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# QUANTUM MINKOWSKI SPACE AND GENERATORS OF QUANTUM LORENTZ GROUP

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# Quantum Minkowski Space and Generators of Quantum Lorentz Group [1]

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## Abstract

From the basic $4\times 4$ $\check{R}$ matrix associated to quantum Lorentz group $SL_q(2,C)$ and its various fusion matrices, the covariant differential calculus on the quantum Minkowski space and the $\check{R}$ commutation relation for the covariant generators of quantum Lorentz group are presented.

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## 1. Introduction

Recently much attention has been payed to the investigation on the quantum groups[1,2,3] and the Yang-Baxter equations [4]. In this talk we will report some recent progress in the research of quantum Minkowski space and quantum Lorentz group.

After giving some preliminary results on $4\times 4$ $\check{R}$ matrix associated to $SL_q(2,C)$ and related quantities, we describe the properties of various matrices fusion from $\check{R}$ in the next section. The covariant differential calculus on quantum Minkowski space $M_q$ and the $\check{R}$ commutation relation for the covariant generators of quantum Lorentz group are discussed in Sec. 3 and 4 respectively.

We start from the quantum $\check{R}$ matrix associated with the quantum group $SL_q(2,C)$

$$||\check{R}|| = \left(\check{R}(q)^{\alpha\beta}{}_{\gamma\delta}\right) = q^{-1/2}\begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix} \tag{1.1}$$

which satisfies the Yang-Baxter equation (YBE)

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{1.2}$$

and the reduction equation

$$(\check{R}-\mu_1)(\check{R}-\mu_2) = 0\ ,\mu_1(q) = q^{1/2},\ \mu_2(q) = -q^{-3/2}. \tag{1.3}$$

The left-acting eigenvectors $t_m(q)^{\alpha\beta}$ and $s(q)^{\alpha\beta}$, as well as the right-acting ones $\bar{t}^m(q)_{\alpha\beta}$ and $\bar{s}(q)_{\alpha\beta}$ are defined by the eigenvalue equations [5]

$$\begin{aligned} &\check{R}(q)^{\alpha\beta}{}_{\gamma\delta}\ t_m(q)^{\gamma\delta} = \mu_2\ t_m(q)^{\alpha\beta}\ ,\quad \check{R}(q)^{\alpha\beta}{}_{\gamma\delta}\ s(q)^{\gamma\delta} = \mu_1\ s(q)^{\alpha\beta}\ . \\ &\bar{t}^m(q)_{\alpha\beta}\check{R}(q)^{\alpha\beta}{}_{\gamma\delta} = \mu_2\ \bar{t}^m(q)_{\gamma\delta}\ ,\quad \bar{s}(q)_{\alpha\beta}\check{R}(q)^{\alpha\beta}{}_{\gamma\delta} = \mu_1\ \bar{s}(q)_{\gamma\delta}\ . \end{aligned} \tag{1.4}$$

Since $\check{R}$ matrix is symmetric under the transposition, the components of left and right eigenvectors can be chosen as the same

$$\begin{aligned} &t_+(q)^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \bar{t}^+(q)_{\alpha\beta},\quad t_3(q)^{\alpha\beta} = \begin{pmatrix} 0 & -q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix}[2]^{-1/2} = \bar{t}^3(q)_{\alpha\beta}\ , \\ &t_-(q)^{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \bar{t}^-(q)_{\alpha\beta},\quad s(q)^{\alpha\beta} = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}[2]^{-1/2} = \bar{s}(q)_{\alpha\beta}, \end{aligned} \tag{1.5}$$

In this choice $t$'s are $q$-symmetric and $s$ $q$-anti-symmetric, i.e.,

$$t_m(q)^{\beta\alpha} = t_m(q^{-1})^{\alpha\beta}, \; s(q)^{\beta\alpha} = -s(q^{-1})^{\alpha\beta} . \tag{1.6}$$

The 2-dim Levi-Civita symbols are related to the singlet eigenvectors

$$\epsilon(q)_{\alpha\beta} = -[2]^{1/2}\bar{s}(q)_{\alpha\beta} , \quad \epsilon(q)^{\alpha\beta} = [2]^{1/2}s(q)^{\alpha\beta} \tag{1.7a}$$

and then

$$\epsilon(q)_{\alpha\beta}\epsilon(q)^{\beta\gamma} = \delta_\alpha{}^\gamma = \epsilon(q)^{\gamma\beta}\epsilon(q)_{\beta\alpha} . \tag{1.7b}$$

For latter use we group $t$'s and $s$ into a matrix-valued four vector

$$\begin{aligned} t_\mu(q) &= (t_0(q), \; t_m(q)) , \quad t_0(q)^{\alpha\beta} = q\; s(q)^{\alpha\beta} ; \\ \bar{t}^\mu(q) &= (\bar{t}^0(q), \; \bar{t}^m(q)) , \quad \bar{t}^0(q)_{\alpha\beta} = q^{-1}\; \bar{s}(q)_{\alpha\beta} . \end{aligned} \tag{1.8}$$

They satisfy the orthogonality condition

$$\bar{t}^\nu(q)_{\alpha\beta}\; t_\mu(q)^{\alpha\beta} = \delta^\nu{}_\mu , \tag{1.9a}$$

and the completeness relation

$$t_\mu(q)^{\alpha\beta}\; \bar{t}^\mu(q)_{\gamma\delta} = \delta^\alpha{}_\gamma\delta^\beta{}_\delta \equiv E^{\alpha\beta}{}_{\gamma\delta} , \tag{1.9b}$$

