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ON QUANTUM WESS-ZUMINO-DE RHAM COMPLEX APPROACH FOR QUANTUM GROUPS

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Abstract

Deformation Ω_q of the usual (classical) de Rham complex Ω has been considered .initiated by Wess-Zumino assumption. The two posible variants of the skew products also had discussed follows the recent Manin formulation.

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1 Introduction .

In small scale, matter reveals itself as quantum and no longer as a continuum.

Does geometry express her figure,in the small scale, no longer as a commutative but as a non-commutative one?

It seems that the most fundamental blocks, vector space and linear groups transform on them, probably must be replaced by the deformed or quantum ones. This is the recently developed quantum planes and quantum groups approach.

After a lot of pioneer work on this aspect, Wess-Zumino[1,2] and their colaborators , developed many important applications in particle physics. Our notes are principally motivated by Wess-Zumino's work and Manin's explanation and

extension. Therefore, we would like to follow the Wess-Zumino's notation as whenever possible.

2 Vector space V, without product .

We consider n-dimensional vector space, mainly n=2. We use the obvious notations like

V $\ni$ (u,v),

V^* $\ni$ l(u,v),linear functions,

C(V) $\ni$ f(u,v),continuous functions, with natural product.

3 Vector space V, with product = plane .

Further algebraic structure is needed to describe the physical systems. For $(x^1, ... x^i, .., x^n) \in V$, this set generates an associative algebra with the axioms of products. In the following, we consider mainly the n=2 case, and use (x,y) instead of $(x^1, ..., x^n)$.

3.1.Plane $A^{2/0}$.

With the commutative product, for (x,y) $\in$ V,

$$xy - yx = 0.$$

Then we can write monomial ; $x^l y^m$, and polynomial; $P_n(x,y) \in K < x, y >$.

3.2.Quantum plane $A_q^{2/0}$.

With the deformed commutative product , for $(x_q, y_q) \in$ V, for simplicity we omit the index q,

$$xy - qyx = 0,$$

then we can write monomial; $x^l y^m$, and polynomial; $P_n(x,y) \in K < x, y >$.

Polynomial function ring

$$F[A_q^{2/0}] = k < x, y > /(xy - qyx).$$

Its transformation is

$$F[A_q^{2/0}] = F[GL_q(2)]F[A_q^{2/0}],$$

with quantum group $GL_q(2)$.

3.3.Super-plane $A^{0/2}$.

With the anti-commutative product, for (ξ, η) $\in$ V,

$$\xi\eta + \eta\xi = 0.$$

So we meet Grassmann algebra.

3.4.Quantum (super-) plane $A_q^{0/2}$.

With the deformed anti-commutative product , for $(\xi_q, \eta_q) \in V$, for simplicity we omit the index q,

$$\xi\eta + q^{-1}\eta\xi = 0.$$

Polynomial function ring

$$F[A_q^{0/2}] = k < \xi, \eta > /(\xi^2, \eta^2, \xi\eta + q^{-1}\eta\xi).$$

Its transformation

$$F[A_q^{0/2}] = F[GL_q(2)]F[A_q^{0/2}],$$

with quantum group $GL_q(2)$.

Generally,the quantum deformation of the super-plane $A^{0/2}$ depends on quantization parameter $p \in C$, which is irrelated with the aforementioned quantization parameter q. So we meet the quantum super-plane $A_p^{0/2}$.In this case ,

$$\xi\eta + p^{-1}\eta\xi = 0.$$

And the polynomial function ring becomes

$$F[A_p^{0/2}] = k < \xi, \eta > /(\xi^2, \eta^2, \xi\eta + p^{-1}\eta\xi).$$

3.5.R-symmetry algebra.

On a given linear (vector) space V, we can define product '.' with suitable axioms. to make it an algebra A. Besides the above-mentioned four examples,the simplest instances are Lie algebras.with Lie products. But in the follows we would like to consider an more generalized instance. the R-symmetry algebra $S_R(V)$, i.e. Yang-Baxter or Zamolodchikov algebra, with R-symmetry product[1]. We use x_1 denote the element of the first vector space V_1, for example x^i, x^k, x^m. And we use x_2 denote the element of the second vector space V_2, for example

x^j, x^l, x^n. Using the standard B or $\hat{R} = PR$ matrix operators, this R-symmetry product can be expressed as

$$x^i x^j = q x^j x^i = B^{ij}_{(12)kl}(q) x^k x^l = q^{-1}\hat{R}^{ij}_{(12)kl}(q) x^k x^l = q x^j x^i$$

$$= q[q^{-1} x^i x^j] = q B^{ji}_{(21)lk}(q^{-1}) x^l x^k = B^{ji}_{(21)lk}(q^{-1})(q x^l x^k),$$

