ,\\ &e_i e_j = e_j e_i\ ,\quad \text{if}\ \ |i-j|\geq 2\ .\end{aligned} \tag{23}$$

Without referring to the braid group, we can see that e_i is commutable with the N-fold comultiplication of $su_q(2)$,

$$\left[e_i\ ,\ \Delta^{(N)}\left(J^{\pm,3}\right)\right] = 0\ , \tag{24}$$

where

$$\begin{aligned}\Delta^{(N)}(J^\pm) &= \sum_{\ell=1}^{N+1} q^{-J^3}\otimes\cdots\otimes q^{-J^3}\otimes J_{(\ell)}^\pm\otimes q^{J^3}\otimes\cdots\otimes q^{J^3}\ ,\\ \Delta^{(N)}(J^3) &= \sum_{\ell=1}^{N+1} 1\otimes\cdots\otimes 1\otimes J_{(\ell)}^3\otimes 1\otimes\cdots\otimes 1\end{aligned} \tag{25}$$

and $J_{(\ell)}^{\pm,3}$ in above equations are located in the ℓ-th position, acting on the ℓ-th spin space.

The Hamiltonian for the anisotropic spin chain is

$$H_{\mathtt{XXZ}}^{\mathtt{1D}} = G\sum_{i=1}^N e_i = G\sum_{i=1}^N\left(\Delta\left(C_q'\right)\right)^{(i,i+1)}\ , \tag{26}$$

which is obviously $su_q(2)$-invariant, i.e.,

$$\left[H_{\mathtt{XXZ}}^{\mathtt{1D}}\ ,\ \bullet\right] = 0\ ,\quad \forall\,\bullet\in su_q(2)^{\otimes N}\ . \tag{27}$$

The Hamiltonian reads explicitly

$$H_{\mathtt{XXZ}}^{\mathtt{1D}} = G\left\{-\sum_{i=1}^N\left[\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{[2]_q}{4}\sigma_i^3\sigma_{i+1}^3\right] + \frac{[2]_q}{2}N1 + \frac{q-q^{-1}}{4}\left(\sigma_1^3 - \sigma_{N-1}^3\right)\right\}\ , \tag{28}$$

which is nothing but the Heisenberg spin chain of XXZ type with boundary terms. The solving of this system can be reduced into simple algebraic calculations using the representations of the TLJ algebra [5], or those of the quantum algebra $su_q(2)$ [4].

2 Structures of T-L-J Algebra

We denote $\mathcal{A}_N$ the T-L-J algebra, which is generated by 1 and e_i with $i = 1, 2, \cdots, N-1$. As is given in the last section, the elements satisfy the relations (23).

A left ideal (denoted I) is a collection of elements such that for any element a of $\mathcal{A}_N$ and i of I, $a\cdot i\in I$. If I has no smaller subcollection, I is called a primitive left ideal. Generally, $a = e_{i_1}e_{i_2}\cdots e_{i_n}$ and is called a word, and 1 is called an empty word. If the length of a can not be reduced by applying (23), a is called a reduced word. Any word is proportional to a reduced word. And the reduced words are called equivalent if they are equal under the application of the third commutation relation of (23), and the set of reduced words span the algebra $\mathcal{A}_N$. The dimensional of $\mathcal{A}_N$ is the number of reduced words, i.e.,

$$\dim(\mathcal{A}_N) = \frac{1}{N+1}\left[\begin{array}{c}2N\\ N\end{array}\right]\ . \tag{29}$$

2.1 Young Operator & Basis of Solution Spaces

There is a natural relationship between the primitive left ideals of $\mathcal{A}_N$ and the irreducible representations of $su_q(2)$, in analogue with the primitive left ideals of permutation algebra and the irreducible representations of $su(2)$. As is well known, the Young operator of a Lie algebra is an idempotent of the primitive left ideals of the permutation algebra. For a quantum Lie algebra, a q-analogous Young operator is constructed [4], and it can be applied in the decomposition of the tensor representations of a quantum Lie algebra. For the spin-$\frac{1}{2}$ chain, we need the Young operator Y_n corresponding to Young pattern of the following type, and explicitly we have

$$Y_n = e_1 e_2 \cdots e_{2n-1} Z_n^N \,, \tag{30}$$

where Z_n^N is composed with $e_{2n+1}, \cdots, e_{N-1}$, and for any e_j such that $j > 2n$ Z_n^N is effectively zero, i.e.,

$$e_j Z_n^N = Z_n^N e_j = 0 \,. \tag{31}$$

Therefore the independent basis of the left ideal should be

$${}^n\xi_{(m)} \triangleq {}^n\xi_{m_1 m_2 \cdots m_n} = C_{m_1 m_2 \cdots m_n} Z_n^N \,, \tag{32}$$

where

$$C_{(m)} \triangleq C_{m_1 m_2 \cdots m_n} = C_1^{m_1} C_3^{m_2} \cdots C_{2n-1}^{m_n} \,, \tag{33}$$

and

$$C_i^{m_i} = e_{m_i} e_{m_i - 1} \cdots e_i \,, \quad 1 \le m_1 < m_2 < \cdots m_n \le N-1 \,, \quad m_i \ge 2i - 1 \,. \tag{34}$$

Different Young patterns describe different irreducible representations. For a certain Young pattern, e.g, the one in Fig. 2.1, the number of irreducible representations is

$$M_n = \begin{bmatrix} N-1 \\ n \end{bmatrix} - \begin{bmatrix} N-1 \\ n-2 \end{bmatrix} \,, \tag{35}$$

which is just the number of independent basis of the primitive left ideal.

1	3	$\cdots$	$2n-1$	$2n+1$	$\cdots$	N
2	4	$\cdots$	$2n$			

Figure 1: *Young pattern* $\mathrm{Y}(N-n, n)$.

2.2 General Scheme for Solution

The solutions of a spin system with its Hamiltonian expressed by the elements of $\mathcal{A}_N$ can be expressed as linear combinations of the independent basis of the primitive left ideal of $\mathcal{A}_N$. Let $|x_1\ x_2\ \cdots\ x_n\rangle$, $1 \le x_1 < x_2 < \cdots < x_n \le N+1$ denote the state with n down spins and x_is denote the locations of the down spins of the chain. The bases for n down spin states with the highest weight j are

$${}^n\phi_{(m)} \triangleq {}^n\xi_{(m)} |1\ 3\ 5\ \cdots\ (2n-1)\rangle = {}^nC_{(m)} Z_n^N |1\ 3\ 5\ \cdots\ (2n-1)\rangle, \tag{36}$$

where $n \le \frac{(N+1)}{2}$ and

$${}^nC_{(m)} = C_1^{m_1} C_3^{m_2} \cdots C_{2n-1}^{m_n}, \quad 1 \le m_1 < m_2 < \cdots < m_n \le N, \quad m_i \ge (2n-1) \tag{37}$$

and

$$C_i^m = e_m e_{m-1} \cdots e_i, \quad j = \frac{1}{2}(N - 2n - 1). \tag{38}$$

Z_n^N is a polynomial of e_i's with the property $e_k Z_n^N = 0$ for $k \ge 2n$. The action of Z_n^N on $|1\ 3\ 5\ \cdots\ (2n-1)\rangle$ gives rise to a constant factor. Therefore the bases of eigenstates for n spins down states are simply

$${}^n\Phi_{(m)} = {}^nC_{(m)} |1\ 3\ 5\ \cdots\ (2n-1)\rangle \,. \tag{39}$$

The eigenstates of the Hamiltonian are the combinations of ${}^n\Phi_{(m)}$ with the same n,

$${}^n\Psi = \sum_{(m)} a_{(m)}\, {}^n\Phi_{(m)}. \tag{40}$$

For every $n \le (N+1)/2$, there are $\begin{bmatrix} N \\ n \end{bmatrix} - \begin{bmatrix} N \\ n-2 \end{bmatrix}$ eigenvalues of energy ${}^nE$ from Hamiltonian equation

$$H\, {}^n\Psi = {}^nE\, {}^n\Psi. \tag{41}$$

The degenerate states with respect to every eigenvalue can be obtained by acting the total spin operator $S_{tot}^- = \Delta^{(N-1)}(J^-)$ on the corresponding eigenstate of the highest weight. All the eigenvalues and eigenstates can be exactly obtained in terms of the above recipe.

The actions of the elements of $\mathcal{A}_N$ on ${}^n\xi_{(m)}$ or ${}^n\Phi_{(m)}$ can be easily automated. Because we freeze the positions of $C(m_i)$ in the expression of ${}^n\xi_{(m)}$ as $\mathtt{C}(m_1)\mathtt{C}(m_2)\cdots\mathtt{C}(m_n)\mathtt{Z}()$, and $\mathtt{C}(m_i) = C_i^{m_i}$, we need not specify i and the indices of operator $\mathtt{Z}()$. In the following, we give the codes in REDUCE for the actions of e_i (encoded $\mathtt{e}(i)$) on $C_i^{m_i}$.

PROCEDURE C

- `operator e,C,Z; noncom e,C,Z;`
- `for all x such that numberp x let e(x)*Z()=0;`

- `for all x,y such that numberp x and numberp y and x-y=-1 or x-y=1`
  `let e(x)*C(y)=C(x);`
- `for all x,y such that numberp x and numberp y and x-y=0`
  `let e(x)*C(y)=2*C(x);`
- `for all x,y such that numberp x and numberp y and x-y>1`
  `let e(x)*C(y)=C(y)*e(x);`
- `for all x,y such that numberp x and numberp y and x-y>-1`
  `let C(x)*C(y)=C(y-2)*C(x);`
- `for all x,y such that numberp x and numberp y and x-y<-1`
  `let e(x)*C(y)=C(x)*D(x,y);`
- `PROCEDURE D(x,y); FOR i:=(x+2):y PRODUCT e(i)$`

2.3 Example: Solutions for $n = 1$ to XXZ Open Chain

Now to show the way to compute the solutions, we turn to a simple example, the open XXZ spin chain, with the Hamiltonian given in (28). We will specify our discussion to n=1 and arbitrary number $(N+1)$ of spins. The eigenstates are simply of the form

$$ {}^1\Psi = \sum_{m=1}^{N} {}^1\Phi_m = \sum_{m=1}^{N} a_m C_1^m |1\rangle \ , \tag{42} $$

as depicted in Fig.2.3. Applying the method encoded in the above, we get the reduction equation

$$ a_{m+1} + ([2]_q - E)\, a_m + a_{m-1} = 0, \quad m = 1, 2, \cdots, N \ . \tag{43} $$

Where

$$ a_0 = 0, \ a_{N+1} = 0 \ . \tag{44} $$

It can be easily solved and we give the solutions explicitly

$$ \begin{aligned} a_m &= (-)^{m-1} \frac{\sin\left(\frac{mk\pi}{N+1}\right)}{\sin\left(\frac{k\pi}{N+1}\right)} \ , \\ E_k &= 2\left(\frac{[2]_q}{2} - \cos\frac{k\pi}{N+1}\right) \ , \quad k = 0, 1, 2, \cdots, N \ . \end{aligned} \tag{45} $$

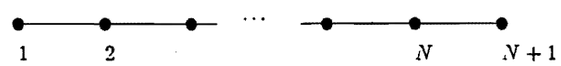


Figure 2:

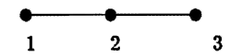


Figure 3:

For each k, there is an energy E_k and a set of coefficients a_m that gives rise to an eigenstate (42) of the highest weight state. The degenerate states with lower weights are obtained by the actions of the lowering operator of the symmetry algebra $su_q(2)$, $J_{tot}^- = \Delta^{(N)}(J_-)$, on the highest weight state.

