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NEW GRADED ALGEBRA RELATED TO NON-STANDARD SOLUTION OF YANG-BAXTER EQUATION

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New Graded Algebra Related to Non-standard Solution of Yang-Baxter Equation [1]

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Abstract

A graded new algebra related to a non-standard R-matrix is presented. Its nontrivial Hopf algebraic structures for both quantum R-matrix and quantum enveloping algebra are investigated.

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Recent years the quantum Yang-Baxter equation (QYBE) [1] has attracted more attention of mathematicians and physicists [2][3] as the QYBE is tightly related to exactly solvable models and the corresponding algebraic structures. For a given non-trivial solution of QYBE one can generally find a neither commutative nor co-commutative Hopf algebra, which is interesting in mathematics. The standard method to construct a new algebra from a solution of QYBE was initiated by Faddeev et al [4]. With this method we have discussed the new algebra related to a non-standard R-matrix with one-parameter [5].

In this letter we investigate the algebra with some supersymmetric properties. It is from a non-standard and one parameter solution of QYBE obtained by using Wu-Elimination [6].

The R-matrix we consider here is

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{1}$$

R satisfies the quantum Yang-Baxter equation [1]

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} . \tag{2}$$

For a given solution of QYBE one can define a new $\hat{R}$ matrix

$$\hat{R} = \eta R \tag{3}$$

which satisfies the graded Yang-Baxter equation

$$\hat{R}_{12}\hat{R}_{13}\hat{R}_{23} = \hat{R}_{23}\hat{R}_{13}\hat{R}_{12} , \tag{4}$$

where

$$\begin{aligned}
\left(\hat{R}_{12}\right)^{j_1 j_2 j_3}_{i_1 i_2 i_3} &= \left(\hat{R}\right)^{j_1 j_2}_{i_1 i_2} \delta^{j_3}_{i_3} , \\
\left(\hat{R}_{13}\right)^{j_1 j_2 j_3}_{i_1 i_2 i_3} &= \left(\hat{R}\right)^{j_1 j_3}_{i_1 i_3} \delta^{j_2}_{i_2} (-1)^{P(j_2)(P(j_3)+P(i_3))} , \\
\left(\hat{R}_{23}\right)^{j_1 j_2 j_3}_{i_1 i_2 i_3} &= \left(\hat{R}\right)^{j_2 j_3}_{i_2 i_3} \delta^{j_1}_{i_1} (-1)^{P(j_1)(P(j_2)+P(j_3)+P(i_2)+P(i_3))}
\end{aligned}$$

and

$$\eta = \sum_{i,j} (-1)^{P(i)P(j)} e_{ii} \otimes e_{jj} \ .$$

$P(i)$ denotes the parity of the ith basic vector. In our case the matrix η is simply

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

First we study the algebra related to the matrix $\hat{R}$, which is generated by the Yang-Baxter equation [4]

$$\hat{R} T_2 T_1 = T_1 T_2 \hat{R} \ , \tag{5}$$

where

$$T = (t_{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

$T_1 = T \otimes I$, I is the 2×2 identity and

$$T_2 = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & -c & d \end{pmatrix} .$$

Equation (5) gives rise to the relations of the algebra elements a, b, c and d

$$\begin{aligned} &ab = qba \ , \quad [d, c] = 0 \ , \quad [a, c] = 0 \ , \quad b^2 = c^2 = 0 \ , \\ &bc = -q^{-1}cb \ , \quad bd = q^{-1}db \ , \quad [a, d] = (1-q)bc \ . \end{aligned} \tag{6}$$

This quantum matrix T can be considered as a linear transformation of plane $\mathcal{A}_q(1,1)$ with coordinates $(x_i) = (\xi, x)$ satisfying $x\xi = q\xi x$ and $\xi^2 = 0$. The proof is straightforward. Considering the coordinates (ξ', x') deduced from the T transformation

$$\begin{pmatrix} \xi' \\ x' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ x \end{pmatrix} ,$$

one can easily find that $x'\xi' = q\xi' x'$ and $\xi'^2 = 0$ by using the following relations

$$t_{ij} x_k = (-1)^{P(t_{ij})P(x_k)} x_k t_{ij} \ ,$$

where

$$P(a) = P(d) = 0 \ , \quad P(b) = P(c) = 1 \ , \quad P(x) = 0 \ , \quad P(\xi) = 1 \ .$$