Define the projection operators

$$Q^{(2)}(q)^{\alpha\beta}{}_{\gamma\delta} = t_m(q)^{\alpha\beta}\bar{t}^m(q)_{\gamma\delta} , \quad Q^{(1)}(q)^{\alpha\beta}{}_{\gamma\delta} = s(q)^{\alpha\beta}\bar{s}(q)_{\gamma\delta} \tag{1.10}$$

then

$$Q^{(i)}Q^{(j)} = \delta^{ij}Q^{(j)} , \quad Q^{(1)} + Q^{(2)} = E . \tag{1.11}$$

Now since $\check{R}$ can be expressed as the linear combination of $Q^{(1)}(q)$ and $Q^{(2)}(q)$

$$\check{R}(q) = \mu_2 Q^{(2)}(q) + \mu_1 Q^{(1)}(q), \tag{1.12}$$

we can obtain

$$Q^{(2)} = \frac{\check{R} - \mu_1 E}{\mu_2 - \mu_1} , \quad Q^{(1)} = \frac{\check{R} - \mu_2 E}{\mu_1 - \mu_2} , \tag{1.13}$$

from which follow the useful relations

$$Q^{(i)}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}Q^{(i)}_{23} , \quad \check{R}_{12}\check{R}_{23}Q^{(i)}_{12} = Q^{(i)}_{23}\check{R}_{12}\check{R}_{23} . \tag{1.14}$$



Quantum group $SL(2,C)$ characterized by quantum matrices $M$ and $\bar{M} = (M^+)^{-1}$. Both of them satisfy the Yang-Baxter relations

$$\check{R}_{12}M_1M_2 = M_1M_2\check{R}_{12}, \; \check{R}_{12}\bar{M}_1\bar{M}_2 = \bar{M}_1\bar{M}_2\check{R}_{12} \tag{1.15a}$$

and another mixed one [6]

$$\check{R}_{12}M_1\bar{M}_2 = \bar{M}_1M_2\check{R}_{12} . \tag{1.15b}$$

For quantum group $SU_q(2)$, $\bar{M} = M$, then three relations coincide into single one. Eq(1.13) allows one to get a relation similar to (1.14)

$$Q^{(i)}_{12}M_1M_2 = M_1M_2Q^{(i)}_{12}, \; Q^{(i)}_{12}\bar{M}_1\bar{M}_2 = \bar{M}_1\bar{M}_2Q^{(i)}_{12} . \tag{1.16}$$

All these relations given above can be represented diagrammatically.

## 2. $\check{R}$ Matrices From Fusion

Various higher dim $\check{R}$ matrices can be obtained from fusion of $\check{R}$ in (1.1). The one associated with the quantum Lorentz group $SO_q(3,1)$ [6,7] is

$$\check{\mathcal{R}}_{(12)(34)} = \check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}^{-1}_{23}, \; \check{\mathcal{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'} = \check{R}^{\beta\gamma}{}_{bc}\check{R}^{\alpha b}{}_{\alpha'b'}\check{R}^{c\delta}{}_{c'\delta'}\check{R}^{-1}(q)^{b'c'}{}_{\beta'\gamma'} \tag{2.1}$$

And another one

$$\check{\mathbf{R}}_{(12)(34)} = \check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}_{23}, \; \check{\mathbf{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'} = \check{R}^{\beta\gamma}{}_{bc}\check{R}^{\alpha b}{}_{\alpha'b'}\check{R}^{c\delta}{}_{c'\delta'}\check{R}^{b'c'}{}_{\beta'\gamma'} \tag{2.2}$$

represents a reducible $\check{R}$ matrix [5]. The reduction can be shown by multiplying $\bar{t}^\mu(q)_{\alpha\beta}$ etc to convert the bispinor indices into four vector ones

$$\begin{aligned} \check{\mathbf{R}}^{\mu\nu}{}_{\kappa\lambda} &\equiv \bar{t}^\mu(q)_{\alpha\beta}\bar{t}^\nu(q)_{\gamma\delta}\check{\mathbf{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'}t_\kappa(q)^{\alpha'\beta'}t_\lambda(q)^{\gamma'\delta'} \\ &= \check{\mathbf{R}}^{00}{}_{00} \oplus \check{\mathbf{R}}^{m0}{}_{0l} \oplus \check{\mathbf{R}}^{0n}{}_{k0} \oplus \check{\mathbf{R}}^{mn}{}_{kl} . \end{aligned} \tag{2.3}$$

Correspondingly the tensor product of two quantum matrices $\Omega_{(12)} = M_1 \otimes M_2$ is converted into

$$\Omega^\mu{}_\nu = \bar{t}^\mu(q)_{\alpha\gamma}M^\alpha{}_\beta M^\gamma{}_\delta t_\nu(q)^{\beta\delta} = \Omega^0{}_0 \oplus \Omega^m{}_n . \tag{2.4}$$

The reduction comes from (1.14) and (1.16).