The solutions of the case with one spin down in the chain, $n = 1$, corresponds to the eigenstates and energies of the ground and lower exciting states of the ferromagnetic chain when G is positive. And when G is negative it corresponds to the solutions of some excited states of antiferromagnetic chains.

For $n \geq 2$ and general N the equation (41) becomes a set of reduction equations. It will be a bit more difficult to solve. However, for a given finite N the problem becomes the solving of a set of linear and homogeneous equations. The energies can be immediately obtained from the condition that the coefficient determinant of the equation set equals to zero.

2.4 Example: Open Chain with 3 Spins

We set $N = 2$ and free boundary condition, so that the Hamiltonian reads simply

$$ H = e_1 + e_2 \tag{46} $$

which we take as the simpliest example to be solved completely by the method we explained in the beginning of this section. We shall consider the isotropic case in the following of the paper with the Temperley-Lieb-Jones element e_i given by formula (8).

When $n = 0$, we immediately get the highest weight vector that reads $|0\rangle = |\uparrow\uparrow\uparrow\rangle = \left|\frac{3}{2}, \frac{3}{2}\right\rangle$. The total spin $J = \frac{3}{2}$, so this solution should be a spin 4-plet, with energy eigenvalue ${}^{n=0}E = 0$. By applying the lowering operator $\Delta^{(2)}(J^-)$, we have

$$ \begin{aligned} \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle &= \Delta^{(2)}(J^-)\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle &&= |\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle \ , \\ \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle &= \tfrac{1}{2}\Delta^{(2)}(J^-)\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle &&= |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle \ , \\ \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle &= \tfrac{1}{3}\Delta^{(2)}(J^-)\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle &&= |\downarrow\downarrow\downarrow\rangle \ . \end{aligned} \tag{47} $$

When $n = 1$, the total spin $J = \frac{1}{2}$, and so this is a doublet. The highest weight vectors are $\chi_1 = C_1^1|1\rangle$ and $\chi_2 = C_1^2|1\rangle$, and actually, as depicted in Fig.2.4, $\chi_1 = |(1,2)_s\rangle\,|\uparrow\rangle_3$ and $\chi_2 = -|\uparrow\rangle_1\,|(2,3)_s\rangle$ where $|(i,j)_s\rangle = |\uparrow\rangle_i\,|\downarrow\rangle_j - |\downarrow\rangle_i\,|\uparrow\rangle_j$ denotes the singlet pair of spins i and j.

We assume that the eigenstate of the Hamiltonian is of the following form

$$\left|\frac{1}{2}\frac{1}{2}\right\rangle = a_1\chi_1 + a_2\chi_2 \ , \tag{48}$$

and therefore applying our PROCEDURE we have

$$E\left|\frac{1}{2}\frac{1}{2}\right\rangle = (2a_1 + a_2)\chi_1 + (a_1 + 2a_2)\chi_2 \ . \tag{49}$$

The solutions exist if and only if

$$\det\begin{bmatrix} 2-E & 1 \\ 1 & 2-E \end{bmatrix} = 0 \tag{50}$$

which yields $E = 1, 3$ and will be denoted ${}^{n=1}E^{k=1,2}$.

1. For ${}^{n=1}E^{k=1} = 1$, we have $a_1 = 1$ and $a_2 = -1$ and the eigenstate reads

$$\left|\frac{1}{2}\frac{1}{2}\right\rangle = |1\rangle - |3\rangle = -|(1,3)_s\rangle\,|\uparrow\rangle_2 \ , \tag{51}$$

which is a highest weight vector and will be remembered as ${}^{n=1}\Phi^{k=1}$. Applying the lowering operator, we have

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = -|(1,3)_s\rangle\,|\downarrow\rangle_2 \ . \tag{52}$$

2. For ${}^{n=1}E^{k=2} = 3$, we have $a_1 = a_2 = 1$ and the eigenstate reads

$$\left|\frac{1}{2}\frac{1}{2}\right\rangle = |1\rangle - 2\,|2\rangle + |3\rangle = -|(1,2)_s\rangle\,|\uparrow\rangle_3 + |\uparrow\rangle_1\,|(2,3)_s\rangle \ , \tag{53}$$

which is also a highest weight vector and will be remembered as ${}^{n=1}\Phi^{k=2}$. Applying the lowering operator, we have

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = -|(1,2)_s\rangle\,|\downarrow\rangle_3 + |\downarrow\rangle_1\,|(2,3)_s\rangle \ , \tag{54}$$

We have been considering the ferromagnetic case, and apparently the order of the energy eigenvalues of these states are ${}^{n=0}E < {}^{n=1}E^{k=1} < {}^{n=1}E^{k=2}$. If we consider the antiferromagnetic case, when one should note that the energy eigenvalues are $-{}^{n}E^{k}$. So the ground state should be ${}^{n=1}\Phi^{k=2}$ which is a spin doublet with degenerated energy eigenvalue -3.

2.5 Antiferromagnetic Ground State Solutions of Closed Chain of 5 Spins

Consider five spins aligned on a circle, as depicted in the Fig.2.5(a). The isotropic

Hamiltonian reads

$$H = G(e_1 + e_2 + e_3 + e_4 + e_{15}) \ , \tag{55}$$

where e_{15} denotes the interaction between spin 5 and spin 1, and

$$e_{15} = e_1 + e_{25} - \{e_1, e_{25}\} \ , e_{25} = e_2 + e_{35} - \{e_2, e_{35}\} \ , e_{35} = e_3 + e_4 - \{e_3, e_4\} \ . \tag{56}$$

For more about the above recursion relation, see the next section.

When $G > 0$ this is a ferromagnetic system, and when $G < 0$ this is a antiferromagnetic system. And we are interested in the antiferromagnetic ground state (AGS), i.e., $G < 0$ and $n = 2$. The quantum angular momentum number for AGS should be 1/2, so it is a spin doublet. The wavefunction is of the following form from formula (39)

$$\Psi = a_{13}\Phi_{13} + a_{14}\Phi_{14} + a_{23}\Phi_{23} + a_{24}\Phi_{24} + a_{34}\Phi_{34} \ , \tag{57}$$

and the action of the Hamiltonian on this state yields the following set of linear equations,

$$\begin{bmatrix} 14-\frac{E}{G} & -8 & 6 & -4 & 6 \\ -8 & 14-\frac{E}{G} & -5 & 4 & -5 \\ 0 & 0 & 3-\frac{E}{G} & 0 & -1 \\ 0 & 0 & 2 & 6-\frac{E}{G} & 2 \\ 0 & 0 & -1 & 0 & 3-\frac{E}{G} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{14} \\ a_{23} \\ a_{24} \\ a_{34} \end{bmatrix} = 0 \ . \tag{58}$$

The existence of non-zero solutions requires that the determinant of the above square matrix vanishes. The determinant is

$$\Delta = -\left(\frac{E}{G}-2\right)\left(\frac{E}{G}-4\right)\left(\frac{E}{G}-6\right)^2\left(\frac{E}{G}-22\right) \ , \tag{59}$$

and take $G = -1$ we get the energy eigenvalue for the AGS, $E = -22$. The corresponding eigenstates are depicted in Figs.2.5(b),2.5(c).

There is an apparent degeneracy for AGS in that the above configurations are in cyclically permutative.

3 Chain with Next-to-Nearest Neighbour Interactions

Now we investigate the isotropic Heisenberg spin-$\frac{1}{2}$ model in one dimension with next-to-nearest neighbour interactions.

3.1 Model and Symmetry

We consider a chain with $N+1$ spins in an open line. Let $e_{i,i+2}$ denote the next-to-nearest neighbour interactions between spins at site i and $i+2$. We find that $e_{i,i+2}$ can

be expressed in terms of the interactions between spins at $\{i, i+1\}$ and $\{i+1, i+2\}$, i.e., e_i and e_{i+1},

$$e_{i,i+2} = e_i + e_{i+1} - \{e_i, e_{i+1}\} \ , \quad i = 1, 2, \cdots, N-1 \ , \tag{60}$$

where $\{\cdot,\cdot\}$ denotes the anticommutator. Therefore the isotropic Heisenberg spin Hamiltonian with next nearest interactions are

$$\begin{aligned} H_{n\to n} &= G\left[\sum_{i=1}^{N} e_i + g\sum_{i=1}^{N-1} e_{i,i+2}\right] \\ &= G\left\{\sum_{i=1}^{N-1}\left[(1+g)e_i + g(e_i + e_{i+1} - e_i e_{i+1} - e_{i+1}e_i)\right] + e_N\right\} \ , \end{aligned} \tag{61}$$

where g is a coupling constant of interactions between next-to-nearest neighbouring spins. For convenience, we omit the factor G in the Hamiltonian in the following.

As this Hamiltonian is composed of the elements of $\mathcal{A}_N$, the eigenstates of this system can also be composed by the independent basis of the left ideal of TLJ algebra. In the following, we will give some detailed results for some special cases.

3.2 Result for $n = 1$ and Arbitrary N

For the case of $N+1$ spins chain with one spin down, i.e., $n = 1$, the eigenstates are simply of the form

$${}^1\Psi = \sum_{m=1}^{N} a_m C_1^m |1\rangle \ . \tag{62}$$

The Hamiltonian equation gives rise to

$$-ga_{m+2} + a_{m+1} + [2(1+g) - E]a_m + a_{m-1} - ga_{m-2} = 0, \quad m = 1, 2, \cdots, N. \tag{63}$$

where we have

$$a_0 = 0, \ a_{-1} = a_1 + a_2, \ a_{N+1} = 0, \ a_{N+2} = a_{N-1} + a_N. \tag{64}$$

The reduction equation (63) can be manipulated by simply setting $a_m = \eta^m$, which gives rise to an equation of η with four solutions,

$$\begin{aligned} \eta_{1,2} &= \frac{\pm 2\sqrt{-A - 2gE + 4g + 1} - A\sqrt{2} + \sqrt{2}}{4\sqrt{2}g} \\ \eta_{3,4} &= \frac{\pm 2\sqrt{A - 2gE + 4g + 1} + A\sqrt{2} + \sqrt{2}}{4\sqrt{2}g} \end{aligned} \tag{65}$$

where $A = \sqrt{16g^2 - 4gE + 8g + 1}$. Therefore $a_m = b_1\eta_1^m + b_2\eta_2^m + b_3\eta_3^m + b_4\eta_4^m$. It is apparent that we can always choose $b_1 = 1$. and the coefficients b_2, b_3, b_4 and energy eigenvalue E are determined by the boundary conditions in (64). Actually,

$$b_2 = -\frac{(\gamma_1 - \gamma_4) - \frac{\alpha_{14}}{\alpha_{34}}(\gamma_3 - \gamma_4)}{(\gamma_2 - \gamma_4) - \frac{\alpha_{24}}{\alpha_{34}}(\gamma_3 - \gamma_4)} \ , \quad b_3 = -\frac{\alpha_{14}}{\alpha_{34}} - b_2\frac{\alpha_{24}}{\alpha_{34}} \ , \tag{66}$$

and from $a_{N+1} = 0$, we give the equation for energy eigenvalues

$$\beta_{14} + \beta_{24}b_2 + \beta_{34}b_3 = 0 \ , \tag{67}$$

where

$$\begin{aligned} \alpha_{ij} &= (\eta_i - \eta_j)\left(\frac{1}{\eta_i\eta_j} + 1 + \eta_i + \eta_j\right) \ , \\ \beta_{ij} &= \eta_i^{N+1} - \eta_j^{N+1} \ , \qquad i, j = 1, 2, 3, 4 \\ \gamma_i &= \eta_i^{N-1}(1 + \eta_i - \eta_i^3) \ . \end{aligned} \tag{68}$$

For $n \geq 2$ and general N the equation (41) becomes a set of reduction equations and will be a bit more difficult to solve. However, for a given finite N the problem becomes the solving of a set of linear and homogeneous equations. The energies can be immediately obtained from the condition that the coefficient determinant of the equation equals zero.