The dual space $\mathcal{A}_q^*(1,1)$ of $\mathcal{A}_q(1,1)$ with coordinates (η, y) satisfy

$$y\eta = q\eta y \ , \quad \eta^2 = 0 \ .$$

Hence it is clear that the quantum plane is different from the one used in definition of $GL_q(1,1)$. And obviously the algebra here defined by (6) is also different to the usual algebra $SU_q(1,1)$.

Let a^{-1} and d^{-1} be the inverses of the elements a and b respectively. The center element of the algebra T is then

$$\delta = ad^{-1} - qd^{-1}bd^{-1}c \ . \tag{7}$$

and the inverse of δ is

$$\delta^{-1} = da^{-1} - ca^{-1}ba^{-1} \ . \tag{8}$$

The element δ is the analogue of the quantum determinant of T and has the following property

$$\delta(TT') = \delta(T)\delta(T') \ ,$$

which can be proved directly by using the commutative relations between elements of T and T',

$$t_{ij} t'_{ij} = (-1)^{P(t_{ij})P(t'_{ij})} t'_{kl} t_{ij} \ .$$

In terms of δ^{-1} we have the inverse of T

$$T^{-1} = \begin{pmatrix} d^{-1} & -d^{-1}bd^{-1} \\ -d^{-1}cd^{-1} & ad^{-2} - (1+q)d^{-1}bd^{-1}c \end{pmatrix} \delta^{-1} \ .$$

And the complete Hopf algebra structure related to T are given by

$$\begin{aligned}
&\Delta(a) = a\otimes a + b\otimes c\,, && \Delta(d) = d\otimes d + c\otimes b\,,\\
&\Delta(b) = a\otimes b + b\otimes d && \Delta(c) = d\otimes c + c\otimes a\,,\\
&S(a) = d^{-1}\,\delta^{-1}\,, && S(b) = -d^{-1}\,b\,d^{-1}\,\delta^{-1}\,,\\
&S(c) = -d^{-1}\,c\,d^{-1}\,\delta^{-1}\,, && S(d) = [a\,d^{-1}\,d^{-1} - (1+q)d^{-1}\,b\,d^{-1}\,c]\,\delta^{-1}\,,\\
&\epsilon(a) = \epsilon(d) = 1\,, && \epsilon(b) = \epsilon(c) = 0\,,
\end{aligned} \tag{9}$$

where Δ is the coproduct, S the antipode and ϵ the counit. The operators Δ and ϵ defined by (9) are algebraic homomorphism and S is graded antihomomorphism,

$$S(t_{ij}\,t_{kl}) = (-1)^{P(t_{ij})P(t_{kl})}S(t_{kl})S(t_{ij})\,.$$

They satisfy the axioms of Hopf algebra such as [7]

$$\begin{aligned}
(\Delta\otimes id)\Delta &= (id\otimes\Delta)\Delta\,,\\
(\epsilon\otimes id)\Delta &= (id\otimes\epsilon)\Delta\,,\\
m(S\otimes id)\Delta &= m(id\otimes S)\Delta\,.
\end{aligned}$$

Hence the algebra given by (6) and (9) is a Hopf algebra. Moreover one can show that

$$\Delta(\delta) = \delta\otimes\delta\,,\quad S(\delta) = \delta^{-1}\,,\quad \epsilon(\delta) = 1\,.$$