It is easy to show that $\check{\mathbf{R}}$ and $\Omega$ satisfy their own YBR and YBE

$$\check{\mathbf{R}}_{(12)(34)}\Omega_{(12)}\Omega_{(34)} = \Omega_{(12)}\Omega_{(34)}\check{\mathbf{R}}_{(12)(34)},$$

$$\check{\mathbf{R}}_{(12)(34)}\check{\mathbf{R}}_{(34)(56)}\check{\mathbf{R}}_{(12)(34)} = \check{\mathbf{R}}_{(34)(56)}\check{\mathbf{R}}_{(12)(34)}\check{\mathbf{R}}_{(34)(56)}. \tag{2.5}$$

Or equivalently, their reduction form $(m, n, \cdots = +, 3, -)$

$$\check{\mathbf{R}}^{mn}{}_{kl}\Omega^k{}_i\Omega^l{}_j = \Omega^m{}_k\Omega^n{}_l\check{\mathbf{R}}^{kl}{}_{ij},$$

$$\check{\mathbf{R}}^{mn}{}_{m'n'}\check{\mathbf{R}}^{n'l}{}_{n''k}\check{\mathbf{R}}^{m'n''}{}_{ij} = \check{\mathbf{R}}^{nl}{}_{n'l'}\check{\mathbf{R}}^{mn'}{}_{in''}\check{\mathbf{R}}^{n''l'}{}_{jk}\ . \tag{2.6}$$

Comparing to the standard form [2], one can see that [5] $\check{\mathbf{R}}^{mn}{}_{kl}$ is nothing but the $\check{R}$ matrix associated with the quantum deformation of $SO(3)$ with parameter $q^2$. Similar to the eigenequation (1.4), $\check{\mathbf{R}}^{mn}{}_{kl}$ has a singlet eigenvector

$$\mathbf{g}(q^2)^{mn} = (\mathbf{g}(q^2)^{+-}, \mathbf{g}(q^2)^{00}, \mathbf{g}(q^2)^{-+}) = (q^{-1}, 1, q) = \mathbf{g}(q^2)_{kl} \tag{2.7}$$

with the property

$$\mathbf{g}(q^2)_{mn}\Omega^m{}_k\Omega^n{}_l = \mathbf{g}(q^2)_{kl},\ \Omega^m{}_k\Omega^n{}_l\mathbf{g}(q^2)^{kl} = \mathbf{g}(q^2)^{mn}\ . \tag{2.8}$$

Therefore $\mathbf{g}(q^2)_{mn}$ can be identified as the deformed Euclidean metric in 3-dim quantum orthogonal space.

In a similar manner $\check{\mathcal{R}}_{(12)(34)}$ in (2.1) and the product $\Lambda_{12} = M_1 \otimes \bar{M}_2$ satisfy the YBR and YBE exactly the same as those in (2.5). They can be converted to the vectorial form as in Eqs (2.3) and (2.4). The $16 \times 16$ matrix $\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}$ has three different eigenvalues

$$\lambda_2(q) = q,\ \lambda_1(q) = -q^{-1},\ \lambda_0(q) = q^{-3}, \tag{2.9}$$

corresponding to a nonet, a sextet and a singlet respectively. The singlet eigenvector

$$g(q)^{\mu\nu} = (g^{+-}, g^{33}, g^{30}, g^{03}, g^{00}, g^{-+}) = (-q^{-1}, -1, 0, 0, 1, -q) = g(q)_{\mu\nu} \tag{2.10}$$

satisfy [7]

$$g(q)^{\mu\lambda}g(q)_{\lambda\nu} = \delta^\mu{}_\nu = g(q)_{\nu\lambda}g(q)^{\lambda\mu} \tag{2.11}$$

$$g(q)_{\mu\nu}\Lambda^\mu{}_\kappa\Lambda^\nu{}_\lambda = g(q)_{\kappa\lambda},\quad \Lambda^\mu{}_\kappa\Lambda^\nu{}_\lambda g(q)^{\kappa\lambda} = g(q)^{\mu\nu}. \tag{2.12}$$

This implies that $\Lambda^\mu{}_\nu$ is the $q$-deformed Lorentz transformation and $g(q)_{\mu\nu}$ the $q$-analogue of the Minkowski metric. The sextet eigenvector $u_{ms}(q)^{\mu\nu}$ $(m = +, 3, -, s = L, R)$ and nonet $w_{mn}(q)^{\mu\nu}$ $(m, n = +, 3, -)$ can also be constructed from the concrete form $\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}$. $u_{ms}$ is $q$-antisymmetric, and the metric q-symmetric

$$u_{ms}(q)^{\nu\mu} = -u_{ms}(q^{-1})^{\mu\nu},\quad g(q)^{\nu\mu} = g(q^{-1})^{\mu\nu}\ . \tag{2.13}$$

Now since $\check{\mathcal{R}}$ is not symmetric under the transformation, its right eigenvectors $\bar{u}^{ms}(q)_{\mu\nu}$ and $\bar{w}^{mn}(q)_{\mu\nu}$ have components different to their left counterparts. But still we can choose them all as orthonormal and complete set. Projection operators can be defined [7]

$$\begin{aligned} Q^{(0)}(q)^{\mu\nu}{}_{\kappa\lambda} &= g(q)^{\mu\nu}\bar{g}(q)_{\kappa\lambda}[2]^{-2}, \\ Q^{(1)}(q)^{\mu\nu}{}_{\kappa\lambda} &= u_{ms}(q)^{\mu\nu}\bar{u}^{ms}(q)_{\kappa\lambda}, \\ Q^{(2)}(q)^{\mu\nu}{}_{\kappa\lambda} &= w_{mn}(q)^{\mu\nu}\bar{w}^{mn}(q)_{\kappa\lambda} \end{aligned} \tag{2.14}$$

with

$$Q^{(i)}Q^{(j)} = \delta^{ij}Q^{(j)},\quad Q^{(0)} + Q^{(1)} + Q^{(2)} = E\ . \tag{2.15}$$