3.3 Approximation at $n = 1$ with Arbitrary N

Assume that $g \sim 0$, i.e., the couplings between next-to-nearest neighbours are very weak. Accurate to the order of $O(g)$, we have

$$\begin{aligned} \eta_{1,2} &\doteq \frac{-(2-E) \pm \sqrt{(2-E)^2 - 4}}{2} + \frac{g}{2}\left(4 - (2-E)^2\right) \ , \\ \eta_3 &\doteq \frac{1}{g} \ , \quad \eta_4 \doteq \delta g \ , \quad \delta = E - E^2/4 \ . \end{aligned} \tag{69}$$

And it is an easy to get

$$\frac{\alpha_{14}}{\alpha_{34}} \doteq \frac{\alpha_{24}}{\alpha_{34}} \doteq \frac{g^2}{\delta} \ , \ \beta_{14} \doteq \eta_1^{N+1} \ , \ \beta_{24} \doteq \eta_2^{N+1} \ , \ \beta_{34} \doteq g^{-N-1} \ . \tag{70}$$

From (66) we have

$$b_2 \doteq -1 + (\gamma_1 - \gamma_2)\,\delta\, g^{N+1} \ , \quad b_3 \doteq 0 - (\gamma_1 - \gamma_2)\, g^{N+3} \ . \tag{71}$$

From (67), we have the equation for energy (accurate to the order of $O(g)$,

$$\eta_1^{N+1} = \eta_2^{N+1} \ , \tag{72}$$

or

$$\frac{-(2-E) + g\left(4 - (2-E)^2\right)}{\sqrt{(2-E)^2 - 4}} \doteq -i\mathrm{ctg}\frac{k\pi}{N+1} \ , \quad k = 0, 1, \cdots N \tag{73}$$

Approximation to the zero-th order, we have $g = 0$ and

$$E_k^{0\text{th}} = 2\left(1 - \cos\frac{k\pi}{N+1}\right) \ . \tag{74}$$

which coincides with the result supplied in eq.(45). Hence the first order approximation yields

$$\frac{-(2-E_k)+g\left(4-\left(2-E_k^{0th}\right)^2\right)}{\sqrt{(2-E_k)^2-4}} \doteq -\mathrm{ictg}\frac{k\pi}{N+1}\,, \tag{75}$$

so we have

$$E_k = 2\left(1-\cos\frac{k\pi}{N+1}\right) - 4g\sin^4\frac{k\pi}{N+1}\,. \tag{76}$$

To get $b_{2,3}$, we supply

$$\begin{aligned} \alpha_{i4} &\doteq \tfrac{1}{\delta}\left[1+\left(\eta_i+\eta_i^2-\eta_i^{-1}\right)\delta\, g\right]\,, \quad (i=1,2) \\ \alpha_{34} &\doteq \tfrac{1}{g^2}\left[1+\left(1+\tfrac{1}{\delta}\right)g\right]\,. \end{aligned} \tag{77}$$

3.4 Complete Solutions for Four Spin Case

Let us consider, for example, the case of four spins ($N=3$). The Hamiltonian (61) is simply

$$H_{n\to n} = (g+1)(e_1+e_3)+(2g+1)e_2 - g(e_1e_2+e_2e_1+e_2e_3+e_3e_2)\,. \tag{78}$$

For $n=0$, the highest weight state of a spin 5-plet,i.e., ${}^0\Psi_{22}=|0\rangle$ with all spins up, ${}^0E=0$ and $J=2$. The get the complete solution space, we will apply the lowering operator $J^-_{\rm tot}=\Delta^{(3)}(J^-)$ repeatedly to ge the states with different weights. We will give the highest weight states first and discuss the complete solution spaces at the end of this section.

The states with $n=1$, are spin triplets, the total spin number is $J=1$. We get three highest such states, with energies ${}^1E^{k=1,2,3}$. The states are the superpositions of the following vectors of primitive left ideal,

$$\begin{aligned} {}^{n=1}\Psi_{11} &= a_1\,{}^1\Phi_1 + a_2\,{}^1\Phi_2 + a_3\,{}^1\Phi_3\,, \\ &= (a_1e_1+a_2e_2e_1+a_3e_3e_2e_1)|1\rangle\,. \end{aligned} \tag{79}$$

where where the subscript $\{11\}$ of Ψ indicates the quantum number $\{J,n\}$ and ${}^1\Phi_i = C_1^i|1\rangle$ and $|i\rangle$ is the state with a spin down at location i.

By use of the PROCEDURE C in section 2.2 and (41) we have

$$\begin{aligned} (g+2-E)a_1 &+ (1+g)a_2 - ga_3 = 0 \\ a_1 &+ (2-E+2g)a_2 + a_3 = 0 \\ -ga_1 &+ (1-g)a_2 + (2+g-E)a_3 = 0 \end{aligned} \tag{80}$$

Therefore we have

1. Solution for $k=1$: for $g\neq 1$

$$\begin{aligned} {}^1E^{k=1} &= 2(g+1)\,, \\ a_2 &= 0\,, \\ a_3 &= -a_1 = -1\,; \end{aligned} \tag{81}$$

with the highest weight state

$${}^1\Psi_{11}^{k=1} = |1\rangle - |2\rangle - |3\rangle + |4\rangle\,, \tag{82}$$

and for $g=1$

$$\begin{aligned} {}^1E^{k=1} &= 4\,, \\ a_2 &= \text{arbitrary}\,, \\ a_3 &= -a_1\,; \end{aligned} \tag{83}$$

with the highest weight state is an arbitrary superposition of ${}^1\Psi_{11}^{k=1}$ and

$${}^1\tilde{\Psi}_{11}^{k=1} = -|2\rangle + |3\rangle \tag{84}$$

so it is degenerate. To be compact, we denote the solutions in above two cases as

$${}^1\Psi_{11}^{k=1} = |1\rangle - (1+a_2)|2\rangle - (1-a_2)|3\rangle + |4\rangle\,, \tag{85}$$

for $a_2=0$, it gives the first case of $g\neq 1$, and for $a_2=$ arbitrary, the second case, i.e., $g=0$.

2. Solution for $k=2$: for $g\neq 1$,

$$\begin{aligned} {}^1E^{k=2} &= 2+g+\sqrt{g^2-2g+2}\,, \\ a_2 &= 2a_1(\sqrt{g^2-2g+2}-g)^{-1}\,, \\ a_3 &= a_1\,, \end{aligned} \tag{86}$$

and for $g=1$,

$$\begin{aligned} {}^1E^{k=2} &= 4\,, \\ a_2 &= 1\,, \\ a_3 &= a_1 = 0\,. \end{aligned} \tag{87}$$

The highest weight states for the above two cases read

$${}^1\Psi_{11}^{k=2} = |1\rangle - (1+a_2)|2\rangle + (1+a_2)|3\rangle - |4\rangle\,, \tag{88}$$

and when a_2 takes the value specified in (86), we get the state for $g\neq 1$, and when $a_2\to\infty$, (i.e., we omit the terms without a_2 as coefficient), we get the state for $g=1$.

3. Solution for $k = 3$,

$$
\begin{aligned}
{}^1E^{k=3} &= 2 + g - \sqrt{g^2 - 2g + 2}\,, \\
a_2 &= 2a_1(\sqrt{g^2 - 2g + 2} + g)^{-1}, \\
a_3 &= a_1\,.
\end{aligned}
\tag{89}
$$

The highest weight state ${}^1\Psi^{k=3}_{11}$ is formally the same as ${}^1\Psi^{k=3}_{1,1}$ in eq.(88), but a_2 is specified as above. There is no degeneracy due to g.

For $n = 2$, we have two spin singlets with angular moment $J = 0$. Their energies are ${}^{n=2}E^{k=1,2}$.

$$
{}^2\Psi = (a_{13}C_1^1C_3^3 + a_{23}C_1^2C_3^3)|13\rangle = (a_{13}e_3e_1 + a_{23}e_2e_3e_1)|13\rangle \tag{90}
$$

Using (41) we get energies for generic g

$$
\begin{aligned}
{}^2E^{k=1} &= 2g + 3 + \sqrt{4g^2 - 6g + 3} \\
{}^2E^{k=2} &= 2g + 3 - \sqrt{4g^2 - 6g + 3}\,,
\end{aligned}
\tag{91}
$$

with

$$
a_{23} = a_{13}\frac{2g - 1}{{}^2E^k - 2(1 + g)} = a_{13}\frac{2g - 1}{1 \pm \sqrt{4g^2 - 6g + 3}} \tag{92}
$$

but there are two singular points for $k = 2$ states

1. If $g = 1$, we have ${}^2E^{k=2} = 4$, $a_{23} = 1, a_{13} = 0$.
2. If $g = 1/2$, we have ${}^2E^{k=2} = 3$ and $a_{23} = 1, a_{13} = 0$.

To write the states into a compact form, we always take $a_{13} = 1$,

$$
{}^2\Psi^k_{00} = -a_{23}|12\rangle + (1 + a_{23})|13\rangle - |14\rangle - |23\rangle + (1 + a_{23})|24\rangle - a_{23}|34\rangle\,, \tag{93}
$$

and for different k and g, a_{23} may be specified as different values as indicated above. In the case of $g = 1$ and $k = 2$, we take $a_{23} = \infty$, and renormalize the state.

Now we give the complete solution spaces with all possible quantum numbers of total spins. The highest weight state are provided early in this section,

1. Spin 5-plet with $n = 0$, $J = 2$ and ${}^0E = 0$.

$$
{}^0\Psi_{22} = |\uparrow\uparrow\uparrow\uparrow\rangle\,, \tag{94}
$$

therefore the descendent states should be

$$
\begin{aligned}
{}^0\Psi_{21} &= \Delta(J^-)|0\rangle = |1\rangle + |2\rangle + |3\rangle + |4\rangle\,, \\
{}^0\Psi_{20} &= \tfrac{1}{2}\Delta(J^-)\cdot\,{}^0\Psi_{21} = |12\rangle + |13\rangle + |14\rangle + |23\rangle + |24\rangle + |34\rangle\,, \\
{}^0\Psi_{2\bar{1}} &= \tfrac{1}{3}\Delta(J^-)\cdot\,{}^0\Psi_{20} = |123\rangle + |124\rangle + |134\rangle + |234\rangle\,, \\
{}^0\Psi_{2\bar{2}} &= \tfrac{1}{4}\Delta(J^-)\cdot\,{}^0\Psi_{2\bar{1}} = |1234\rangle = |\downarrow\downarrow\downarrow\downarrow\rangle\,,
\end{aligned}
\tag{95}
$$

2. Spin triplets with $n = 1$, $J = 1$ and ${}^1E^{k=1,2,3}$,

(a) For $k = 1$, the highest weight state is ${}^1\Psi^{k=1}_{11}$ in eq.(85). The descendent states for general g are

$$
\begin{aligned}
{}^1\Psi^{k=1}_{10} &= \Delta(J^-)\cdot\,{}^1\Psi^{k=1}_{11} \\
&= -a_2|12\rangle + a_2|13\rangle + 2|14\rangle - 2|23\rangle - a_2|24\rangle + a_2|34\rangle\,, \\
{}^1\Psi^{k=1}_{1\bar{1}} &= \tfrac{1}{2}\Delta(J^-)\cdot\,{}^1\Psi^{k=1}_{10} \\
&= -|123\rangle + (1 - a_2)|124\rangle + (1 + a_2)|134\rangle - |234\rangle\,.
\end{aligned}
\tag{96}
$$

When $g \neq 1$, $a_2 = 0$ and so

$$
\begin{aligned}
{}^1\Psi^{k=1}_{10} &= 2(|24\rangle - |23\rangle)\,, \\
{}^1\Psi^{k=1}_{1\bar{1}} &= -|123\rangle + |124\rangle + |134\rangle - |234\rangle\,,
\end{aligned}
\tag{97}
$$

and when $g = 1$, a_2 is an arbitrary complex number, hence the ${}^1\Psi^{k=1}_{10}$ should be an arbitrary superposition of $|14\rangle - |23\rangle$ and $-|12\rangle + |13\rangle - |24\rangle + |34\rangle$, and the state ${}^1\Psi^{k=1}_{1\bar{1}}$ should be an arbitrary superposition of $|124\rangle + |134\rangle$ and $-|124\rangle + |124\rangle + |134\rangle - |234\rangle$.

