AS-ITP-92-56
Fermilab Library
0 1160 0019892 3

ASITP

INSTITUTE OF THEORETICAL PHYSICS ACADEMIA SINICA

AS–ITP–92–56
Dec. 1992

TEMPERLEY-LIEB ALGEBRAIC REPRESENTATIONS FROM ISOTROPIC HIGHER SPIN CHAIN MODELS

Shao–Ming FEI

P.O.Box 2735, Beijing 100080, The People's Republic of China

Telefax : (086)-1-2562587 Telephone : 2568348

Telex : 22040 BAOAS CN Cable : 6158

ASITP-92-56

Temperley-Lieb Algebraic Representations From Isotropic Higher Spin Chain Models [1]

Shao-Ming Fei

CCAST (World Laboratory)
P. O. Box 8730, Beijing 100080, China

and

Institute of Theoretical Physics, Academia Sinica
P. O. Box 2735, Beijing 100080, China[2]

Abstract

We investigate the Temperley-Lieb algebraic structures in isotropic higher spin chain models. The Temperley-Lieb algebraic constructions for spins from $\frac{3}{2}$ to $\frac{7}{2}$ are presented. Their related spin chain Hamiltonians with $SU(2)$ symmetry and the corresponding solutions of Yang-Baxter equation are also discussed.

[1]Supported in part by the National Natural Science Foundation of China and LWTZ-1298 of Chinese Academy of Sciences.

[2]Mailing address

The spin chain models, such as the Heisenberg model, are of importance in studying the ferromagnetic and antiferromagnetic spin chains. So far only the spin-$\frac{1}{2}$ chain models have been extensively investigated, though experimentally spin chains to spin $j = \frac{7}{2}$ can be constructed in principle with higher spin atomic cluster at every site of the chain. On exactly solving the spin-$\frac{1}{2}$ chain models there is the well known Bethe ansatz [1] method by using the $\check{R}$-matrix, the related solution of Yang-Baxter equation [2] that stands for the integrability of the system. The Temperley-Lieb (T-L) algebraic approach is also an effective method for finite spin chains, in which the Hamiltonian equation are reduced into a set of coupled linear algebraic equations and the problem to solve the model becomes the usual eigenvalue problem of a matrix with finite dimensions [3].

In this letter we study the T-L algebraic structures and the related $\check{R}$-matrices in the isotropic chain models of spin values from $\frac{3}{2}$ to $\frac{7}{2}$. We find that there is a T-L algebraic construction for every chain with certain spin value. The corresponding spin chain Hamiltonian are of $SU(2)$ symmetry. And with respect to every T-L algebraic representation there are two $\check{R}$ matrices.

We consider a chain of $N+1$ atomic clusters with spin value j respectively. The spin operators S_i, $i = 1, 2, 3$, for every atomic cluster constitute the $SU(2)$ algebra,

$$[S_i, S_j] = \epsilon_{ijk} S_k \ , \quad i, j, k = 1, 2, 3. \tag{1}$$

The corresponding matrix representations are given by

$$\begin{aligned} (S_+)_{mm'} &= < jm|S_+|jm' > = \sqrt{(j-m')(j+m'+1)}\delta_{m,m'+1} \ , \\ (S_-)_{mm'} &= < jm|S_-|jm' > = \sqrt{(j+m')(j-m'+1)}\delta_{m,m'-1} \ , \\ (S_3)_{mm'} &= < jm|S_3|jm' > = m\delta_{m,m'} \ , \qquad -j \le m, m' \le j \end{aligned} \tag{2}$$

where $S_\pm = S_1 \pm S_2$ and $|jm>$ are the usual representation of spin. The total spin operator for the spin chain is

$$J_\alpha = \sum_{i=1}^{N+1} 1^{(1)} \otimes 1^{(2)} \otimes \cdots \otimes 1^{(i-1)} \otimes S_\alpha^{(i)} \otimes 1^{(i+1)} \otimes \cdots \otimes 1^{(N+1)} \ , \quad \alpha = 1, 2, 3$$

satisfying

$$[J_i, J_j] = i\epsilon_{ijk} J_k \ , \quad i,j,k = 1,2,3,$$

where $1^{(i)}$ is a $(2j+1)\times(2j+1)$ identity matrix at site i and $\otimes$ denotes the usual matrix tensor.

A N-state T-L algebra is described by the element e_i satisfying the T-L algebraic relations [4],

$$\begin{aligned} &e_i e_{i\pm1} e_i = e_i \ , \\ &e_i e_j = e_j e_i \ , \quad \text{if} \quad |i-j| \geq 2 \ , \end{aligned} \tag{3}$$

and

$$e_i^2 = \beta e_i \ , \tag{4}$$

where β is a constant and $i = 1, 2, \cdots, N$.