As a matter of fact, we can construct various different "fusion" matrices from quartic product of basic $\check{R}(q)$. Let us define

$$\check{R}_{(12)(34)}(N) = \check{R}(a, b, c, d) = \check{R}^{(a)}_{(23)}\check{R}^{(b)}_{12}\check{R}^{(c)}_{34}\check{R}^{(d)}_{23} \tag{2.16}$$

with $a, b, c, d = \pm$, $\check{R}^{(+)} = \check{R}, \check{R}^{(-)} = \check{R}^{-1}$. There are totally 16 of them, $N = 1$ to 16.

$$\begin{aligned} &\check{R}(1) = \check{R}(++++), && \check{R}(2) = \check{R}(+++-), \\ &\check{R}(3) = \check{R}(++-+), && \check{R}(4) = \check{R}(++--), \\ &\check{R}(5) = \check{R}(+-++), && \check{R}(6) = \check{R}(+-+-), \\ &\check{R}(7) = \check{R}(+--+), && \check{R}(8) = \check{R}(+---), \\ &\check{R}(9) = \check{R}(-+++), && \check{R}(10) = \check{R}(-++-), \\ &\check{R}(11) = \check{R}(-+-+), && \check{R}(12) = \check{R}(-+--), \\ &\check{R}(13) = \check{R}(--++), && \check{R}(14) = \check{R}(--+-), \\ &\check{R}(15) = \check{R}(---+), && \check{R}(16) = \check{R}(----) \end{aligned} \tag{2.17}$$

We see immediately that $\check{\mathbf{R}}_{(12)(34)} = \check{R}(1)$, $\check{\mathcal{R}}_{(13)(24)} = \check{R}(2)$. Pairs of these matrices are inverse of each other

$$\begin{aligned} &\check{R}(1)^{-1} = \check{R}(16),\quad \check{R}(2)^{-1} = \check{R}(8),\quad \check{R}(3)^{-1} = \check{R}(14),\quad \check{R}(4)^{-1} = \check{R}(6), \\ &\check{R}(5)^{-1} = \check{R}(12),\quad \check{R}(7)^{-1} = \check{R}(10),\quad \check{R}(9)^{-1} = \check{R}(15),\quad \check{R}(11)^{-1} = \check{R}(13), \end{aligned} \tag{2.18}$$

and some of them are connected by transposition

$$\check{R}(2)^t = \check{R}(9),\quad \check{R}(4)^t = \check{R}(11),\quad \check{R}(6)^t = \check{R}(13),\quad \check{R}(8)^t = \check{R}(15) \tag{2.19}$$

and others are symmetrical. These 16 matrices fall into six groups, namely

$$(1,16);\ (3,14);\ (5,12);\ (7,10);\ (2,8,9,15);\ (4,6,11,13).$$

A further look shows that some of the matrices have same topology, e.g., $\check{R}(2)$, $\check{R}(14)$, $\check{R}(9)$ and $\check{R}(12)$ have same twine but different direction diagrammatically. They may represent relevant processes, processes with cross symmetry. We have already noticed that $\check{R}(1)$ and $\check{R}(16)$ describe $SO_{q^2}(3)$ quantum group, and $\check{R}(2)$ and $\check{R}(8)$ quantum group $SO_q(3,1)$. One may imagine that the relevant matrices $\{\check{R}(9),\check{R}(15)\}$, $\{\check{R}(3),\check{R}(14)\}$ and $\{\check{R}(5),\check{R}(12)\}$ also have something to do with quantum Lorentz group.

Of all these 16 matrices, $\check{R}(7)$ and $\check{R}(10)$ are special. All others but these two satisfy their own YBE, i.e., for $N \neq 7,\ 10$

$$\check{R}_{(12)(34)}(N)\check{R}_{(34)(56)}(N)\check{R}_{(12)(34)}(N) = \check{R}_{(34)(56)}(N)\check{R}_{(12)(34)}(N)\check{R}_{(34)(56)}(N) \quad (2.20)$$

We have another kind of generalized YBE for $N \neq N'$, i.e.,

$$\check{R}_{(12)(34)}(N)\check{R}_{(34)(56)}(N)\check{R}_{(12)(34)}(N') = \check{R}_{(34)(56)}(N')\check{R}_{(12)(34)}(N)\check{R}_{(34)(56)}(N) \quad (2.21a)$$

$$\check{R}_{(12)(34)}(N)\check{R}_{(34)(56)}(N')\check{R}_{(12)(34)}(N') = \check{R}_{(34)(56)}(N')\check{R}_{(12)(34)}(N')\check{R}_{(34)(56)}(N) \quad (2.21b)$$

Various combinations of $N$ and $N'$ are allowed. And the ones with N, N' =2,4,6,8 play an important role in discussing the quantum Lorentz group.