(b) $k = 2$. For generic g we give the descending states as

$$
\begin{aligned}
{}^1\Psi^{k=2}_{10} &= -a_2|12\rangle + (2 + a_2)|13\rangle - (2 + a_2)|24\rangle + a_2|34\rangle\,, \\
{}^1\Psi^{k=2}_{1\bar{1}} &= |123\rangle - (1 - a_2)|124\rangle + (1 + a_2)|134\rangle - |234\rangle\,,
\end{aligned}
\tag{98}
$$

and for $g = 1$ we have

$$
\begin{aligned}
{}^1\Psi^{k=2}_{11} &= -|2\rangle + |3\rangle\,, \\
{}^1\Psi^{k=2}_{10} &= -|12\rangle + |13\rangle - |24\rangle + |34\rangle\,, \\
{}^1\Psi^{k=2}_{1\bar{1}} &= -|124\rangle + |134\rangle\,.
\end{aligned}
\tag{99}
$$

(c) When $k = 3$ there is no degeneracy due to g, therefore

$$
\begin{aligned}
{}^1\Psi^{k=3}_{10} &= -a_2|12\rangle + (2 + a_2)|13\rangle - (2 + a_2)|24\rangle + a_2|34\rangle\,, \\
{}^1\Psi^{k=3}_{1\bar{1}} &= |123\rangle - (1 + a_2)|124\rangle + (1 + a_2)|134\rangle - |234\rangle\,,
\end{aligned}
\tag{100}
$$

3. Spin singlets, with $n = 2$, total angular momentum $J = 0$ and energies ${}^2E^{k=1,2}$.

In this way we get all the eigenstates and energy levels of the 1-D Heisenberg spin system ($N = 3$) with next to nearest interactions. It is worthwhile to note that when $g = 0$, we have

$$
{}^0E <^1 E^{k=3} <^2 E^{k=2} <^1 E^{k=1} <^1 E^{k=2} <^2 E^{k=1}\,, \tag{101}
$$

and when $g = 1/2$ there is degeneracy and

$$
{}^0E <^1 E^{k=3} <^1 E^{k=2} <^1 E^{k=1} =^2 E^{k=2} <^2 E^{k=1}\,, \tag{102}
$$

and when $g = 1$ there are degeneracies

$$ {}^0E <^1 E^{k=3} <^2 E^{k=2} =^1 E^{k=1} =^1 E^{k=2} <^2 E^{k=1} , \tag{103} $$

and when $g = 2$, we have

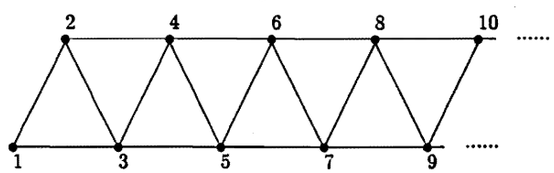
$$ {}^0E <^1 E^{k=3} <^2 E^{k=2} <^1 E^{k=2} <^1 E^{k=1} <^2 E^{k=1} , \tag{104} $$

and the readers may notice that there are level-overlappings when g changes from 0 to 2, and these changes are due to the competition between the nearest and next-to-nearest neighbour interactions. That is, the increasing of the next-to-nearest neighbour interactions causes two levels to meet each other and exchanges positions.

Here the boundaries of Hamiltonian (61) are open. The cyclic boundary conditions may be also considered by adding a boundary term $e_{1,4}$ representing interactions between spin 1 and 4, which is easily deduced from relation (60)

$$ \begin{aligned} e_{1,4} &= e_{1,3} + e_3 - \{e_{1,3}, e_3\} \\ &= e_1 + e_2 + e_3 - \{e_1, e_2\} - \{e_2, e_3\} - \{e_1, e_3\} \\ &\quad + \{e_1, \{e_2, e_3\}\} . \end{aligned} \tag{105} $$

It should be noted that the lower dimensional Heisenberg spin model with higher order non-nearest interactions corresponds to higher dimensional ones with lower order non-nearest interactions. Here the 1-D chain with next to nearest interactions is the same as the 2-D strip with triangular nearest interactions, i.e.,



Different selections of non-nearest interactions of 1-D chain will correspond to different kinds of 2-D or higher dimension lattice interactions. Similarly higher dimensional Heisenberg spin models may be exactly solved by reducing them to be 1-D problems. In any way the Hamiltonians are always the polynomials of the elements of Temperley-Lieb-Jones algebra and hence their eigenstates are of the form (40).

4 2D Isotropic Triangular Lattice

4.1 2-Fold Co-Casimir

Let's observe the 2-fold co-multiplication of Casimir (or the 2-nd rank co-Casimir) of $su(2)$ case, which well illustrates the idea (the generalization to the quantum algebra

$su_q(2)$ is straightforward, but involves more computations, hence we leave it to a separate discussion):

$$ \begin{aligned} \Delta^{(2)}(C'') &= \tfrac{3}{2} 1 \otimes 1 \otimes 1 \\ &\quad - [\sigma^+ \otimes \sigma^- \otimes 1 + \sigma^- \otimes \sigma^+ \otimes 1 \\ &\quad + \sigma^+ \otimes 1 \otimes \sigma^- + \sigma^- \otimes 1 \otimes \sigma^+ \\ &\quad + 1 \otimes \sigma^+ \otimes \sigma^- + 1 \otimes \sigma^- \otimes \sigma^+ \\ &\quad + \tfrac{1}{2} (\sigma^3 \otimes \sigma^3 \otimes 1 + \sigma^3 \otimes 1 \otimes \sigma^3 + 1 \otimes \sigma^3 \otimes \sigma^3)] , \end{aligned} \tag{106} $$

where $C'' = \dfrac{15}{4} - C$ and $\Delta^{(2)} = (\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

Applying the Pauli matrices, we finally obtain

$$ \Delta^{(2)}(C'') = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \tag{107} $$

which can be written as

$$ \Delta^{(2)}(C'') = 2e_1 + 2e_2 - e_1 e_2 - e_2 e_1 , \tag{108} $$

where

$$ e_1 = \Delta(C') \otimes 1 , \quad e_2 = 1 \otimes \Delta(C') , \tag{109} $$

are just the elements of a two dimensional TLJ algebra, where C' is the limit of C'_q at $q \to 1$.

Denoting

$$ w_i = 1^{(1)} \otimes 1^{(2)} \otimes \cdots \otimes 1^{(i-1)} \otimes \Delta^{(2)}(C')^{(i,i+1,i+2)} \otimes 1^{(i+3)} \otimes \cdots \otimes 1^{(N+1)} , \tag{110} $$

or simply

$$ w_i = \Delta^{(2)}(C')^{(i,i+1,i+2)} , \tag{111} $$

we have

$$ w_i = 2e_i + 2e_{i+1} - e_i e_{i+1} - e_{i+1} e_i , \quad i = 1, 2, \cdots, N-1 , \tag{112} $$

with the corresponding representation in (22).

It is interesting to note that the operators w_i span a closed algebra with following relations

$$\begin{aligned} & w_i^2 = 3w_i \ , \\ & w_i w_j = w_j w_i \ , \quad \text{if} \ \ |i-j| \geq 3 \ , \\ & w_i w_{i+1} w_i + w_{i+1} = w_{i+1} w_i w_{i+1} + w_i \ , \\ & 4w_i w_{i+2} - 5\left(w_i w_{i+2}\right)^2 + \left(w_i w_{i+2}\right)^3 = \\ & 4w_{i+2} w_i - 5\left(w_{i+2} w_i\right)^2 + \left(w_{i+2} w_i\right)^3 \ . \end{aligned} \tag{113}$$

In stead of e_i in (15) now the Hamiltonian for this spin system is given by w_i

$$H^{1\mathrm{D}}_{\mathrm{n}\to\mathrm{n}} = \sum_{i=1}^{N-1} w_i = \sum_{i=1}^{N} \left(\Delta^{(2)}(C'')\right)^{(i,i+1,i+2)} \ , \tag{114}$$

which is obviously $su(2)$-invariant, i.e.,

$$\left[H^{1\mathrm{D}}_{\mathrm{n}\to\mathrm{n}} \ , \ \bullet\right] = 0 \ , \quad \forall \bullet \in su(2)^{\otimes N} \ . \tag{115}$$

The elements of $su(2)^{\otimes N}$ are

$$J^{\alpha}_{\mathrm{tot}} = \sum_{i=1}^{N+1} \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_{i-1} \otimes J^{\alpha}_i \otimes \mathbf{1}_{i+1} \otimes \cdots \otimes \mathbf{1}_{N+1} \ , \tag{116}$$

where $\alpha = \pm, 3$.

It can be seen that the Heisenberg spin chain with next-to-nearest neighbour (n→n) interaction is recovered, when g is specified as 1. Explicitly,

$$\begin{aligned} H^{1\mathrm{D}}_{\mathrm{n}\to\mathrm{n}} \ = \ & \frac{3}{2}N\mathbf{1} - \sum_{i=1}^{N-1} \Big[\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{1}{2}\sigma_i^3\sigma_{i+1}^3 + \sigma_i^+\sigma_{i+2}^- + \\ & \sigma_i^-\sigma_{i+2}^+ + \tfrac{1}{2}\sigma_i^3\sigma_{i+2}^3 + \sigma_{i+1}^+\sigma_{i+2}^- + \sigma_{i+1}^-\sigma_{i+2}^+ + \tfrac{1}{2}\sigma_{i+1}^3\sigma_{i+2}^3\Big] \ . \end{aligned} \tag{117}$$

4.2 T-L-J Algebra on Triangular Lattice

Now we consider the 2-fold co-multiplication of Casimir of $su(2)$, which leads to the Hamiltonian of 2D Heisenberg model (with nearest interactions) rather than the 1D Heisenberg model including the next-to-nearest neighbour interactions. As a typical 2D problem, we consider $N \times M - 1$ spins placed on a triangular lattice of N rows and M columns. Each spin is labeled by its row and column numbers (i,j), where $1 \leq i \leq M$, and $1 \leq j \leq N$. Every three spins at (i,j), $(i,j+1)$ and $(i+1,j)$ form a triangle that is denoted (i,j) simply in the following. The triangles are printed as rectangular ones (hence the spin at (M,N) is missing), jut for the convenience of printing.