or more compactly as

$$x_1 x_2 = B_{(12)}(q) x_1 x_2 = q^{-1}\hat{R}_{(12)}(q) x_1 x_2.$$

Similarly,the expression

$$x^j x^i = q^{-1} x^i x^j = B^{ji}_{(21)lk}(q^{-1}) x^l x^k,$$

$$x^l x^k = q^{-1} x^k x^l = B^{lk}_{(21)nm}(q^{-1}) x^n x^m.$$

can be expresed as

$$x_2 x_1 = B_{(21)}(q^{-1}) x_2 x_1.$$

Hence,

$$B_{(12)}(q) B_{(21)}(q^{-1}) = I.$$

This means the defining equation of the R-symmetry product

$$(E_{(12)} - B_{(12)}(q)) x_1 x_2 = 0,$$

transforms to itself, under the permutation of x_1 with x_2 and q with q^{-1}. Here,$E_{(12)}$ is the unity matrix operator in tensor product space $V_1 \otimes V_2$.

Similarly, for the case of ξ, the defining equation of product

$$(E_{(12)} - q^2 B_{(12)}(q)) \xi_1 \xi_2 = 0,$$

transforms to itselves, under the permutation of ξ_1 with ξ_2 and q with q^{-1} .

But, contrary to this situation.the screw product expression is not invariant under this permutation, because the "intermediary" equation

$$x_1 \xi_2 - B_{(12)}(q) \xi_1 x_2 = 0,$$

is not invariant, but transforms to

$$x_2 \xi_1 - q^{-2} B^{-1}_{(21)}(q) \xi_2 x_1 = 0.$$

That means the screw product has two possible variants.

On the other hand, this can be checked directly as

$$B_{(12)}(q)B_{(21)}(q^{-1}) =$$

$$q^{-1}\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & (q-q^{-1}) & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}\begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & (q^{-1}-q) & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I.$$

In the following, we concentrated our consideration mainly on such R-symmetry algebra $S_R(V)$, which is combined with its two given sub-algebras $S_{R'}(V^t)$ and $S_{R''}(V^{tt})$, here

$$V^t \bigoplus V^{tt} = V.$$

And we denote their coordinates respectively as x and ξ

4 de Rahm complexes.

4.1.(Clasical) de Rham complex $\Omega(V)$.

Let $x^1, ..., x^n$ be the linear coordinats on V', and $\xi^1, ..., \xi^n$ be the linear coordinates on V". We denote $\Omega(V) = F[V^t] \otimes F[V^{tt}]$. And we say it has a de Rham structure if there exist an "exterior differential" d (mapping);

$$V^t \ni x^i \longrightarrow \xi^i \in V^{tt},$$

such that define V" to be the algebra, generated by $dx^i = \xi^i$, with an anti-commutative product

$$(dx^i)^2 = 0,$$

$$dx^i dx^j = -dx^j dx^i, i \neq j.$$

For example, the (Clasical) de Rham complex $\Omega[A^{2/0}]$ in the book of Bott-Tu[5],

$$\Omega = \Omega[A^{2/0}] = F[A^{2/0}]F[A^{0/2}].$$

4.2. Quantum Wess-Zumino-de Rahm complex $\Omega[A_q^{2/0}]$.

Now, we consider the combined space

$$\Omega = (x \bigotimes y) \bigotimes (\xi \bigotimes \eta).$$

Manin[4], following Wess-Zumino, extended this de Rham complex to the quantum ones with the quantization parameter q, as follows

$$\Omega_q = \Omega[A_q^{2/0}] = F[A_q^{2/0}]F[A_q^{0/2}].$$

We call this quantum Wess-Zumino-de Rham complex. Of course, some game on the permutation of alphabet, as quantum de Rham W-Z complex.is equally well. Because the classical de Rham complex is a well known fundamental device in algebraic geometry and related topics, it is worthwhile to research how is its deformed ones.

In this case, we replace x for x_q, ξ for ξ_q and d for the corresponding exterior differential d_q such that, the image $d_q x_q$ of x_q equals ξ_q, which are assumed implicitly in Wess-Zumino[1,2] , and explicitly in Manin's paper [4].

1.We consider the eigen-value problems, in the tensor product space $V \otimes V$. For the case n=2,we specified to V=$\Omega_q = \Omega[A_q^{2/0}]$,with its two subspaces $V' = A_q^{2/0}$ and $V'' = A_q^{0/2}$. That is the tensor product space $V \otimes V$ decomposed as

$$\begin{pmatrix} A_q^{2/0} \otimes A_q^{2/0}, & A_q^{2/0} \otimes A_q^{0/2} \\ A_q^{0/2} \otimes A_q^{2/0}, & A_q^{0/2} \otimes A_q^{0/2} \end{pmatrix}.$$

and the corresponding product is

$$\begin{pmatrix} xx & xy, & x\xi & x\eta \\ yx & yy & y\xi & y\eta \\ \xi x & \xi y & \xi\xi & \xi\eta \\ \eta x & \eta y & \eta\xi & \xi\eta \end{pmatrix}.$$

We shall prove that the operator $\hat{R}$ has two eigenvaues q and ($-q^{-1}$), in the tensor product algebra

$$A_q^{2/0} \bigotimes A_q^{2/0},$$

and

$$A_q^{0/2} \bigotimes A_q^{0/2},$$

respectively.