It can be easily shown that $\check{R}(4)$, as well as $\check{R}(2) = \check{\mathcal{R}}$, satisfy the same YBR with the quantum Lorentz matrix $\Lambda_{12} = M_1 \otimes \bar{M}_2$

$$\check{R}(N)_{(12)(34)}\Lambda_{(12)}\Lambda_{(34)} = \Lambda_{(12)}\Lambda_{(34)}\check{R}_{(12)(34)}(N),\ for\ N = 2,4,6,8 \quad (2.22)$$

The 16× 16 matrix $\check{R}(4)^{\mu\nu}{}_{\kappa\lambda}$ has three different eigenvalues, $-q^2$, 1 and $-q^{-2}$. It can be shown that the eigenvectors $g$, $u$ and $w$ appearing in (2.10) and (2.13) are also the eigenvectors of $\check{R}(4)$ .

$$\begin{aligned}
\check{R}(4)^{\mu\nu}{}_{\kappa\lambda}g(q)^{\kappa\lambda} &= g(q)^{\mu\nu}\\
\check{R}(4)^{\mu\nu}{}_{\kappa\lambda}u_{mL}(q)^{\kappa\lambda} &= -q^2u_{mL}(q)^{\mu\nu}\\
\check{R}(4)^{\mu\nu}{}_{\kappa\lambda}u_{mR}(q)^{\kappa\lambda} &= -q^{-2}u_{mR}(q)^{\mu\nu}\\
\check{R}(4)^{\mu\nu}{}_{\kappa\lambda}w_{mn}(q)^{\kappa\lambda} &= w_{mn}(q)^{\mu\nu}
\end{aligned} \quad (2.23)$$



So if we decompose the $q$-antisymmetric projection operator $Q^{(1)}(q)$ into two parts

$$Q^{(1)}(q) = Q^{(1L)}(q) + Q^{(1R)}(q)$$

$$Q^{(1L)}(q)^{\mu\nu}{}_{\kappa\lambda} = u_{mL}(q)^{\mu\nu}\bar{u}^{mL}(q)_{\kappa\lambda},\quad Q^{(1R)}(q)^{\mu\nu}{}_{\kappa\lambda} = u_{mR}(q)^{\mu\nu}\bar{u}^{mR}(q)_{\kappa\lambda}, \quad (2.24)$$

we have

$$\check{R}(4) = Q^{(0)} - q^2Q^{(1L)} - q^{-2}Q^{(1R)} + Q^{(2)} \quad (2.25a)$$

whereas

$$\check{R}(2) = q^{-3}Q^{(0)} - q^{-1}Q^{(1L)} - q^{-1}Q^{(1R)} + qQ^{(2)} \quad (2.25b)$$

All these eigenvectors can also be constructed by the basic $\check{R}$ matrix (1.1) and basic eigenvectors $t_\mu(q)^{\alpha\beta}$ and $s(q)^{\alpha\beta}$. The results are

$$\begin{aligned}
g(q)^{\mu\nu} &= \bar{t}(q)_{\alpha\dot{\beta}}\ \bar{t}(q)_{\gamma\dot{\delta}}\ q^{1/2}\ \check{R}(q)^{\dot{\beta}\gamma}{}_{\beta\dot{\gamma}}\ \epsilon(q)^{\alpha\beta}\epsilon(q)^{\dot{\gamma}\dot{\delta}},\\
g(q)_{\mu\nu} &= \epsilon(q)_{\alpha\beta}\ \epsilon(q)_{\dot{\gamma}\dot{\delta}}\ q^{-1/2}\ \check{R}^{-1}(q)^{\beta\dot{\gamma}}{}_{\dot{\beta}\gamma}\ t_\mu(q)^{\alpha\dot{\beta}}\ t_\nu(q)^{\gamma\dot{\delta}},\\
u_{kL}(q)^{\mu\nu} &= \bar{t}(q)_{\alpha\dot{\beta}}\ \bar{t}(q)_{\gamma\dot{\delta}}\ q^{1/2}\ \check{R}(q)^{\dot{\beta}\gamma}{}_{\beta\dot{\gamma}}\ t_k(q)^{\alpha\dot{\beta}}\ s(q)^{\dot{\gamma}\dot{\delta}},\\
u_{kR}(q)^{\mu\nu} &= \bar{t}(q)_{\alpha\dot{\beta}}\ \bar{t}(q)_{\gamma\dot{\delta}}\ q^{1/2}\ \check{R}(q)^{\dot{\beta}\gamma}{}_{\beta\dot{\gamma}}\ s(q)^{\alpha\dot{\beta}}\ t_k(q)^{\dot{\gamma}\dot{\delta}},\\
w_{mn}(q)^{\mu\nu} &= \bar{t}(q)_{\alpha\dot{\beta}}\ \bar{t}(q)_{\gamma\dot{\delta}}\ q^{1/2}\ \check{R}(q)^{\dot{\beta}\gamma}{}_{\beta\dot{\gamma}}\ t_m(q)^{\alpha\dot{\beta}}\ t_n(q)^{\dot{\gamma}\dot{\delta}}.
\end{aligned} \quad (2.26)$$

Various properties of $g$, $u_{ks}$ and $w_{mn}$ can be obtained by these basic relations.