For the spin at cite (i,j) there is an internal spin space $V_{1/2}$. Each spin interacts with its nearest neighbours, i.e., spin (i,j) interacts with spins at $(i,j\pm1)$, $(i\pm1,j)$,

$(i-1,j+1)$ and $(i+1,j-1)$ etc.. For simplicity, we write the tensorized space V as follows,

$$V = \begin{bmatrix} \left(V_{1/2}\right)_{(1,N)} & \left(V_{1/2}\right)_{(2,N)} & \cdots & \left(V_{1/2}\right)_{(M-1,N)} & \\ \left(V_{1/2}\right)_{(1,N-1)} & \left(V_{1/2}\right)_{(2,N-1)} & \cdots & \left(V_{1/2}\right)_{(M-1,N-1)} & \left(V_{1/2}\right)_{(M,N-1)} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \left(V_{1/2}\right)_{(1,1)} & \left(V_{1/2}\right)_{(2,1)} & \cdots & \left(V_{1/2}\right)_{(M-1,1)} & \left(V_{1/2}\right)_{(M,1)} \end{bmatrix} , \tag{118}$$

where each spin space $\left(V_{1/2}\right)_{(i,j)}$ in the i-th column and j-th row tensors all other spin spaces. V will be referred as the lattice of tensorized spin spaces.

To construct the Hamiltonian, we start from the 2-fold co-Casimir of $su(2)$ given in (106) again. However here we put the three spin spaces on a three point lattice, i.e., they are tensorized as followings

$$V = \begin{bmatrix} V_{1/2} & \\ V_{1/2} & V_{1/2} \end{bmatrix} . \tag{119}$$

The co-Casimir $\Delta^{(2)}(C'')$ acting on these three spaces is denoted

$$\begin{aligned} \Delta^{(2)}(C'') \ = \ & \tfrac{3}{2}\begin{bmatrix} \mathbf{1} & \\ \mathbf{1} & \mathbf{1} \end{bmatrix} - \Bigg[\begin{bmatrix} \sigma^+ & \\ \sigma^- & \mathbf{1} \end{bmatrix} + \begin{bmatrix} \sigma^- & \\ \sigma^+ & \mathbf{1} \end{bmatrix} + \begin{bmatrix} \sigma^+ & \\ \mathbf{1} & \sigma^- \end{bmatrix} \\ & + \begin{bmatrix} \sigma^- & \\ \mathbf{1} & \sigma^+ \end{bmatrix} + \begin{bmatrix} \mathbf{1} & \\ \sigma^+ & \sigma^- \end{bmatrix} + \begin{bmatrix} \mathbf{1} & \\ \sigma^- & \sigma^+ \end{bmatrix} \\ & + \tfrac{1}{2}\left(\begin{bmatrix} \sigma^3 & \\ \sigma^3 & \mathbf{1} \end{bmatrix} + \begin{bmatrix} \sigma^3 & \\ \mathbf{1} & \sigma^3 \end{bmatrix} + \begin{bmatrix} \mathbf{1} & \\ \sigma^3 & \sigma^3 \end{bmatrix}\right)\Bigg] \ , \end{aligned} \tag{120}$$

Each of the three elements in $\lfloor \ \ \rfloor$ tensors all other two elements.

Now we put the co-Casimir onto the lattice of tensorized spin spaces with N rows and M columns at an arbitrary triangle (i,j) and denote this element $W_{(i,j)}$, explicitly,

$$W_{(i,j)} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & * & \mathbf{1} & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & *_{(i,j)} & * & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\ \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & \mathbf{1} \end{bmatrix} , \tag{121}$$

where triangle $\begin{bmatrix} * & \\ *_{(i,j)} & * \end{bmatrix}$ at position (i,j), $(i,j+1)$ and $(i+1,j)$ (with $1 \leq i \leq M-1$ and $1 \leq j \leq N-1$) denotes the position of the co-Casimir (120).

Apparently $W_{i,j}$, $1 \le i \le N$ and $1 \le j \le M$, form a closed algebra corresponding to (113) in the case of 1D Heisenberg model including interactions between second nearest neighbours. Explicitly,

$$
\begin{aligned}
& W_{i,j}^2 = 3W_{i,j} \ , \\
& 4W_{i,j}W_{i',j'} - 5\left(W_{i,j}W_{i',j'}\right)^2 + \left(W_{i,j}W_{i',j'}\right)^3 = \\
& 4W_{i',j'}W_{i,j} - 5\left(W_{i',j'}W_{i,j}\right)^2 + \left(W_{i',j'}W_{i,j}\right)^3 \\
& \quad \text{if} \ \ (i',j') = (i \pm 1, j) \ \text{ or } \ (i, j \pm 1) \ \text{ or } \ (i \mp 1, j \pm 1), \\
& W_{i,j}W_{i',j'} = W_{i',j'}W_{i,j} \ , \text{otherwise} \ .
\end{aligned}
\tag{122}
$$

Nevertheless it should be emphasized that the W_{ij} algebra given in (122) can be reduced to TLJ algebra. Let

$$
f_{i,j} = \begin{bmatrix}
1 & 1 & \cdots & 1 & & \cdots & 1 & \\
\cdot & \cdot & \cdots & \cdot & & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & *_{(i,j+1)} & & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & *_{(i,j)} & & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & & \cdots & \cdot & \cdot \\
1 & 1 & \cdots & 1 & & \cdots & 1 & 1
\end{bmatrix} \ , \tag{123}
$$

where the two cites marked $*$ are (i,j) and $(i,j+1)$, and

$$
e_{i,j} = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & *_{(i,j)} & *_{(i+1,j)} & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1
\end{bmatrix} \ , \tag{124}
$$

where the cites marked by $*$ are (i,j) and $(i+1,j)$. It can be found that

$$
\Delta^{(2)}(C'') = 2f_{1,1} + 2e_{1,1} - e_{1,1}f_{1,1} - f_{1,1}e_{1,1} \ , \tag{125}
$$

where

$$
\begin{aligned}
f_{1,1} &= \Delta(C') \otimes 1 = \begin{bmatrix} * & \\ * & 1 \end{bmatrix} \\
e_{1,1} &= 1 \otimes \Delta(C') = \begin{bmatrix} 1 & \\ * & * \end{bmatrix}
\end{aligned}
\tag{126}
$$

are just the elements of a two dimensional TLJ algebra. Generally we have

$$
W_{i,j} = 2f_{i,j} + 2e_{i,j} - e_{i,j}f_{i,j} - f_{i,j}e_{i,j} \ , \tag{127}
$$

where $i = 1, 2, \cdots, M-1$, $\quad j = 1, 2, \cdots, N-1$, with the representation in (118). The algebra spanned by $f_{i,j}$ and $e_{i,j}$ is nothing but the 2D lattice version of the TLJ algebra, with relations

$$
\begin{aligned}
& e_{i,j}^2 = 2e_{i,j} \\
& f_{i,j}^2 = 2f_{i,j} \\
& e_{i,j}e_{i',j'} = e_{i',j'}e_{i,j} \quad j' \ne j \ \text{ or } \ j' = j \ \text{ but } \ |i'-i| \ge 2 \\
& f_{i,j}f_{i',j'} = f_{i',j'}f_{i,j} \quad i' \ne i \ \text{ or } \ i' = i \ \text{ but } \ |j'-j| \ge 2 \\
& e_{i,j}e_{i\pm1,j}e_{i,j} = e_{i,j} \\
& f_{i,j}f_{i,j\pm1}f_{i,j} = f_{i,j} \\
& e_{i,j}f_{\overline{i,j}}e_{i,j} = e_{i,j} \ , \ \ \overline{i,j} = (i,j) \ , \text{ or } \ (i+1,j) \ , \text{ or } \ (i,j-1) \ , \text{ or } \ (i+1,j-1) \\
& f_{i,j}e_{\underline{i,j}}f_{i,j} = f_{i,j} \ , \ \ \underline{i,j} = (i,j) \ , \text{ or } \ (i,j+1) \ , \text{ or } \ (i-1,j) \ , \text{ or } \ (i-1,j+1) \ .
\end{aligned}
\tag{128}
$$

It is easily seen that if the Hamiltonian for this spin system is

$$
H^{2D} = \sum_{j=1}^{N-1}{}' \sum_{i=1}^{M-1}{}' W_{i,j} \tag{129}
$$

or simply

$$
H^{2D} = \sum_{j=1}^{N-1}{}' \sum_{i=1}^{M-1}{}' \begin{bmatrix} * & \\ *_{(i,j)} & * \end{bmatrix} \tag{130}
$$

where the symbol $\sum'$ denotes a summation neglecting the point of $(i,j) = (M,N)$, then the system is obviously $su(2)$-invariant, i.e.,

$$
\left[H^{2D} \ , \ \bullet\right] = 0 \ , \quad \forall \bullet \in su(2)^{\otimes(NM-1)} \ . \tag{131}
$$

The elements in $su(2)^{\otimes(NM-1)}$ are constructed as follows

$$
\Delta^{(NM-1)}(J^\alpha) = \sum_{j=1}^{N-1}{}' \sum_{i=1}^{M-1}{}' \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & * & 1 & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & *_{(i,j)} & * & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot & \cdot \\
1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1
\end{bmatrix} \ , \tag{132}
$$

where $\alpha = \pm, 3$ and the triangle $\begin{bmatrix} * & \\ *_{(i,j)} & * \end{bmatrix}$ at positions (i,j), $(i,j+1)$ and $(i+1,j)$ (with $1 \le i \le M-1$ and $1 \le j \le N-1$) denotes

$$
\begin{bmatrix} J^\alpha & \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & \\ J^\alpha & 1 \end{bmatrix} + \begin{bmatrix} 1 & \\ 1 & J^\alpha \end{bmatrix} \ . \tag{133}
$$

Although we treated with a triangular lattice of regular configuration, the treatment is actually true to the triangular lattice of any irregular forms. Above all by using the coproduct operation, the Heisenberg type spin models of $su(2)$ group symmetry and TLJ algebraic structures in high dimensions and with higher order nonnearest neighbour interactions can be easily constructed. And every higher dimensional Heisenberg Hamiltonian may give rise to some lower dimensional ones with high order nearest spin interactions.

4.3 A Simple Example

We set $N = 2$ and free boundary condition, so that the Hamiltonian reads simply

$$H = e_1 + e_2 + e_{1,3} \ , \tag{134}$$

which we take as the simpliest example to be solved completely by the method we explained in the beginning of this section. Because

$$e_{1,3} = e_1 + e_3 - \{e_1, e_2\} \ , \tag{135}$$

we have

$$H = 2e_1 + 2e_2 - \{e_1, e_2\}. \tag{136}$$

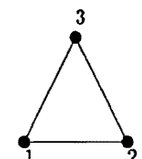


Fig.4.3.3(a)

This model is slightly different from the three spin chain discussed in the previous section 2.4.

When $n = 0$, the discussion is the same as for the open chain model in 2.4.

When $n = 1$, the total spin $J = \frac{1}{2}$, and the state is a doublet. The highest weight vectors are $\chi_1 = C_1^1|1\rangle$ and $\chi_2 = C_1^2|1\rangle$, as depicted in Fig.4.3(b)(c). The eigenstate of the Hamiltonian is of the following form

$$\left|\frac{1}{2}\frac{1}{2}\right\rangle = a_1\chi_1 + a_2\chi_2 \ , \tag{137}$$

and therefore applying our PROCEDURE, we have

$$E\left|\frac{1}{2}\frac{1}{2}\right\rangle = 3a_1\chi_1 + 3a_2\chi_2 \ . \tag{138}$$

The solution exists if and only if that $E = 3$, and the eigenstates are doubly degenerated,

$$\left|\frac{1}{2}\frac{1}{2}\right\rangle = \chi_1 \ \text{ or } \ \chi_2 \ , \tag{139}$$

i.e., any supercomposition of the two basis vectors can be the solution.