For the given matrix operator $\hat{R}$ the eigen-value problems, for eigenvalue Q $\in C$, is as follows

$$(\hat{R} - QE)v \bigotimes v = 0,$$

here $\hat{R}$ is related to our familiar operators R and B as follows

$$\hat{R} = qB = PR,$$

P is a permutation matrix operator and we denote the unity operator matrix as I or E_{12}. Now for the details. It is well known that

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & (q-q^{-1}) & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & (q-q^{-1}) & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1}(q-q^{-1}) & q^{-1} & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, the eigen-value problem becomes

$$Q\begin{pmatrix} v_1 \otimes v_1 \\ v_1 \otimes v_2 \\ v_2 \otimes v_1 \\ v_2 \otimes v_2 \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & (q-q^{-1}) & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}\begin{pmatrix} v_1 \otimes v_1 \\ v_1 \otimes v_2 \\ v_2 \otimes v_1 \\ v_2 \otimes v_2 \end{pmatrix}$$

$$= \begin{pmatrix} qv_1 \otimes v_1 \\ (q-q^{-1})v_1 \otimes v_2 + v_2 \otimes v_1 \\ v_1 \otimes v_2 \\ qv_2 \otimes v_2 \end{pmatrix}.$$

The equation of the second row,in the above matrix equations, gives

$$[Q-(q-q^{-1})]v_1 \bigotimes v_2 = v_2 \bigotimes v_1,$$

while the equation of the third row gives

$$v_1 \bigotimes v_2 = Qv_2 \bigotimes v_1 = Q[Q-(q-q^{-1})]v_1 \bigotimes v_2,$$

so we get the following eigen value equation,an algebraically quadratic equation on Q

$$QQ - Q(q-q^{-1}) - 1 = 0.$$

It is easy to get their two solutions

$$Q = q$$

and

$$Q = -q^{-1}.$$

2.As for the eigen spaces,we consider its correspondingly in the following three cases. Case 1.For Q=q, We consider the tensor product algebra $A_q^{2/0} \otimes A_q^{2/0}$, with the aditional requirement of some kind of the unknown deformed (commutative) product, in this case we denote x^i in place of v^i. The above-mentioned eigen-value equation becomes

$$Q\begin{pmatrix} xx \\ xy \\ yx \\ yy \end{pmatrix} = \begin{pmatrix} qxx \\ (q-q^{-1})xy + yx \\ xy \\ qyy \end{pmatrix}.$$

The first and fourth row are satisfied automatically,without any additional requirment. We notice, in this Q=q case, the equation of the third row requires

$$xy = qyx,$$

that is just mean a deformed commutative product.

Case 2.For Q= - q^{-1},

We consider the tensor product algebra $A_q^{0/2} \otimes A_q^{0/2}$, with the aditional requirement on some kind of unknown deformed (anti-commutative) product. In this case we denote ξ in place of v^i. We notice , in this $Q = -q^{-1}$ case, the equation of the third row requires

$$\xi\eta = -q^{-1}\eta\xi.$$

hence

$$\xi^i\xi^i = 0.$$

In our case that is

$$\xi\xi = 0, \eta\eta = 0.$$

The above-mentioned eigen-value equation becomes

$$Q\begin{pmatrix}\xi\xi\\ \xi\eta\\ \eta\xi\\ \eta\eta\end{pmatrix}=\begin{pmatrix}q\xi\xi\\ (q-q^{-1})\xi\eta+\eta\xi\\ \xi\eta\\ q\eta\eta\end{pmatrix}.$$

The equation of the first row is now

$$-q^{-1}\xi\xi=q\xi\xi.$$

which requires

$$\xi\xi=0.$$

While the equation of the fourth row is

$$-q^{-1}\eta\eta=q\eta\eta,$$

which requires

$$\eta\eta=0.$$

The equation of the third row requires

$$\xi\eta=-q^{-1}\eta\xi.$$

To sum up , it means we have defined a deformed anti-commutative product with ξ,η variables.

Case 3. We consider the tensor product algebra $A_q^{2/0}\otimes A_q^{0/2}\in\Omega_q$.

As for the cross products, we have two solutions as follows.