The T-L algebras related to the isotropic spin chain are of $SU(2)$ symmetry. We find that the representation of such a T-L algebraic element e_i is of the form

$$e_i = 1^{(1)} \otimes 1^{(2)} \otimes \cdots \otimes 1^{(i-1)} \otimes E \otimes 1^{(i+2)} \otimes \cdots \otimes 1^{(N+1)} \ , \tag{5}$$

where E is a $(2j+1)^2 \times (2j+1)^2$ matrix satisfying

$$E^2 = \beta E \ . \tag{6}$$

Let

$$\vec{S} \dot{\otimes} \vec{S} = S_1 \otimes S_1 + S_2 \otimes S_2 + S_3 \otimes S_3 \ .$$

Then the matrix representation E in (5) is uniquely given by the element $\vec{S} \dot{\otimes} \vec{S}$. Obviously such a matrix E commutes with the coproducts of $SU(2)$ algebraic generators

$$[E, \ \Delta S_i] = 0 \ , \quad i = 1,2,3,$$

where $\Delta S_i = S_i \otimes 1 + 1 \otimes S_i$. While the T-L element e_i commutes with the total $SU(2)$ generators of the chain,

$$[e_i, \ J_\alpha] = 0 \ , \quad \alpha = 1,2,3.$$

Therefore the Hamiltonians of spin-j chain models will have $SU(2)$ symmetry if it is constituted of the T-L algebraic element e_i. For instance,

$$H = J \sum_{i=1}^{N} e_i \ , \tag{7}$$

where J is a constant referring to the ferromagnetic or antiferromagnetic properties of the chain. Obviously

$$[H, J_i] = 0 \ , \quad i = 1,2,3 \ .$$

It can be deduced from the T-L algebraic relations that a T-L algebra representation always gives rise to solutions of the Yang-Baxter equation [2]

$$\check{R}_{12} \check{R}_{23} \check{R}_{12} = \check{R}_{23} \check{R}_{12} \check{R}_{23} \ , \tag{8}$$

where $\check{R}_{12} = \check{R} \otimes 1$ and $\check{R}_{23} = 1 \otimes \check{R}$. Accounting to the condition (6) and that the solution $\check{R}$ may be multiplied by an arbitrary non-zero constant, we know that $\check{R}$ should have the form $\check{R} = E + C\, 1 \otimes 1$, where C is a constant. Substituting it into equation (8) and taking the T-L algebraic relation (3) into account we have

$$(C^2 - \beta C + 1)(E \otimes 1 + 1 \otimes E) = 0 \ .$$

Which gives rise to $C = \frac{\beta \pm \sqrt{\beta^2 - 4}}{2}$ and thus

$$\check{R} = E + \frac{\beta \pm \sqrt{\beta^2 - 4}}{2} 1 \otimes 1 \ . \tag{9}$$

The followings are some results about the T-L algebra representation for higher spin chains.

For spin-$\frac{3}{2}$ chain, i.e., $j = \frac{3}{2}$, we have the T-L algebra element e_i with

$$E = \frac{33}{32} 1 \otimes 1 + \frac{31}{24} \vec{S} \dot{\otimes} \vec{S} - \frac{5}{18} (\vec{S} \dot{\otimes} \vec{S})^2 - \frac{2}{9} (\vec{S} \dot{\otimes} \vec{S})^3 \ , \tag{10}$$

where the corresponding spin matrix representations of S_i, $i = 1,2,3$ are given by equations (2) with $j = \frac{3}{2}$. In this case $\beta = 4$ and the relation (4) turns out to be

$$e_i^2 = 4 e_i \ .$$

From equation (9) we have the solutions of Yang-Baxter equation

$$\check{R} = E - (2 \pm \sqrt{3})\, 1 \otimes 1 \,.$$

The related spin chain Hamiltonian (7) is integrable and can be exactly solved in principle.

Similarly for the case of $j = 2$, we have

$$E = -\frac{1}{36}\left[60\, \vec{S} \dot{\otimes} \vec{S} + 17(\vec{S} \dot{\otimes} \vec{S})^2 - 4(\vec{S} \dot{\otimes} \vec{S})^3 - (\vec{S} \dot{\otimes} \vec{S})^4\right]$$

with $\beta = 5$ and

$$e_i^2 = 5 e_i \,.$$

The R-matrix is

$$\check{R} = E - \frac{5 \pm \sqrt{21}}{2}\, 1 \otimes 1 \,.$$

For the case of $j = \frac{5}{2}$, we have

$$\begin{aligned} E \;=\; & -\frac{1}{460800}\Big[980375\, 1 \otimes 1 - 287660\, \vec{S} \dot{\otimes} \vec{S} - 371872\, (\vec{S} \dot{\otimes} \vec{S})^2 \\ & -33152\, (\vec{S} \dot{\otimes} \vec{S})^3 + 8960\, (\vec{S} \dot{\otimes} \vec{S})^4 + 1024\, (\vec{S} \dot{\otimes} \vec{S})^5\Big] \,. \end{aligned}$$

While $\beta = 6$ and

$$e_i^2 = 6 e_i \,.$$

The corresponding R-matrix is

$$\check{R} = E - (3 \pm 2\sqrt{2})\, 1 \otimes 1 \,.$$

And for the case of $j = 3$, we have

$$\begin{aligned} E \;=\; & \frac{1}{8100}\Big[32076\, 1 \otimes 1 + 13608\, \vec{S} \dot{\otimes} \vec{S} - 3879\, (\vec{S} \dot{\otimes} \vec{S})^2 \\ & -1464\, (\vec{S} \dot{\otimes} \vec{S})^3 - 38\, (\vec{S} \dot{\otimes} \vec{S})^4 + 16\, (\vec{S} \dot{\otimes} \vec{S})^5 + (\vec{S} \dot{\otimes} \vec{S})^6\Big] \,. \end{aligned}$$