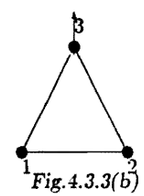


Fig.4.3.3(b)

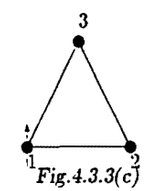


Fig.4.3.3(c)

Fig.4.3(b) $(1,2)$ *are in spin singlet pair.*
Fig.4.3(c) $(2,3)$ *are in spin singlet pair.*

5 2D Isotropic Square Lattice

Consider the 2D Heisenberg spin-$\frac{1}{2}$ model with nearest neighbour interactions on a finite square lattice of N rows and M columns (see Fig. 5.0),

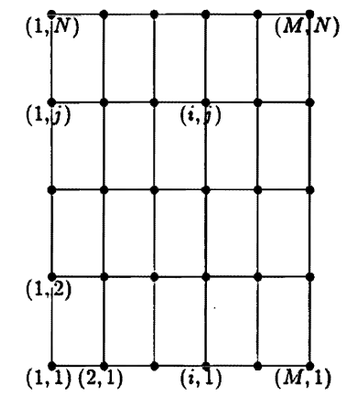


Fig.5.0

Each spin in i-th column and j-th row will be labelled (i,j). The Hamiltonian reads

$$H^{2D} = -G\sum_{i,i'}\sum_{j,j'}\left(\sigma^x_{i,j}\sigma^x_{i',j'} + \sigma^y_{i,j}\sigma^y_{i',j'} + \sigma^z_{i,j}\sigma^z_{i',j'}\right) \ , \tag{140}$$

where $(i',j') = (i,j+1)$, or $(i+1,j)$, and i runs through 1 to $M-1$, j through 1 to $N-1$ and $\sigma^{x,y,z}$ are Pauli matrices.

To show that this model is exactly solvable is to show that the energy eigenvalues, and eigenstates can be obtained without making any approximation. The discussion will be separated into three main steps. First, we show that the 2D problem (with nearest neighbour interactions only) is equivalent to a 1D problem (with selected near neighbour interactions). Second, we show that the near neighbour interactions can be decomposed into nearest neighbour interactions. Third, the eigenvalues and eigenstates of the Hamiltonian of 1D spin model, which includes near neighbour interactions but decomposed into nearest neighbour interactions, is exactly solvable.

5.1 The First Step

Assume that N =even, and relabel the lattice points as shown in Fig. 5.1(a).

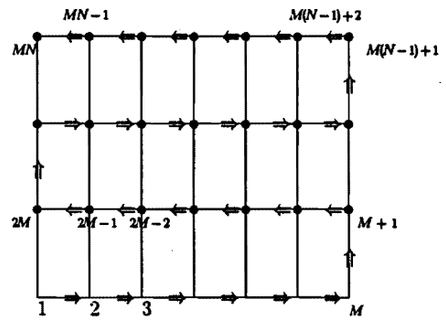


Fig.5.1(a). The arrow ⇒ indicates the way we relabel the sites.

In this way, the 2D problem is reduced into a 1D problem, but with non-nearest neighbour interactions, i.e.,

$$H^{2D} = -2G \sum_{[i,i']} \vec{S}_i \cdot \vec{S}_{i'} \tag{141}$$

where $[i, i']$ denote the following combinations,

1. $\langle i, i' \rangle$, $i' = i \pm 1$, i.e., nearest neighbour interactions,
2. Let $N = 2m$, and m is an positive integer. For $0 \le \ell \le m$, there are interactions between the following pairs of spins,

$$\begin{cases} i = 2(\ell+1)M - k \ , \\ i' = 2\ell M + 1 + k \ , \end{cases} \quad \text{where } k = 0, 1, 2, \cdots, M-1 \tag{142}$$

or explicitly see Fig.5.1(b)

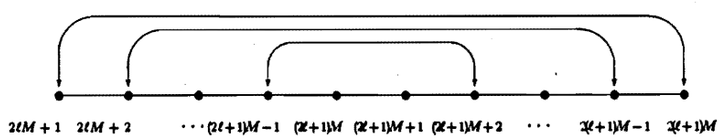


Fig.5.1(b). Each overbrace denotes a non-nearest interaction between two indicated spins.

3. For $1 \le \ell \le m$, there are interactions between the following pairs of spins,

$$\begin{cases} i = (2\ell+1)M - k \ , \\ i' = (2\ell-1)M + 1 + k \ , \end{cases} \quad \text{where } k = 0, 1, 2, \cdots, M-1 \tag{143}$$

or explicitly see Fig.5.1(c),

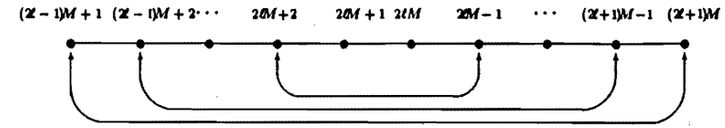


Fig.5.1(c). An underbrace also denotes a nonnearest interaction

If N =odd, then the route is shown in Fig.5.1(d).

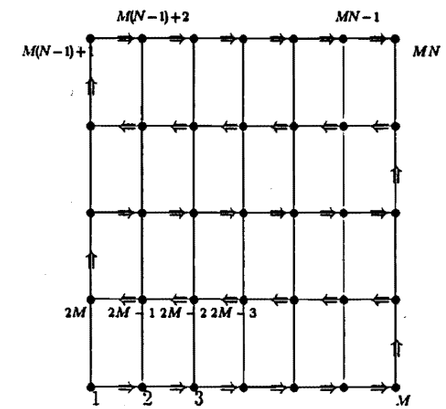


Fig.5.1(d). The arrow ⇒ indicates the way we relabel the sites.

The interactions can also be similarly classified into the three items listed for even M. And the items 2 and 3 repeat after every $2M$ spins, to form cycles, but the first M and last $2M$ spins are out of the cycles. There are interactions of item 2 in the first M spins and interactions of item 3 in the last $2M$ spins.

5.2 The Second Step

We are to show that for the 1D case the near neighbour interactions can be decomposed into nearest neighbour interactions.

Let's go back to the 2D version of the problem, and consider a cell at the positions (i, j), $(i, j+1)$, $(i+1, j)$ and $(i+1, j+1)$ denoted

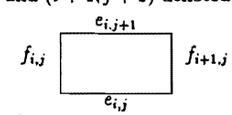


Fig.5.2.

where $e_{i,j}$ and $f_{i,j}$ denote the terms of Hamiltonian (140) at these positions [5],

$$e_{i,j} = -\frac{1}{2}\left(\sigma^x_{i,j}\sigma^x_{i+1,j} + \sigma^y_{i,j}\sigma^y_{i+1,j} + \sigma^z_{i,j}\sigma^z_{i+1,j}\right) \tag{144}$$

and

$$f_{i,j} = -\frac{1}{2}\left(\sigma^x_{i,j}\sigma^x_{i,j+1} + \sigma^y_{i,j}\sigma^y_{i,j+1} + \sigma^z_{i,j}\sigma^z_{i,j+1}\right) , \tag{145}$$

so that

$$H^{2D} = G\sum_{i=1}^{M-1}\sum_{j=1}^{N-1}(e_{i,j} + f_{i,j}) . \tag{146}$$

It is easy to find that $e_{i,j}$ and $f_{i,j}$ satisfy the algebraic relations (128) and form an algebra. This algebra can be reduced to the TLJ algebra [1] at $q = 1$ since

$$\begin{aligned} f_{i,j} &= e_{i,j+1} + f_{i+1,j} + e_{i,j} \\ &\quad - \{e_{i,j+1}, f_{i+1,j}\} - \{f_{i+1,j}, e_{i,j}\} - \{e_{i,j+1}, e_{i,j}\} \\ &\quad + e_{i,j+1}f_{i+1,j}e_{i,j} + e_{i,j+1}e_{i,j}f_{i+1,j} + f_{i+1,j}e_{i,j+1}e_{i,j} + e_{i,j}f_{i+1,j}e_{i,j+1} , \end{aligned} \tag{147}$$

where $\{a, b\} = ab + ba$. An exchange of $f_{i,j}$ and $f_{i+1,j}$ gives a similar reducing relation. By applying these two reducing relations, any off route interactions in Fig. 5.0 can be expressed by the interactions on route. In other words, an outer overbrace in Fig.5.2(c) (or an underbrace in Fig.5.2(d) can be decomposed into a summation of products of a nearest inner overbrace (or a nearest inner underbrace in Fig.5.2(d) and nearest neighbour interactions, therefore the outer brace disappears. Repeating this procedure, all braces disappear, i.e., one can decompose all braces into summations of products of nearest neighbour interactions.

5.3 The Third Step

We relabel the indices of all the $e_{i,j}$ and $f_{i,j}$ on route in the way denoted in Fig.5.2(a) or Fig. 5.2(e), so that the necessary algebraic elements are e_i with $i = 1, 2, \cdots, MN - 1$, and we arrive at the $MN - 1$ dimensional TLJ algebra at the special case of $q = 1$. Therefore the Hamiltonian (140) can be expressed as a polynomial of these elements. The remaining problem is to find eigenvalues and eigenstates for this 1D Hamiltonian.

5.4 Example: 2×2 Square Lattice

The Hamiltonian reads

$$H = G(e_1 + e_2 + e_3 + e_{14}) . \tag{148}$$

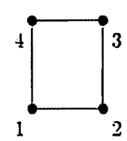


Fig.5.4(a).

First, the spin 5-plet is the ferromagnetic ground state, with $\frac{E}{G} = 0$, and $n = 0$, where all four spins are aligned upward.

When $n = 1$, i.e., there is one spin pointing downward in the highest weight state, the total angular momentum is $J = 1$. Upon the actions of J^-_{tot}, we get four other descendent states to span a solution space. To give the highest weight state(s), we apply the primitive ideal basis and the state(s) should be of the following form

$$\Psi = \left(a_1C^1_1 + a_2C^2_1 + a_3C^3_1\right)|1\rangle , \tag{149}$$

and the Hamiltonian equation yields

$$\begin{array}{llll} (3 - \frac{E}{G})a_1 & +a_2 & +a_3 & = 0 , \\ & (2 - \frac{E}{G})a_2 & & = 0 , \\ a_1 & +a_2 & +(3 - \frac{E}{G})a_3 & = 0 . \end{array} \tag{150}$$

The solutions of the these coupled linear equations yields

$$\left\{ \begin{array}{rcl} \frac{E}{G} &=& 2 , \\ a_1 + a_2 + a_3 &=& 0 ; \end{array} \right. \left\{ \begin{array}{rcl} \frac{E}{G} &=& 4 , \\ a_2 &=& 0 , \\ a_3 &=& a_1 ; \end{array} \right. \tag{151}$$

There is degeneracy of order three for the energy $E = 2$.

When $n = 2$, we have spin singlet states. We apply the basis of Φ_{13}, Φ_{23}, and $\Psi = \Phi_{13} + \Phi_{23}$. We have the solutions

$$\left\{ \begin{array}{rcl} \frac{E}{G} &=& 3 , \\ a_{23} &=& -a_{13} , \end{array} \right. \left\{ \begin{array}{rcl} \frac{E}{G} &=& 6 , \\ a_{23} &=& a_{13} . \end{array} \right. \tag{152}$$

It is easy to see that when $G < 0$, e.g., $G = -1$, we have the antiferromagnetic ground states with energy eigenvalue $E = -6$. The two basis vectors Φ_{13} and Φ_{23} are explicitly depicted in the following figure.