First solution

$$x^i\xi^j=q^2B_{(12)}(q)\xi^kx^l,$$
$$\xi^ix^j=q^{-2}B^{-1}_{(12)}(q)x^k\xi^l.$$

Second solution

$$x^i\xi^j=q^{-2}B^{-1}_{(21)}(q)\xi^kx^l,$$
$$\xi^ix^j=q^2B_{(21)}(q)x^k\xi^i.$$

We would like to prove these cross-commutation relations by the requirements about thier transformation invariance.in contrast to Wess-Zumino's symmetry consideration.

The quantum planes $A_q^{2/0}$ trasforms as

$$A_q^{2/1}\ni\begin{pmatrix}x\\ y\end{pmatrix}\to\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}a&b\\ c&d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}ax+by\\ cx+dy\end{pmatrix}\in A_q^{2/1}.$$

The requirement of the invariance of the non-commutativity

$$xy=qyx,\qquad and\qquad x'y'=qy'x',$$

deduce that

$$acxx+adxy+bcyx+bdyy=q(caxx+cbxy+dayx+dbyy),$$

hence

$$ac=qca,\quad bd=qdb.$$

and

$$ad-da=qcb-q^{-1}bc.$$

While the quantum super-plane $A_q^{0/2}$ transforms as

$$A_q^{0/2}\ni\begin{pmatrix}\xi\\ \eta\end{pmatrix}\to\begin{pmatrix}\xi'\\ \eta'\end{pmatrix}=\begin{pmatrix}a&b\\ c&d\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}=\begin{pmatrix}a\xi+b\eta\\ c\xi+d\eta\end{pmatrix}\in A_q^{0/2}.$$

The requirement of the invariance of the non-commutativity

$$0=\xi\xi=\xi'\xi',$$

deduce that

$$ab\xi\eta+ba\eta\xi=(ab-qba)\xi\eta=0$$

hence

$$ab=qba.$$

Similarly, from

$$0=\eta\eta=\eta'\eta'$$

we deduce that

$$cd\xi\eta+dc\eta\xi=(cd-qdc)\xi\eta,$$

hence we get

$$cd=qdc.$$

And from

$$\xi'\eta'=-q^{-1}\eta'\xi'.$$

we get

$$ac\xi\xi + ad\xi\eta + bc\eta\xi + bd\eta\eta = -q^{-1}(ca\xi\xi + cb\xi\eta + da\eta\xi + db\eta\eta).$$

Hence we have

$$ad - da = -q^{-1}cb + qbc,$$

and

$$bc = cb.$$

To sum up, the quantum group

$$M_q(2) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

generated by even variables a,b,c,d with the following non-commutativity relations

$$ab = qba; \qquad ac = qca; \qquad ad = da + (q - q^{-1})cb;$$

$$bc = cb; \qquad bd = qdb; \qquad ca = qac.$$

Now , we would like utilize the (exterior) differential d, and admitting aditional with $dx = \xi, dy = \eta$.Here,we consider x and ξ as independent variables as in some cotangent bundle, and the exterior differential d is defined on this cotangent bundle.not constrained just on the base space.

Firt of all, we differentiate the one relation we have know,i.e. $(xy - qyx) = 0$, and get

$$0 = d(xy - qyx) = \xi y + x\eta - q\eta x - qy\xi = (\xi y - qy\xi) + (x\eta - q\eta x).$$

Comparing dimension, we see that we must find a system of four cross- commutation relations expresing $x\xi, x\eta, y\xi, y\eta$ linear via $\xi x, \xi y, \eta x, \eta y$ as follows

$$\begin{pmatrix} x\xi \\ x\eta \\ y\xi \\ y\eta \end{pmatrix} = \begin{pmatrix} C_1^1 & C_2^1 & C_3^1 & C_4^1 \\ C_1^2 & C_2^2 & C_3^2 & C_4^2 \\ C_1^3 & C_2^3 & C_3^3 & C_4^3 \\ C_1^4 & C_2^4 & C_3^4 & C_4^4 \end{pmatrix} \begin{pmatrix} \xi x \\ \xi y \\ \eta x \\ \eta y \end{pmatrix}.$$

And from the first differential relation ,we know

$$c_1^2 = c_4^2 = 0 = c_1^3 = c_4^3$$

and

$$1 - qc_2^3 + c_2^2 = 0, -qc_3^3 + c_3^2 - q = 0.$$

Secondly, we differentiate the above matrix equation

$$\begin{pmatrix} xd\xi \\ \xi\eta + xd\eta \\ \eta\xi + yd\xi \\ yd\eta \end{pmatrix} = \begin{pmatrix} C_1^1 & C_2^1 & C_3^1 & C_4^1 \\ C_1^2 & C_2^2 & C_3^2 & C_4^2 \\ C_1^3 & C_2^3 & C_3^3 & C_4^3 \\ C_1^4 & C_2^4 & C_3^4 & C_4^4 \end{pmatrix} \begin{pmatrix} (d\xi)x \\ (d\xi)y \\ (d\eta)x \\ (d\eta)y \end{pmatrix} - \begin{pmatrix} C_2^1\xi\eta + C_3^1\eta\xi \\ C_2^2\xi\eta + C_3^2\eta\xi \\ C_2^3\xi\eta + C_3^3\eta\xi \\ C_2^4\xi\eta + C_3^4\eta\xi \end{pmatrix},$$