## 3. Covariant Differential Calculus on $M_q$

The covariant differential calculus on the quantum plane was first discussed by Wess and Zumino (for the A type case) [8] and then generalized to the cases of quantum orthogonal plane (B,D type) [9,10] or quantum symplectic plane (C type) [10]. For quantum spinor $u^\alpha$, their components are noncommutative. The same is true for their derivatives $\partial_\alpha = \frac{\partial}{\partial u^\alpha}$

$$u^1\ u^2 = q\ u^2\ u^1,\quad \partial_1\partial_2 = q^{-1}\partial_2\partial_1\ . \quad (3.1)$$

Raising the indices by the Levi-Civita symbol $\partial^\alpha = \epsilon(q)^{\alpha\beta}\partial_\beta$, these two relations can be written as the vanishing of their $q$-anti-symmetric combination, i.e.,

$$\epsilon(q)_{\alpha\beta}u^\alpha u^\beta = \epsilon(q)_{\alpha\beta}\partial^\alpha\partial^\beta = 0\ . \quad (3.2)$$

The cross commutation relations between the coordinates $u^\alpha$ and derivatives $\partial^\alpha$ determined by the consistency [8] can be put into the form

$$\partial^\alpha u^\beta + \mu_1(q)\check{R}(q)^{\alpha\beta}{}_{\gamma\delta} u^\gamma \partial^\delta = \epsilon(q)^{\alpha\beta} \tag{3.3}$$

or, equivalently

$$\begin{aligned} &\bar{t}^m(q)_{\alpha\beta}[\partial^\alpha u^\beta + \mu_1(q)\mu_2(q) u^\alpha \partial^\beta] = 0, \\ &\epsilon(q)_{\alpha\beta}[\partial^\alpha u^\beta + \mu_1(q) u^\alpha \partial^\beta] = \epsilon(q)_{\alpha\beta}\epsilon(q)^{\alpha\beta} = -[2] \ . \end{aligned} \tag{3.4}$$

And from the relation of $q$-antisymmetric projection operator in (1.13), (3.2) turn out to be

$$[E - \mu_2(q)\check{R}(q)]^{\alpha\beta}{}_{\gamma\delta}\ u^\gamma u^\delta = [E - \mu_2(q)\check{R}(q)]^{\alpha\beta}{}_{\gamma\delta}\ \partial^\gamma \partial^\delta = 0. \tag{3.5}$$

In a similar way for quantum Minkowski coordinates $x^\mu$ and their derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$, we define $\partial^\mu = g(q)^{\mu\nu}\partial_\nu$. Again their $q$-antisymmetric combination should vanish.

$$\bar{u}^{ms}(q)_{\mu\nu} x^\mu x^\nu = \bar{u}^{ms}(q)_{\mu\nu}\partial^\mu \partial^\nu = 0 \ . \tag{3.6}$$

The consistency condition yields relations between coordinates and derivatives [7]

$$\begin{aligned} &g(q)_{\mu\nu}[\partial^\mu x^\nu + \lambda_1^{-1}(q)\lambda_0^{-1}(q) x^\mu \partial^\nu] = g(q)_{\mu\nu} g(q)^{\mu\nu} = [2]^2 \ , \\ &\bar{u}^{ms}(q)_{\mu\nu}[\partial^\mu x^\nu + \lambda_1^{-2}(q) x^\mu \partial^\nu] = 0 \\ &\bar{w}^{mn}(q)_{\mu\nu}[\partial^\mu x^\nu + \lambda_1^{-1}(q)\lambda_2^{-1}(q) x^\mu \partial^\nu] = 0 \end{aligned} \tag{3.7}$$

which can be put into the form similar to (3.3)

$$\partial^\mu x^\nu + \lambda_1^{-1}(q)\check{\mathcal{R}}^{-1}(q)^{\mu\nu}{}_{\kappa\lambda} x^\kappa \partial^\lambda = g(q)^{\mu\nu} \ . \tag{3.8}$$

But now the relations in Eq(3.6) can not be put into the form similar to (3.5) with linear dependence on the matrix $\check{\mathcal{R}} = \check{R}(2)$, since the $q$-antisymmetric projection operator for Lorentz group is quadratic but not linear in $\check{\mathcal{R}}$. However they can be put into the form

$$[E^{\mu\nu}{}_{\kappa\lambda} - \check{R}(4,q)^{\mu\nu}{}_{\kappa\lambda}] x^\kappa x^\lambda = 0, \quad [E^{\mu\nu}{}_{\kappa\lambda} - \check{R}(4,q)^{\mu\nu}{}_{\kappa\lambda}]\partial^\kappa \partial^\lambda = 0 \tag{3.9}$$

from (2.25a). The consistency among (3.9) and (3.8) can be checked by making use of (2.21) relations for $N = 2, 4, 6, 8$. The details of the differential calculus can be found in our recent paper [7].



## 4. Covariant Generators of Quantum Lorentz Group

Now similar to the classical case we can define the generators of the $SL_q(2,C)$ in the quantum spinor space as [11]

$$L^{\alpha\beta} = u^\alpha \partial^\beta \ . \tag{4.1}$$

Using the commutation relations given in (3.4) and (3.5), we can obtain

$$\begin{aligned} &L^{\alpha\beta} L^{\gamma\delta} - q^{-1}\check{\mathcal{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'} L^{\alpha'\beta'} L^{\gamma'\delta'} \\ &= q^{1/2}\ \check{R}^{-1}(q)^{\alpha\beta}{}_{\alpha'\beta'}\epsilon(q)^{\beta'\gamma'}\check{R}^{-1}(q)^{\gamma\delta}{}_{\gamma'\delta'}\check{R}^{\alpha'\delta'}{}_{\sigma\tau} L^{\sigma\tau} + \epsilon(q)^{\beta\gamma} L^{\alpha\delta} \end{aligned} \tag{4.2}$$