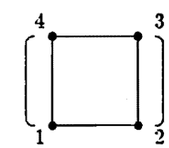


Fig. 5.4(b)

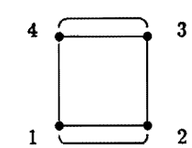


Fig. 5.4(c)

Fig.5.4(b)(c). The braces indicate that the pointed spin pairs are in spin-singlet pairs.

5.5 Another Example: 2×3 Square Lattice

The Hamiltonian reads

$$H = G(e_1 + e_2 + e_3 + e_4 + e_5 + e_{14} + e_{36}) \ , \qquad (153)$$

and the non-nearest neighbour interactions e_{14} and e_{36} can be reduced to nearest neighbour interactions, or the elements of T-L-J algebra $\mathcal{A}_5$, according to our earlier discussions. Further solutions will be routine.

According to our discussions in the first three subsections, this small lattice can be equivalently deformed into the following linear chain of six spin aligned along a line and with non-nearest neighbour interactions shown in Fig.5.5(b).

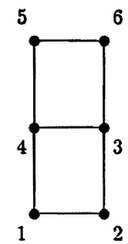


Fig.5.5(a).

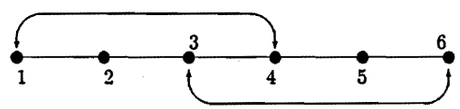


Fig.5.5(b). Each overbrace denotes a nonnearest interaction between two spins.

First, the spin 7-plet is the ferromagnetic ground state, with $\frac{E}{G} = 0$, and $n = 0$, where all spins are aligned upward. When $n = 1$, i.e., in the highest weight state, there is one spin pointing downward, the total angular momentum is $J = 2$. Upon the actions of $J^-_{\rm tot}$, we get four other descendent states spanning a solution space. To give the highest weight state(s), we apply the primitive ideal basis and the state(s) should be of the following form

$$\Psi = (a_1 C_1 + a_2 C_2 + a_3 C_3 + a_4 C_4 + a_5 C_5)\,|1\rangle \ , \qquad (154)$$

and the Hamiltonian equation yields

$$\begin{array}{lllll} (3-\frac{E}{G})a_1 & +a_2 & +a_3 & +a_4 & & = 0 \ , \\ & (2-\frac{E}{G})a_2 & & -a_4 & & = 0 \ , \\ a_1 & +2a_2 & +(4-\frac{E}{G})a_3 & +2a_4 & +a_5 & = 0 \ , \\ & -a_2 & & +(2-\frac{E}{G})a_4 & & = 0 \ , \\ & a_2 & +a_3 & +a_4 & +(3-\frac{E}{G})a_5 & = 0 \ . \end{array} \qquad (155)$$

The solutions of the these coupled linear equations yields

$$\left\{\begin{array}{l} \frac{E}{G} = 1 \ , \\ a_2 = -2a_1 \ , \\ a_3 = 2a_1 \ , \\ a_4 = -2a_1 \ , \\ a_5 = a_1 \ ; \end{array}\right. \quad \left\{\begin{array}{l} \frac{E}{G} = 2 \ , \\ a_2 = 0 \ , \\ a_3 = -a_1 \ , \\ a_4 = 0 \ , \\ a_5 = a_1 \ ; \end{array}\right. \quad \left\{\begin{array}{l} \frac{E}{G} = 3 \ , \\ a_3 = 0 \ , \\ a_4 = -a_2 \ , \\ a_5 = -a_1 \ ; \end{array}\right. \quad \left\{\begin{array}{l} \frac{E}{G} = 5 \ , \\ a_2 = 0 \ , \\ a_3 = 2a_1 \ , \\ a_4 = 0 \ , \\ a_5 = a_1 \ . \end{array}\right. \qquad (156)$$

There is degeneracy of order two for the energy $E = 3$.

When $n = 2$, we apply the basis of Φ_{13}, Φ_{23}, Φ_{14}, Φ_{24}, Φ_{34}, Φ_{15}, Φ_{25}, Φ_{35} and Φ_{45}. let us affix same indices to coeficients a_{ij} as Φ_{ij}, we have finally the following solutions

$$\begin{array}{l} \frac{E}{G} = 3 \ , \\ a_{14} = 2a_{13} \ , \\ a_{15} = -2a_{13} \ , \\ a_{23} = 0 \ , \\ a_{24} = 0 \ , \\ a_{25} = 2a_{13} \ , \\ a_{34} = -4a_{13} \ , \\ a_{35} = a_{13} \ , \\ a_{45} = 0 \ ; \end{array} \quad \begin{array}{l} \frac{E}{G} = 5 \ , \\ a_{14} = -2a_{13} \ , \\ a_{15} = 0 \ , \\ a_{23} = 0 \ , \\ a_{24} = 0 \ , \\ a_{25} = 2a_{13} \ , \\ a_{34} = 0 \ , \\ a_{35} = -a_{13} \ , \\ a_{45} = 0 \ ; \end{array} \quad \begin{array}{l} \frac{E}{G} = \frac{9\pm\sqrt{17}}{2} \doteq \begin{array}{l} 5.823 \\ 3.177 \end{array} \\ a_{14} = 0 \ , \\ a_{15} = 0 \ , \\ a_{23} = \pm\frac{\sqrt{17}\mp 1}{4} a_{13} \ , \\ a_{24} = 0 \ , \\ a_{25} = 0 \ , \\ a_{34} = 0 \ , \\ a_{35} = -a_{13} \ , \\ a_{45} = \mp\frac{\sqrt{17}\mp 1}{4} a_{13} \ ; \end{array} \quad \begin{array}{l} \frac{E}{G} = 6 \ , \\ a_{14} = -a_{13} \ , \\ a_{15} = a_{13} \ , \\ a_{23} = 0 \ , \\ a_{24} = 0 \ , \\ a_{25} = -a_{13} \ , \\ a_{34} = -a_{13} \ , \\ a_{35} = a_{13} \ , \\ a_{45} = 0 \ . \end{array} \qquad (157)$$

When $n = 3$, we have fewer basis, explicitly, Φ_{135}, Φ_{145}, Φ_{235}, Φ_{245} and Φ_{345}, and we give the solutions

$$\left\{\begin{array}{l} \frac{E}{G} = 4 \ , \\ a_{145} = a_{135} \ , \\ a_{235} = -a_{135} \ , \\ a_{245} = 2a_{135} \ , \\ a_{345} = -a_{135} \ , \end{array}\right. \quad \left\{\begin{array}{l} \frac{E}{G} = 6 \ , \\ a_{135} = 0 \ , \\ a_{235} = -a_{145} \ , \\ a_{245} = 0 \ , \\ a_{345} = 0 \ . \end{array}\right. \qquad (158)$$

They are spin singlets.

At last about this example, we remark that when $G < 0$, e.g., $G = -1$, we have the antiferromagnetic ground states with energy eigenvalue $E = -6$, which is of order two degeneracy, with $n = 2$ and $n = 3$. But generaly, it is believed that $n = 2$ state comes into AGS because of boundary effect. The spin singlet AGS is more interesting, and we write it explicitly

$$^3\Psi = \Phi_{145} - \Psi_{235} \ , \qquad (159)$$

where

$$\begin{array}{rcl} \Phi_{145} & = & C_1^1 C_3^4 C_5^5 = e_1 e_4 e_3 e_5 |1\ 3\ 5\rangle \ , \\ & = & 32|(1,2)_s(3,6)_s(4,5)_s\rangle \ , \\ \Phi_{235} & = & C_1^2 C_3^3 C_5^5 = e_2 e_1 e_3 e_5 |1\ 3\ 5\rangle \ , \\ & = & 32|(1,4)_s(2,3)_s(5,6)_s\rangle \ , \end{array} \qquad (160)$$

see Fig.5.5(c)(d).

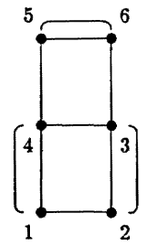


Fig.5.5(c).

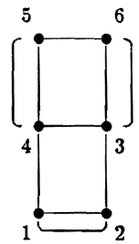


Fig.5.5(d).

Fig.(5.5)(c),(d) The brace indicated spins pairs are in spin-singlet pairs. Shown in Fig.(c) is the solution Φ_{145}, while Fig.(d) represents Φ_{235}.

6 Auxiliary Discrete Symmetry

We have shown how to reduce the two dimensional lattices into chain without changing physical properties in that the latter are easier to manage. However, the way to reduce the lattice into chain is not unique. Actually any random relabelling works. Relabellings do not change the Hamiltonian of the system, and such relabellings are implicit discrete symmetries of this Hamiltonian. We will show that making use of such discrete symmetries will simplify the problem greatly. Let's first prove an useful theorem

Theorem For a system with $N+1$ spins, the $n = n_0$ eigenstates are exhausted in the eigenstates of any possible relabelling symmetry.

To prove this theorem, we suppose that there is a relabelling operation $\hat{\tau}$, and the eigenstates of Hamiltonian H are Φ_i with $i \in S(= \{1, 2, \cdots\})$. Some of Φ_i are coincidently the eigenstates of $\hat{\tau}$, and we forget them. We suppose Φ_i with $i \in S' \subseteq S\})$ are not eigenstates of $\hat{\tau}$, and all these states are collected in set $\mathcal{E}$. Then we claim that $\hat{\tau}\Phi_i \in \mathcal{E}$, and we must have a Hilbert space of $\hat{\tau}$, given by Φ_{i_1}, Φ_{i_2}, $\cdots$, and Φ_{i_m} such that

$$\begin{aligned} \hat{\tau}\Phi_{i_1} &\sim \Phi_{i_2} , \\ \hat{\tau}\Phi_{i_2} &\sim \Phi_{i_3} , \\ \cdots \quad & \quad \cdots \\ \hat{\tau}\Phi_{i_m} &\sim \Phi_{i_1} , \end{aligned} \tag{161}$$

and the sign $\sim$ can be replaced by $=$ if they are properly renormalized. This claim is correct because a relabelling can not change physics, and for the same reason, we claim that all Φ_{i_k} listed above must have the same energy eigenvalue. Therefore we are always able to superpose them to get the eigenstates of $\hat{\tau}$-operation. And it is easy to see that the eigenvalues of $\hat{\tau}$ are $\exp\left(1\frac{2\pi}{m}\right)$.

In the following, we show, by some definite examples, that relabellings are nothing but rotations, projections in most cases.

6.1 The Triangle with 3 Spins

This is the example given in section 4.3 and we reconsider it in the case of $n = 1$ in view of the symmetries of relabellings. And by a simple observation, we can see that any relabelling is an operation of the D_3 symmetric about the center of this triangle (assumed standard).

It is known that its basis for $n = 1$ states are χ_1 and χ_2. We will add one more vector

$$\chi_3 = |(3,1)_s\rangle = |\downarrow\uparrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle , \tag{162}$$

which is a superposition of the two basis vectors,

$$\chi_1 + \chi_2 + \chi_3 = 0 . \tag{163}$$

The three vectors are depicted in the following graphs.

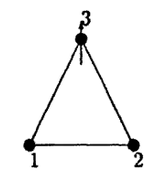


Fig.6.1(a) χ_1

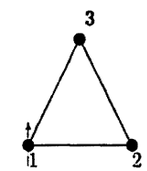


Fig.6.1(b) χ_2

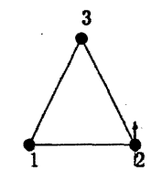


Fig.6.1(c) χ_3

As the lattice, and therefore the Hamiltonian H, is D_3 symmetric, so we have

$$[\,\hat{\tau}, H\,] = 0 , \quad \text{where } \hat{\tau} \in D_3 \tag{164}$$

which means that the operators $\hat{\tau}$ and H have simultaneous the same eigenstates. So we can begin our searching for the eigenstates of H by searching for the eigenstates of $\hat{\tau}$.