hence we get

$$\begin{pmatrix} 0 \\ \xi\eta \\ \eta\xi \\ 0 \end{pmatrix} = -\begin{pmatrix} (C_2^1 - qC_3^1)\xi\eta \\ (C_2^2 - qC_3^2)\xi\eta \\ (-q^{-1}C_2^3 + C_3^3)\eta\xi \\ (C_2^4 - qC_3^4)\xi\eta \end{pmatrix},$$

i.e.

$$-C_2^1 + qc_3^1 = 0, -C_2^2 + qC_3^2 = 1, q^{-1}C_2^3 - C_3^3 = 1, -C_2^4 + qC_3^4 = 0.$$

We choose C_2^1 =0, then we have C_3^1 =0.We choose C_2^4 =0, then we have C_3^4 =0.This can be explaned in the forthcoming paper [6]. If, we require the transformed cross products preserve the same non- commutativity, then we get

$$\begin{pmatrix} x'\xi' \\ x'\eta' \\ y'\xi' \\ y'\eta' \end{pmatrix} = \begin{pmatrix} C_1^1 & C_2^1 & C_3^1 & C_4^1 \\ C_1^2 & C_2^2 & C_3^2 & C_4^2 \\ C_1^3 & C_2^3 & C_3^3 & C_4^3 \\ C_1^4 & C_2^4 & C_3^4 & C_4^4 \end{pmatrix} \begin{pmatrix} \xi'x' \\ \xi'y' \\ \eta'x' \\ \eta'y' \end{pmatrix}.$$

So , we get on one side

$$x'\xi' = aax\xi + abx\eta + bay\xi + bby\eta = aa(C_1^1\xi x + C_4^1\eta y) + ab(C_1^2\xi x + C_2^2\xi y + C_3^2\eta x) +$$

$$ba(C_1^3\xi x + C_2^3\xi y + C_3^3\eta x + C_4^3\eta y) + bb(C_1^4\xi x + C_2^4\xi y + C_2^4\xi y + C_3^4\eta x + C_4^4\eta y),$$

and on the other side , we have

$$x'\xi' = C_1^1[aa\xi x + ab\xi y + ba\eta x + bb\eta y] + 0 + 0 + C_4^1[cc\xi x + cd\xi y + dc\eta x + dd\eta y].$$

Hence, we have

$$C_1^4 bb = C_4^1 cc \Rightarrow C_1^4 = 0 = C_1^1.$$

$$abC_2^2 + baC_2^3 = C_1^1 ab + C_4^1 cd \Rightarrow qC_2^2 + C_2^3 = C_1^1 q,$$
$$abC_3^2 + baC_3^3 = C_1^1 ba + C_4^1 dc \Rightarrow qC_3^2 + C_3^3 - C_1^1 = 0,$$
$$aaC_4^1 + bbC_4^4 = C_1^1 bb + C_4^1 dd \Rightarrow C_1^1 = C_4^4.$$

To sum up we have five unknown coefficients $C_1^1, C_2^2, C_3^2, C_2^3, C_3^3$, which satisfy four equations

$$-C_2^2 + qC_3^2 = 1, \quad q^{-1}C_2^3 - C_3^3 = 1, \quad qC_2^2 + C_2^3 = qC_1^1, \quad qC_3^2 + C_3^3 - C_1^1 = 0.$$

It is easy to get an one parameter r solution,

$$C_1^1 = (q+q^{-1})r-1, \quad C_4^4 = (q+q^{-1})r-1, \quad C_2^2 = rq-1, \quad C_3^2 = r = C_2^3, \quad C_3^3 = q^{-1}r-1.$$

Hence we get

$$x\xi = C_1^1\xi = (q+q^{-1})r-1]\xi x; \quad x\eta = C_2^2\xi y + C_3^2\eta x = (rq-1)\xi y + r\eta x;$$

$$y\xi = C_2^3\xi y + C_3^3\eta x = r\xi y + (q^{-1}r-1)\eta x; \quad y\eta = C_4^4\eta y = [(q+q^{-1})r-1]\eta y.$$

Using the associativity of (the product of) our algebra.

$$x\eta\xi = x(\eta\xi) = (x\eta)\xi,$$

we get the equations

$$x(\eta\xi) = -q[(q+q^{-1})r-1]r(\xi\eta x), (x\eta)\xi = [(rq-1)(q^{-1}r-1)-r[(q+q^{-1})r-1]q](\xi\eta x).$$