by a tedious but straightforward derivation. Then by changing the bispinor indices $(\alpha, \beta)$ into 4-vector index $\mu$ and defining

$$L^\mu = \bar{t}^\mu(q)_{\alpha\beta} L^{\alpha\beta} \tag{4.3}$$

we obtain

$$\begin{aligned} &L^\mu L^\nu - q^{-1}\check{\mathcal{R}}(q)^{\mu\nu}{}_{\kappa\lambda} L^\kappa L^\lambda = f^{\mu\nu}{}_\rho L^\rho \\ &= [2]^{1/2}[u_{kL}(q)^{\mu\nu} + u_{kR}(q)^{\mu\nu}] L^k + [2]^{-1/2}(q - q^{-1}) g(q)^{\mu\nu} L^0 \ . \end{aligned} \tag{4.4}$$

In a similar way, the generators in the conjugate spinor space are defined as

$$\bar{L}^{\dot\alpha\dot\beta} = \bar{u}^{\dot\alpha}\bar{\partial}^{\dot\beta} \tag{4.5}$$

which satisfy relations similar to those in Eq(4.2)

$$\begin{aligned} &\bar{L}^{\dot\alpha\dot\beta}\bar{L}^{\dot\gamma\dot\delta} - q^{-1}\check{\mathcal{R}}^{\dot\alpha\dot\beta,\dot\gamma\dot\delta}{}_{\dot\alpha'\dot\beta',\dot\gamma'\dot\delta'}\bar{L}^{\dot\alpha'\dot\beta'}\bar{L}^{\dot\gamma'\dot\delta'} \\ &= q^{1/2}\ \check{R}^{-1}(q)^{\dot\alpha\dot\beta}{}_{\dot\alpha'\dot\beta'}\epsilon(q)^{\dot\beta'\dot\gamma'}\check{R}^{-1}(q)^{\dot\gamma\dot\delta}{}_{\dot\gamma'\dot\delta'}\check{R}^{\dot\alpha'\dot\delta'}{}_{\dot\delta\dot\tau}\bar{L}^{\dot\delta\dot\tau} + \epsilon(q)^{\dot\beta\dot\gamma}\bar{L}^{\dot\alpha\dot\delta} \ . \end{aligned} \tag{4.6}$$

Then if we introduce

$$\bar{L}^\mu = \bar{t}^\mu(q^{-1})_{\dot\beta\dot\alpha}\bar{L}^{\dot\alpha\dot\beta} \tag{4.7}$$

we have

$$\begin{aligned} &\bar{L}^\mu \bar{L}^\nu - q^{-1}\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}\bar{L}^\kappa \bar{L}^\lambda = f^{\mu\nu}{}_\rho \bar{L}^\rho \\ &= [2]^{1/2}[u_{kL}(q)^{\mu\nu} + u_{kR}(q)^{\mu\nu}]\bar{L}^k + [2]^{-1/2}(q - q^{-1}) g(q)^{\mu\nu}\bar{L}^0 \end{aligned} \tag{4.8}$$

The same is true for replacing $q^{-1}\check{\mathcal{R}} = q^{-1}\check{R}(2)$ by $\check{R}(4)$ in Eqs. (4.2), (4.4), (4.6), (4.8). In above definition, the conjugate spinor $\bar{u}_{\dot{\alpha}} \equiv (u^{\alpha})^*$ transforms according to the hermitian conjugate (complex transpose) of the quantum matrix $M$.

$$u^{\alpha} \longrightarrow M(q)^{\alpha}{}_{\beta} u^{\beta} \ , \bar{u}_{\dot{\alpha}} \longrightarrow \bar{u}_{\dot{\beta}} M(q)^{+\dot{\beta}}{}_{\dot{\alpha}} \ . \tag{4.9}$$

The conjugate derivative is $\frac{\partial}{\partial \bar{u}_{\dot{\alpha}}} = \partial^{\dot{\alpha}}$. Raising the dotted index of $\bar{u}_{\dot{\alpha}}$, we see

$$\bar{u}^{\dot{\alpha}} \equiv \bar{u}_{\dot{\beta}} \epsilon(q)^{\dot{\beta}\dot{\alpha}} \longrightarrow \tilde{M}^{\dot{\alpha}}{}_{\dot{\gamma}} \ \bar{u}^{\dot{\gamma}} \ . \tag{4.10}$$

The commutation relation among $\bar{u}^{\dot{\alpha}}$ and $\bar{\partial}^{\dot{\alpha}}$ are found to be the same as those for $u^{\alpha}$ and $\partial^{\alpha}$ in (3.3)and (3.5):

$$\begin{aligned} \bar{u}^{\dot{\alpha}} \bar{u}^{\dot{\beta}} &= q^{-1/2} \check{R}^{\dot{\alpha}\dot{\beta}}{}_{\dot{\gamma}\dot{\delta}} \bar{u}^{\dot{\gamma}} \bar{u}^{\dot{\delta}} \ , \\ \bar{\partial}^{\dot{\alpha}} \bar{\partial}^{\dot{\beta}} &= q^{-1/2} \check{R}^{\dot{\delta}\dot{\gamma}}{}_{\dot{\beta}\dot{\alpha}} \bar{\partial}^{\dot{\gamma}} \bar{\partial}^{\dot{\delta}} \\ \bar{\partial}^{\dot{\alpha}} \bar{u}^{\dot{\beta}} &= \epsilon(q)^{\dot{\alpha}\dot{\beta}} + q^{-3/2} \check{R}(q)^{\dot{\alpha}\dot{\beta}}{}_{\dot{\gamma}\dot{\delta}} \bar{u}^{\dot{\gamma}} \bar{\partial}^{\dot{\delta}} \ . \end{aligned} \tag{4.11}$$