Now we study the quantum enveloping algebra related to the R matrix (1). It is determined by the following Yang-Baxter relations [4],

$$\begin{aligned}
\hat{R}\,L_1^{\pm}\,L_2^{\pm} &= L_2^{\pm}\,L_1^{\pm}\,\hat{R}\,,\\
\hat{R}\,L_1^{+}\,L_2^{-} &= L_2^{-}\,L_1^{+}\,\hat{R}\,,
\end{aligned} \tag{11}$$

where $L_1^{\pm} = L^{\pm}\otimes I$,

$$L^{+} = \begin{pmatrix} k & x\\ 0 & l\end{pmatrix}\,,\quad L^{-} = \begin{pmatrix} m & 0\\ y & n\end{pmatrix}\,,$$

and

$$L_2^{+} = \begin{pmatrix} k & x & 0 & 0\\ 0 & l & 0 & 0\\ 0 & 0 & k & -x\\ 0 & 0 & 0 & l\end{pmatrix}\,,\quad L_2^{-} = \begin{pmatrix} m & 0 & 0 & 0\\ y & n & 0 & 0\\ 0 & 0 & m & 0\\ 0 & 0 & -y & n\end{pmatrix}\,,$$

From (1) and (11) we have the following algebraic relations

$$\begin{aligned}
&kx = qxk\,,\quad ky = q^{-1}yk\,,\quad lx = qxk\,,\\
&ly = q^{-1}yl\,,\quad x^2 = y^2 = 0\,,\quad qxy + yx = (1-q)(lm - nk)\,.
\end{aligned} \tag{12}$$

Here we have omitted all the commutative relations. It is clear that the algebra element m and n are the centers. Generally not all the elements are independent. We can assume that

$$l = q^{H-\frac{K}{2}}\,,\quad l = q^{H+\frac{K}{2}}\,,\quad n = q^{2K}\,,\quad m = q^{-K}$$

and

$$X^{+} = \frac{1}{\sqrt{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q-q^{-1})}}k^{-1/2}x\,,\quad X^{-} = \frac{1}{\sqrt{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(q-q^{-1})}}k^{-1/2}y\,.$$

Then the algebraic relations become

$$\begin{aligned}
&[H, X^{\pm}] = \pm X^{\pm}\,,\quad [K, \bullet] = 0\,,\\
&\{X^{+}, X^{-}\} = \frac{q^{2K} - q^{-2K}}{q - q^{-1}} = [2K]\,.
\end{aligned} \tag{13}$$

The corresponding Hopf structures of algebra (13) can be defined by

$$\begin{aligned}
&\Delta(H) = H\otimes I + I\otimes H\,,\\
&\Delta(K) = K\otimes I + I\otimes K\,,\\
&\Delta(X^{+}) = q^{\frac{H}{2}+\frac{K}{4}}\otimes X^{+} + X^{+}\otimes q^{\frac{H}{2}-\frac{3K}{4}}\,,\\
&\Delta(X^{-}) = q^{-\frac{H}{2}+\frac{7K}{4}}\otimes X^{-} + X^{-}\otimes q^{-\frac{H}{2}-\frac{5K}{4}}\,,\\
&\epsilon(H) = \epsilon(K) = \epsilon(X^{\pm}) = 0\,,\quad \epsilon(1) = 1\,,\\
&S(H) = -H\,,\quad S(K) = -K\,,\quad S(X^{\pm}) = -q^{\frac{1}{2}}(q)^{\mp H\pm\frac{K}{2}}X^{\pm}\,.
\end{aligned} \tag{14}$$

It is straightforward to show that above definition of Hopf operations satisfies the axioms of the Hopf algebra (10).

So far we have presented the whole algebra structures, the algebra of quantum matrix and the quantum enveloping algebra, related to the solution (1) of QYBE.

With neither commutative nor co-commutative Hopf algebraic structures, this algebra is different to the usual q-deformed algebra $GL_q(1,1)$. As the algebra (13) gives rise to $GL(1,1)$ algebra at q approaches one, this algebra is another kind q-deformation of $GL(1,1)$. In addition we would like to indicate that, different to the case of new q-deformation of $GL(2)$ in [5], both the matrices (1) and (3) give rise to no Temperley-Lieb algebra representations. This can be understood as these R-matrices have two different non-zero eigenvalues satisfying a quadratic equation respectively.

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