Hence we got the quadratic polynomial equation on r

$$r^2 - (q+q^{-1})r + 1 = 0.$$

Their two solutions are $r = q$ *or* $r = q^{-1}$. Corresponds to these, the net result is First solution

$$x\xi = q^2\xi x; \quad x\eta = (q^2-1)\xi y + q\eta x; \quad y\xi = q\xi y; \quad y\eta = q^2\eta y.$$

The second solution

$$x\xi = q^{-2}\xi x; \quad x\eta = q^{-1}\eta x; \quad y\xi = q^{-1}\xi y + (q^{-2}-1)\eta x; \quad y\eta = q^{-2}\eta y.$$

4.3. Quantum Wess-Zumino-de Rahm complex $\Omega_{p,q'} = \Omega[A_{q'}^{2/0}, A_p^{0/2}]$.

1.Of course the deformation or quantization parameter of the second space $A^{0/2}$ need not be choose as the same of the first space $A^{2/0}$, they may take an independent value p in place of q. And we generalize correspondingly the above tensor product space $V \otimes V$, from V $= \Omega_q$ to V$= \Omega_{p,q'} = \Omega\ [A_{q'}^{2/0}, A_p^{0/2}]$,with its two subspace $V^t = A_{q'}^{2/0}$ and $V^{tt} = A_p^{2/0}$. We use q' here , just because the different notation convention in Wess-Zumino and Manin's paper[4].When $q' = q^{-1}, p = q^{-1}$, our complex $\Omega_{p,q'}$,go back to $\Omega_q = \Omega[A_q^{2/0}]$ of paragraphy 4.2.

This is the tensor product space

$$\begin{pmatrix} A_{q'}^{2/0} \otimes A_{q'}^{2/0} & A_{q'}^{2/0} \otimes A_p^{0/2} \\ A_p^{0/2} \otimes A_{q'}^{2/0} & A_p^{0/2} \otimes A_p^{0/2} \end{pmatrix},$$

and the corresponding product is

$$\begin{pmatrix} x_{q'}x_{q'} & x_{q'}y_{q'} & x_{q'}\xi_p & x_{q'}\eta_p \\ y_{q'}x_{q'} & y_{q'}y_{q'} & y_{q'}\xi_p & y_{q'}\eta_x \\ \xi_p x_{q'} & \xi_p y_{q'} & \xi_p\xi_p & \xi_p\eta_p \\ \eta_p x_{q'} & \eta_p y_{q'} & \eta_p\xi_p & \eta_p\xi_p \end{pmatrix}.$$

But for simplicity ,we omit the index p as well as q'.

2.In this case , the quantum planes and quantum groups have new pattern. For $A_{q'}^{2/0}$.

$$A_{q'}^{2/0} \ni \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \in A_{q'}^{2/0}.$$

The requirement of the invariance of the non-commutativity,

$$xy = q'^{-1}yx \qquad and \qquad x'y' = q'^{-1}y'x',$$

deduce that

$$acxx + adxy + bcyx + bdyy = q'^{-1}(caxx + cbxy + dayx + dbyy),$$

hence

$$ac = q'^{-1}ca. \qquad bd = q'^{-1}db,$$

and

$$ad - da = q'^{-1}cb - q'bc.$$

For $A_p^{0/2}$,

$$A_p^{0/2} \ni \begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \begin{pmatrix} \xi^t \\ \eta^t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in A_p^{0/2}.$$

The requirement of the invariance of the non-commutativity,

$$0 = \xi\xi = \xi^t\xi^t$$

deduce that

$$ab\xi\eta + ba\eta\xi = (ab - p^{-1}ba)\xi\eta,$$

hence

$$ab = p^{-1}ba.$$

Similarly, from

$$0 = \eta\eta = \eta^t\eta^t$$

we deduce that

$$cd\xi\eta + dc\eta\xi = (cd - p^{-1}dc)\xi\eta,$$

hence we get

$$cd = p^{-1}dc.$$

From

$$\xi^t\eta^t = -p\eta^t\xi^t,$$

we get

$$ac\xi\xi + ad\xi\eta + bc\eta\xi + bd\eta\eta = -p(ca\xi\xi + cb\xi\eta + da\eta\xi + db\eta\eta),$$

hence

$$ad - da = -pcb + p^{-1}bc.$$

Due to

$$ad - da = q^{t-1}cb - q^t bc$$

so

$$bc = pq^{t-1}cb$$

. To sum up,the quantum group

$$M_{p,q^t}(2) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

generated by even variables a,b,c,d with the following non-commutativity relatons .

$$ab = p^{-1}ba; \quad ac = q^{t-1}ca; \quad ad = da + (q^{t-1} - p)cb;$$

$$bc = pq^{t-1}cb; \quad bd = q^{t-1}db; \quad cd = p^{-1}dc.$$

3.Now, we would like utilize the (exterior) differential d ,and admitting aditional with $dx = \xi$, $dy = \eta$.