Here $\beta = 7$, i.e.,

$$e_i^2 = 7 e_i \,.$$

The related R-matrix is

$$\check{R} = E - \frac{7 \pm 3\sqrt{5}}{2}\, 1 \otimes 1 \,.$$

For the case of $j = \frac{7}{2}$, we get

$$\begin{aligned} E \;=\; & -\frac{1}{3251404800}\Big[8332038351\, 1 \otimes 1 + 12364158492\, \vec{S} \dot{\otimes} \vec{S} \\ & +1374000048\, (\vec{S} \dot{\otimes} \vec{S})^2 - 454121280\, (\vec{S} \dot{\otimes} \vec{S})^3 - 74101504\, (\vec{S} \dot{\otimes} \vec{S})^4 \\ & +21504\, (\vec{S} \dot{\otimes} \vec{S})^5 + 430080\, (\vec{S} \dot{\otimes} \vec{S})^6 + 16384\, (\vec{S} \dot{\otimes} \vec{S})^7\Big] \end{aligned}$$

and $\beta = 8$. That is

$$e_i^2 = 8 e_i \,.$$

The concerned R-matrix is

$$\check{R} = E - (4 \pm \sqrt{15})\, 1 \otimes 1 \,.$$

The T-L algebra representations can be investigated for general spin value j according to (2). In fact, for arbitrary j, the $(2j+1)^2 \times (2j+1)^2$ matrix E is simply of the form

$$(E)_{n,m} = (-1)^{k+l} \delta_{2jk+1,2jl+1}\, \delta_{n,2jk+1}\, \delta_{m,2jl+1} \qquad k, l \le 2j+1 \,. \tag{11}$$

And $\beta = 2j+1$. That is

$$e_i^2 = (2j+1) e_i \,.$$

The corresponding solutions of Yang-Baxter equation are

$$\check{R} = E + \frac{2j+1 \pm \sqrt{(2j+1)^2 - 4}}{2}\, 1 \otimes 1 \,. \tag{12}$$

Here we would like to indicate that the solutions of Yang-Baxter equation given by (9) are not necessary independent because of the uniqueness of the universal R-matrix [5]. As an example we take $j = \frac{1}{2}$. For the isotropic case from equations (11) and (9) we get only one R-matrix as $C = 0$, $\check{R} = -P$. P is the permutation

matrix. However for the anisotropic case that the related spin chain is of the q-deformed algebra $SU_q(2)$ symmetry [6],

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

and $\beta = [2] = q + q^{-1}$ [7]. Formula (9) gives rise to two solutions

$$\check{R}_1 = \begin{pmatrix} -q & 0 & 0 & 0 \\ 0 & q^{-1}-q & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -q \end{pmatrix}, \qquad \check{R}_2 = \begin{pmatrix} -q^{-1} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & q-q^{-1} & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

Since E commutes with the coproducts of $SU_q(2)$ elements, both $\check{R}_1$ and $\check{R}_2$ also commute with the coproducts of $SU_q(2)$ elements. Nevertheless they are equivalent since $\check{R}_1 = P \cdot \check{R}_2 \cdot P|_{q\to q^{-1}}$. For $j > \frac{1}{2}$ formula (9) gives rise to two different R-matrices even for the isotropic cases. The equivalence between these pairs of R-matrices can be similarly discussed.

The T-L algebra representations concerned in this letter are related to the isotropic spin chain with $SU(2)$ symmetry. The spin chain with T-L algebraic structure and $SU_q(2)$ symmetry can also be investigated like the case of $j = 1$ [8].

References

[1] R.J. Baxter, *J. Stat. Phys.* **28**, (1982) 1; and *Exactly Solved Models in Statistical Mechanics* (Academic Press, New York, 1982).

[2] C.N. Yang, Phys. Rev. Lett. **19**(1967)1312-1314.

[3] D. Levy, *Phys. Rev. Lett.* **64** (1990), 499.

[4] H. N. V. Temperley and E. Lieb, *Proc. Roy. Soc. London* A **322** (1971), 251.

[5] M. Jimbo, Lectures at NanKai Institute of Mathematics in Tianjin, 1991.

[6] M.Jimbo, Lett. Math. Phys. 10(1985)63; 11(1986)247; Comm. Math. Phys. 102(1986)537.

L.D. Faddeev, in Les Houches Lectures, 1982.

P.P. Kulish and E.K. Sklyanin, Lecture Notes in Physics, Vol. 151, 61.

[7] B.-Y. Hou, B.-Y. Hou and Z.-Q. Ma, *J. Phys.* A: *Math. and Gen.* **24** (1991) 2847.

[8] Shao-Ming Fei, *Temperley-Lieb Algebraic Structures in Spin-1 Chain Models*, ASITP-92-54.