And the relations between undotted quantities with dotted ones are

$$\begin{aligned} u^{\alpha} \bar{u}^{\dot{\alpha}} &= q^{1/2} \check{R}^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}} \bar{u}^{\dot{\beta}} u^{\beta}, \quad & \partial^{\alpha} \bar{\partial}^{\dot{\alpha}} &= q^{1/2} \check{R}^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}} \bar{\partial}^{\dot{\beta}} \partial^{\beta}, \\ \partial^{\alpha} \bar{u}^{\dot{\alpha}} &= q^{-1/2} \check{R}^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}} \bar{u}^{\dot{\beta}} \partial^{\beta}, \quad & u^{\alpha} \bar{\partial}^{\dot{\alpha}} &= q^{1/2} \check{R}^{\alpha\dot{\alpha}}{}_{\beta\dot{\beta}} \bar{\partial}^{\dot{\beta}} u^{\beta}, \end{aligned} \tag{4.12}$$

from which we obtain

$$L^{\alpha\beta} \bar{L}^{\dot{\alpha}\dot{\beta}} = \check{\mathbf{R}}^{\alpha\beta,\dot{\alpha}\dot{\beta}}{}_{\dot{\gamma}\dot{\delta},\gamma\delta} \bar{L}^{\dot{\gamma}\dot{\delta}} L^{\gamma\delta} \tag{4.13}$$

Here notice that, the $\check{R}$ matrix is the $\check{\mathbf{R}} = \check{R}(1)$ appearing in (2.4). Transfer to the vector index we see

$$\begin{aligned} \check{\mathbf{R}}(q)^{\mu\nu}{}_{\kappa\lambda} &= \check{\mathbf{R}}^{00}{}_{00} \oplus \check{\mathbf{R}}^{m0}{}_{0l} \oplus \check{\mathbf{R}}^{0n}{}_{k0} \oplus \check{\mathbf{R}}^{mn}{}_{kl} \\ &= 1 \oplus \delta^{m}{}_{l} \oplus \delta_{k}{}^{n} \oplus \check{\mathbf{R}}(q)^{mn}{}_{kl} \end{aligned} \tag{4.14}$$

$\check{\mathbf{R}}^{mn}{}_{kl}$ is the $\check{R}$ matrix for $SO_{q^2}(3)$.

More explicitly, (4.4) can be read as

$$\begin{aligned} &[L^0, L^m] = 0 \ , \\ &q(L^3 - L^0)L^+ - q^{-1} L^+ (L^3 - L^0) = q[2]^{1/2} L^+ \ , \\ &qL^-(L^3 - L^0) - q^{-1}(L^3 - L^0)L^- = q[2]^{1/2} L^- \ , \\ &L^+L^- - L^-L^+ + (q - q^{-1})L^3(L^3 - L^0) = q[2]^{1/2} L^3 \ , \\ &-q^{-1}L^+L^- - L^3L^3 + L^0L^0 - qL^-L^+ = q^2[2]^{1/2} L^0 \ . \end{aligned} \tag{4.15}$$



and similar for $\bar{L}^0$ and $\bar{L}^m$. And similarly, (4.13) gives

$$[L^0, \bar{L}^0] = [L^n, \bar{L}^0] = [L^0, \bar{L}^m] = 0 \tag{4.16}$$

$$\begin{aligned} &L^+\bar{L}^+ = q^2 \bar{L}^+ L^+ \ , \\ &L^+\bar{L}^3 = \bar{L}^3 L^+ \ , \\ &L^+\bar{L}^- = q^{-2} \bar{L}^- L^+ \ , \\ &L^3\bar{L}^+ = \bar{L}^+ L^3 + (q^2 - q^{-2}) \bar{L}^3 L^+ \ , \\ &L^3\bar{L}^3 = -q^{-1}(q^2 - q^{-2}) \bar{L}^- L^+ + \bar{L}^3 L^3 \ , \\ &L^3\bar{L}^- = \bar{L}^- L^3 \ , \\ &L^-\bar{L}^+ = q^{-2} \bar{L}^+ L^- - q^{-1}(q^2 - q^{-2}) \bar{L}^3 L^3 + (1 - q^{-2})(q^2 - q^{-2}) \bar{L}^- L^+ \ , \\ &L^-\bar{L}^3 = (q^2 - q^{-2}) \bar{L}^- L^3 + \bar{L}^3 L^- \ , \\ &L^-\bar{L}^- = q^2 \ \bar{L}^- L^- \ . \end{aligned} \tag{4.17}$$

It is easily seen that in the limit $q \to 1$, $\bar{L}^{\mu}$ and $L^{\nu}$ became two sets of commutative angular momentum with $L^0$ and $\bar{L}^0$ being their Casimir operators.

All the cross commutation relations given above are explicitly covariant and take the $\check{R}$ commutation form just as those proposed by Woronowicz [3]. All the structure constants are comprised by the relevant $\check{R}$ matrices and their character quantities. The consistency are easily to prove.

Along the same line we can discuss the generators of quantum Poincare group.

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