First of all , we differential the one relation we have know, i.e. $xy - q^{t-1}yx = 0$, and get

$$0 = d(xy - q^{t-1}yx) = \xi y + x\eta - q^{t-1}\eta x - q^{t-1}y\xi = (\xi y - q^{t-1}y\xi) + (x\eta - q^{t-1}\eta x).$$

4.Comparing dimension, we see that we must find a system of four cross-commutation relations expresing $x\xi, x\eta, y\xi, y\eta$ linearly via $\xi x, \xi y, \eta x, \eta y$ as follows

$$\begin{pmatrix} x\xi \\ x\eta \\ y\xi \\ y\eta \end{pmatrix} = \begin{pmatrix} C_1^1 & C_2^1 & C_3^1 & C_4^1 \\ C_1^2 & C_2^2 & C_3^2 & C_4^2 \\ C_1^3 & C_2^3 & C_3^3 & C_4^3 \\ C_1^4 & C_2^4 & C_3^4 & C_4^4 \end{pmatrix} \begin{pmatrix} \xi x \\ \xi y \\ \eta x \\ \eta y \end{pmatrix}.$$

From the above mentioned equation $d(xy - q^{t-1}yx) = 0$, we get

$$C_1^2 = C_4^2 = C_1^3 = C_4^3 = 0.$$

5.Secondly, we differentiate the above matrix equation,and get

$$\begin{pmatrix} xd\xi \\ \xi\eta + xd\eta \\ \eta\xi + yd\xi \\ yd\eta \end{pmatrix} = \begin{pmatrix} C_1^1 & C_2^1 & C_3^1 & C_4^1 \\ C_1^2 & C_2^2 & C_3^2 & C_4^2 \\ C_1^3 & C_2^3 & C_3^3 & C_4^3 \\ C_1^4 & C_2^4 & C_3^4 & C_4^4 \end{pmatrix} \begin{pmatrix} (d\xi)x \\ (d\xi)y \\ (d\eta)x \\ (d\eta)y \end{pmatrix} - \begin{pmatrix} C_2^1\xi\eta + C_3^1\eta\xi \\ C_2^2\xi\eta + C_3^2\eta\xi \\ C_2^3\xi\eta + C_3^3\eta\xi \\ C_2^4\xi\eta + C_3^4\eta\xi \end{pmatrix}.$$

So, we get

$$\begin{pmatrix} 0 \\ -p\eta\xi \\ \eta\xi \\ o \end{pmatrix} = -\begin{pmatrix} -pC_2^1 + C_3^1 \\ -pC_2^2 + C_3^2 \\ -pC_2^3 + C_3^3 \\ -pC_2^4 + C_3^4 \end{pmatrix} \eta\xi,$$

i.e.

$$-pC_2^1 + C_3^1 = 0, -pC_2^2 + C_3^2 = p, -pC_2^3 + C_3^3 = -1, -pC_2^4 + C_3^4 = 0.$$

We choose $C_2^1 = 0$, then we have $C_3^1 = 0$. We choose $C_2^4 = 0$, then we have $C_3^4 = 0$.

6.We require the transformed cross products preserve the same non-commutativity , then we get

$$\begin{pmatrix} x^t\xi^t \\ x^t\eta^t \\ y^t\xi^t \\ y^t\eta^t \end{pmatrix} = \begin{pmatrix} C_1^1 & C_2^1 & C_3^1 & C_4^1 \\ C_1^2 & C_2^2 & C_3^2 & C_4^2 \\ C_1^3 & C_2^3 & C_3^3 & C_4^3 \\ C_1^4 & C_2^4 & C_3^4 & C_4^4 \end{pmatrix} \begin{pmatrix} \xi^t x^t \\ \xi^t y^t \\ \eta^t x^t \\ \eta^t y^t \end{pmatrix}.$$

So, we get on one side

$$x^t\xi^t = aax\xi + abx\eta + bay\xi + bby\eta = aa(C_1^1\xi x + 0 + C_4^1\eta y)$$

$$+ab(0 + C_2^2\xi y + C_3^2\eta x) + ba(0 + C_2^3\xi y + C_3^3\eta x) + bb(C_1^4\xi x + 0 + C_4^4\eta y),$$

and on the other side, we have

$$x^t\xi^t = C_1^1[aa\xi x + ab\xi y + ba\eta x + bb\eta y] + C_4^1[cc\xi x + cd\xi y + dc\eta x + dd\eta y].$$