Suppose that Φ is an eigenstate of $\hat{\tau}$, we expect that Φ is a superposition of χ_i $(i = 1, 2, 3)$, or explicitly

$$\Phi = \chi_1 + a_1\chi_2 + a_2\chi_3 , \tag{165}$$

where a_1 and a_2 are coefficients to be determined. As the action of $\hat{\tau}$ is a rotation of 120° anti-clockwise, the result is

$$\tau\Phi = \chi_3 + a_1\chi_1 + a_2\chi_2 . \tag{166}$$

From

$$\hat{\tau}\Phi = \tau\Phi , \tag{167}$$

we have

$$\tau^3 = 1 , \quad a_1 = \tau^2 , \quad a_2 = \tau , \tag{168}$$

therefore

$$\begin{aligned} \Phi_1 &= \chi_1 + e^{i\frac{2\pi}{3}}\chi_2 + e^{i\frac{\pi}{3}}\chi_3 , \\ \Phi_2 &= \chi_1 + e^{i\frac{\pi}{3}}\chi_2 + e^{i\frac{2\pi}{3}}\chi_3 , \end{aligned} \tag{169}$$

which are degenerated in energy eigenvalues. It is easy to see that χ_1 and χ_2 are both solutions with energy eigenvalue degenerated. This is just the result we obtained in section 4.3.

Generaly, if the configuration of the lattice gives symmetry (discrete) group G to its Hamiltonian H, i.e.,

$$[\,\hat{g}, H\,] = 0\,, \quad \text{where } \hat{g} \in G, \tag{170}$$

the eigenstates of H can be found in the eigenstates of operators $\hat{g}$.

6.2 The Square with 4 Spins

We reconsider the square solved in 5.4 in view of the symmetry D_4 of its configuration. As we pointed out in the basis for $n = 2$ states are Φ_{13} and Φ_{23} as depicted in Figs. 5.4(b)(c).

Now it is apparent that

$$[\,\hat{\tau}, H\,] = 0\,, \quad \text{where } \hat{\tau} \in D_4\,, \tag{171}$$

so we classify the basis vectors of the Hilbert space by their D_4-symmetry properties. Suppose that Φ is an eigenstate of $\hat{\tau}$, we expect that Φ is a superposition of Φ_i ($i = 13, 23$), or explicitly

$$\Phi = a_1\Phi_{13} + a_2\Phi_{23}\,, \tag{172}$$

where a_i are coefficients to be determined, and

$$\hat{\tau}\Phi = \tau\Phi\,. \tag{173}$$

As the action of $\hat{\tau}$ is a rotation of 90° anti-clockwise, the result of such an action is

$$\tau\Phi = a_2\Phi_1 + a_1\Phi_2\,. \tag{174}$$

From eq.(173) we have

$$\tau^2 = 1\,, \quad a_1 = \pm a_2 = \pm 1\,, \tag{175}$$

and so

$$\Phi_{1,2} = \Phi_{13} \pm \Phi_{23}\,. \tag{176}$$

This is just the result we obtained in section 5.4.

6.3 The Tetraploid with 4 Spins

We consider a new example, a lattice with interactions depicted in the Fig.6.3(a).

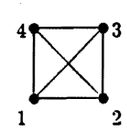


Fig.6.3(a).

This is actually the simpliest 3-dimensional spin model, a tetraploid, as redepict in Fig.6.3(b). The Hamiltonian reads

$$H = e_1 + e_2 + e_3 + e_{1,3} + e_{1,4} + e_{2,4}\,, \tag{177}$$

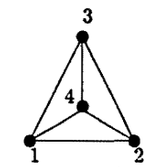


Fig.6.3(b)

and according to our discussion in section 5.4, there are two basis vectors, Φ_{13} and Φ_{23}, to span the Hilbert space and we now add one more vector χ to make it clear that the three vectors span the space of the operators $\hat{\tau} \in D_4$:

$$\chi = |(31)_s(24)_s\rangle = -|34\rangle + |23\rangle + |14\rangle - |12\rangle\,, \tag{178}$$

It is easy to show that

$$\chi + \Phi_{13} + \Phi_{23} = 0\,. \tag{179}$$

We depict the three vectors in Fig.6.3(c)(d)(e).

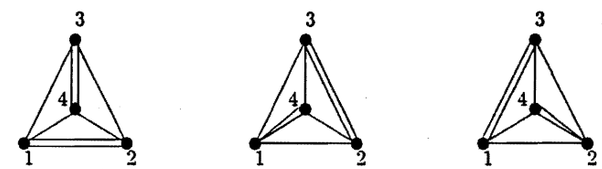


Fig.6.3(c). Φ_{13} *Fig.6.3(d). Φ_{23}* *Fig.6.3(d).χ*

Fig.6.3(c)(d)(e). The spins connected by double lines are in singlets.

Now we suppose that the eigenstates of the group operators $\hat{\tau} \in D_4$ are of the following form

$$\Phi \triangleq [a_1, a_2, a_3] \triangleq a_1\Phi_{13} + a_2\Phi_{23} + a_3\chi\,, \tag{180}$$

where a_i are coefficients to be determined. Perform the rotation operation (120° anti-clockwise) $\hat{\tau}_1$ we get

$$\Phi \triangleq [a_3, a_1, a_2]\,, \tag{181}$$

and from

$$\hat{\tau}_1\Phi = \tau_1\Phi\,, \tag{182}$$

we have

$$\tau_1^3 = 1\,, \quad a_1 = 1\,, \quad a_2 = \tau_1^{-1}\,, \quad a_3 = \tau_1\,. \tag{183}$$

Hence we have the eigenstates for $\hat{\tau}_1$ (other D_4-operations are automatically satisfied)

$$\begin{aligned} \Phi_1 &= \Phi_{13} - e^{i\frac{\pi}{3}}\Phi_{23}\,, \\ \Phi_2 &= \Phi_{13} - e^{-i\frac{\pi}{3}}\Phi_{23}\,, \end{aligned} \tag{184}$$

which are eigenstates with different eigenvalues $e^{\pm i\frac{\pi}{3}}$. These two states should be super-composed to form the eigenstates of Hamiltonian. And the result is: any superposition of Φ_1 and Φ_2 is the eigenstate, as they have degenerated energy eigenvalues.

6.4 The Tetraploid with 6 Spins

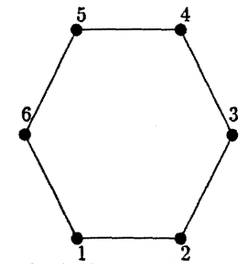


Fig.6.4(a)

We consider a circular linear chain with 6 spins, Fig.6.4(a), with the Hamiltonian reads

$$H = \sum_{i=1}^{5} e_i + e_{1,6} \,. \qquad (185)$$

According to the theory (see section 2), there are five highest weight vectors, $\Phi_{m_1 m_2 m_3} = C_{m_1 m_2 m_3}|135\rangle$ as depicted in Figs.(b)-(f).

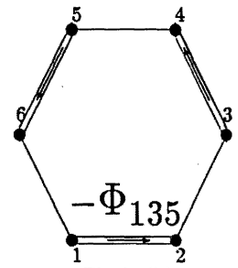


Fig.6.4(b).

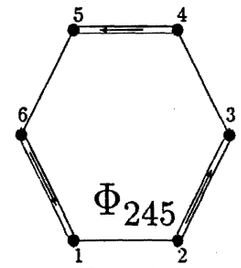


Fig.6.4(c).

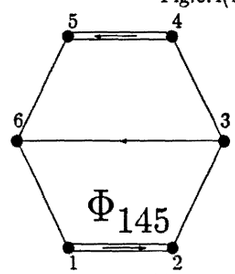


Fig.6.4(d).

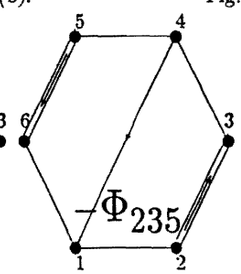


Fig.6.4(e).

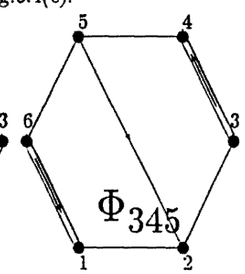
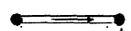


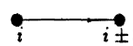
Fig.6.4(f).

where



$\triangleq |(i, i+1)_s\rangle$,

and



$\triangleq |(i, i \pm 3)_s\rangle$.

The eigenstates of the Hamiltonian H should be of the following form

$$\Phi = \sum_{m_1, m_2, m_3} a_{m_1, m_2, m_3} \Phi_{m_1, m_2, m_3} \,, \qquad (186)$$

and will be denoted as

$$\Phi = \begin{bmatrix} a_{135} & a_{245} & \\ a_{145} & a_{235} & a_{345} \end{bmatrix} . \qquad (187)$$

And now we consider the symmetry of this lattice system, the anti-clockwise rotation of 60° (which can also be viewed as a relabelling of spin indices of $1 \to 6$, $6 \to 5$, $\cdots$, $5 \to 4$ and will be denoted as operator $\hat{r}$). According to our theorem, we have

$$\hat{r}\Phi = -\begin{bmatrix} a_{245} & a_{135} & \\ a_{345} & a_{145} & a_{235} \end{bmatrix} = -r\Phi \,, \qquad (188)$$

and easily we have the following solutions

$$\begin{aligned} r_1 &= -1 \,, \quad \Phi_1 &= \begin{bmatrix} 1 & -1 & \\ 0 & 0 & 0 \end{bmatrix} , \\ r_2 &= 1 \,, \quad \Phi_2 &= \begin{bmatrix} a_1 & a_1 & \\ b_1 & b_1 & b_1 \end{bmatrix} , \\ r_{3,4} &= e^{\pm i\frac{2\pi}{3}} \,, \quad \Phi_{3,4} &= \begin{bmatrix} 0 & 0 & \\ 1 & e^{\mp i\frac{2\pi}{3}} & e^{\pm i\frac{2\pi}{3}} \end{bmatrix} . \end{aligned} \qquad (189)$$

By checking directly, we see that Φ_1 is really a solution with energy eigenvalue $E = 6$, while Φ_2 is not. $\Phi_{3,4}$ are solutions with identical energy eigenvalues $E = 4$.

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Figure 4: *The pairs of the spins indicated by the over ovals are spin singlet pairs.*

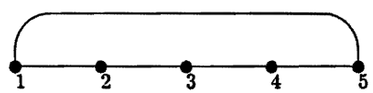

Figure 5: *Circular chain consisted of five spins in nearest neighbour interactions.*

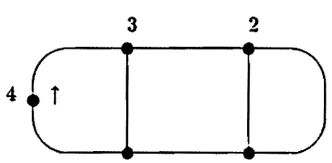

Figure 6: *Spins 1,2 and 3,5 are in singlet pairs, and spin 4 is arbitrary. This state is a spin doublet.* Fig.2.5 (b)

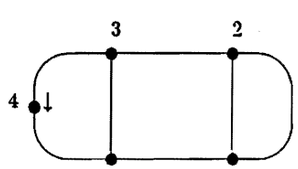

Figure 7: *Spins 1,2 and 3,5 are in singlet pairs, and spin 4 is arbitrary. This state is a spin doublet.* Fig.2.5 (c)