Hence we get

$$C_1^4 bb = C_4^1 cc \Rightarrow C_1^4 = 0 = C_4^1$$

$$abC_2^2 + baC_2^3 = C_1^1 ab + C_4^1 cd \Rightarrow p^{-1}C_2^2 + C_2^3 = C_1^1 p^{-1},$$

$$abC_3^2 + baC_3^3 = C_1^1 ba + C_4^1 dc \Rightarrow (p^{-1}C_3^2 + C_3^3 - C_1^1) = 0.$$

$$aaC_4^1 + bbC_4^4 = C_1^1 bb + C_4^1 dd \Rightarrow C_1^1 = C_4^4.$$

To sume up we have five unknown coefficients $C_1^1, C_2^2, C_3^2, C_2^3, C_3^3$, which satisfy four equations.

$$-pC_2^2 + C_3^2 = p. \quad -pC_2^3 + C_3^3 = -1.$$

$$p^{-1}C_2^2 + C_2^3 = p^{-1}C_1^1, p^{-1}C_3^2 + C_3^3 - C_1^1 = 0.$$

It is easy to get an one parameter r solution, which is the same as get by Manin using some abstract method.

$$C_1^1 = [(p + q^{t-1})r - 1], \quad C_4^4 = [(p + q^{t-1})r - 1].$$

$$C_2^2 = (rq^{t-1} - 1), \quad C_3^2 = pq^{t-1}r, \quad C_2^3 = r, \quad C_3^3 = (pr - 1).$$

7.Corresponing to the case 1,case 2 and case 3 of Ω_q^t, now in our Ω_{p,q^t},we generalize it to the following case 1', case 2' and case 3'.

Case 1'.We consider now the tensor product algebra $A_{q^t}^{2I0} \otimes A_{q^t}^{2I0}$. In this case $x, y \in A_{q^t}^{2I0}$, and we meet

$$xy = q^t yx.$$

Case 2'.We consider now the tensor product algebra $A_p^{0I2} \otimes A_p^{0I2}$. In this case $\xi, \eta \in A_p^{0I2}$, and we meet

$$\xi\eta = -p\eta\xi.$$

Case 3'..We consider now the tensor product algebra $A2I0_{q^t} \otimes A_p^{0I2}$. As for the cross products, we have

$$x\xi = [(p + q^{t-1})r - 1]\xi x; \quad x\eta = (rq^{t-1} - 1)\xi y + pq^{t-1}r\eta x;$$

$$y\xi = r\xi y + (pr - 1)\eta x; \quad y\eta = [(p + q^{t-1})r - 1]\eta y.$$

Using the associativity of the product of our algebra,

$$x\eta\xi = x(\eta\xi) = (x\eta)\xi,$$

we get the equations

$$x(\eta\xi) == -p^{-1}x\xi\eta = -p^{-1}[(p + q^{t-1})r - 1]\xi[rq^{t-1} - 1)\xi y + pq^{t-1}r\eta x]$$

$$= -[(p + q^{t-1})r - 1]q^{t-1}r(\xi\eta x),$$

$$(x\eta)\xi = [(rq^{t-1} - 1)\xi y + pq^{t-1}r\eta x]\xi = (rq^{t-1} - 1)\xi y\xi + pq^{t-1}r\eta(x\xi)$$

$$= [(rq^{t-1} - 1)(0 + pr - 1) + (pq^{t-1}r[(p + q^{t-1})r - 1](-p^{-1})(\xi\eta x).$$

Hence,we have the quadratic polynomial equation on r

$$(q^{t-1}r - 1)(pr - 1) = 0.$$

i.e.

$$r^2 - (p + q^{t-1})r + pq^{t-1} = 0.$$

Their two solutions are $r = p^{-1}$ and $r = q^t$. Corresponding to these, the net result is;

First solution

$$x\xi = (pq^t)^{-1}\xi x; \quad x\eta = (p^{-1}q^{t-1} - 1)\xi y + q^{t-1}\eta x; \quad y\xi = p^{-1}\xi y; \quad y\eta = (pq^t)^{-1}\eta y.$$

Second solution

$$x\xi = (pq^t)\xi x; \quad x\eta = p\eta x; \quad y\xi = q^t\xi y + ((pq^t) - 1)\eta x; \quad y\eta = (pq^t)\eta y.$$

As for the transformation on the quantum complex

$$\Omega[A_q^{2f0}] = F[GL_q(2)]\Omega[A_q^{2f0}],$$

is extended to

$$\Omega[A_{q^t}^{2f0}] = \Omega[GL_{q^t}(2)]\Omega[A_{q^t}^{2f0}].$$

A lot of consistency conditions related to differential d and ∂, for example $\partial_i x^i$, $\partial_i \xi^j$, $\partial_i \partial_j$, can be verified in the same way as in Wess-Zumino[1].

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