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Oct. 6, 1992

TEMPERLEY-LIEB ALGEBRAIC STRUCTURES IN SPIN–1 CHAIN MODELS

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Temperley-Lieb Algebraic Structures in Spin-1 Chain Models [1]

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Abstract

We show that there are two Temperley-Lieb algebraic structures in spin-1 chain models. The related Hamiltonians are of $SU(2)$ and $SU_q(2)$ symmetries respectively. The corresponding solutions of Yang-Baxter equation are also discussed.

[1]Supported in part by the National Natural Science Foundation of China and LWTZ-1298 of Chinese Academy of Sciences.

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The exact solutions of the Heisenberg spin-$\frac{1}{2}$ chain models have been thoroughly investigated in terms of both Bethe ansatz [1] and Temperly-Lieb (T-L) algebraic structures of the models [2]. The Bethe ansatz method gives rise to the general forms of the energy and eigenstates with a Bethe ansatz equation which is difficult to solve. While in the T-L algebraic approach the Hamiltonian equation are reduced into a set of coupled linear algebraic equations and the problem to solve the model becomes the usual eigenvalue problem of a matrix with finite dimensions [2-3].

As is well known, for the case of spin-$\frac{1}{2}$ chain, there are two integrable Heisenberg spin models with $SU(2)$ and $SU_q(2)$ symmetries respectively. The isotropic one is the XXX model with $SU(2)$ symmetry. Its related T-L algebra element e_i satisfies $e_i^2 = 2e_i$. The anisotropic one is the XXZ model. With a suitable boundary term this XXZ model is of the q-deformed $SU_q(2)$ symmetry [4] and its T-L algebra element satisfies $e_i^2 = [2]e_i$, where $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$. e_i represents the nearest interactions between spins at site i and $i+1$. Both XXX and XXZ models can be exactly solved according to the T-L algebraic properties and the symmetry algebras $SU(2)$ and $SU_q(2)$. Especially for the isotropic case, we find that the higher dimensional Heisenberg spin model with non-nearest interactions can also be described by the T-L algebra and exactly solved [5].

For the case of spin-1, the Heisenberg model can no longer be simply manipulated. This letter is dedicated to investigate the T-L algebraic structures of the spin-1 chain models. We find that, similar to the case of spin-$\frac{1}{2}$, there are also two T-L algebraic structures. One is related to the isotropic model with $SU(2)$ symmetry. And the q-deformed one is related to the anisotropic model with $SU_q(2)$ symmetry. With respect to these T-L algebraic representations there are braid group representations and solutions of the Yang-Baxter equation [6].

We consider a chain with $N+1$ spins. The spin operators $S_\pm$ and S_3 of every spin constitute the $SU(2)$ algebra

$$[S_+, S_-] = 2S_3 \,, \qquad [S_3, S_\pm] = \pm S_\pm \,, \tag{1}$$

where $S_\pm = S_1 \pm iS_2$ and

$$S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The total spin operator for the spin chain is then

$$J_\alpha = \sum_{i=1}^{N+1} 1^{(1)} \otimes 1^{(2)} \otimes \cdots \otimes 1^{(i-1)} \otimes S_\alpha^{(i)} \otimes 1^{(i+1)} \otimes \cdots \otimes 1^{(N+1)},$$

where $1^{(i)}$ is a 3×3 identity matrix at site i, $\otimes$ denotes matrix tensor and $\alpha = \pm, 3$ or $1, 2, 3$. J_α satisfy the $SU(2)$ algebraic relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad i, j, k = 1, 2, 3.$$

We find that the representation of T-L algebraic element with $SU(2)$ symmetry is given by

$$e_i = 1^{(1)} \otimes 1^{(2)} \otimes \cdots \otimes 1^{(i-1)} \otimes E \otimes 1^{(i+2)} \otimes \cdots \otimes 1^{(N+1)}, \tag{2}$$

where

$$E = ((S_1 \otimes S_1 + S_2 \otimes S_2 + S_3 \otimes S_3)^2 - 1 \otimes 1) \stackrel{def.}{=} (\vec{S} \dot{\otimes} \vec{S})^2 - 1 \otimes 1.$$

e_i satisfies the T-L algebraic relations [7],

$$\begin{aligned} e_i^2 = 3e_i, \quad & e_i e_{i\pm 1} e_i = e_i, \\ e_i e_j = e_j e_i, \quad & \text{if } |i-j| \geq 2 \end{aligned} \tag{3}$$

and commutes with $SU(2)$ generators of the chain,

$$[e_i, J_\alpha] = 0, \quad \alpha = \pm, 3.$$

Therefore the Hamiltonian of spin-1 chain model will have $SU(2)$ symmetry if it is constituted of T-L algebraic element e_i. Here is a spin-1 chain model with Hamiltonian

$$H = J\sum_{i=1}^{N} e_i, \tag{4}$$

where J is a constant referring to the ferromagnetic or antiferromagnetic properties of the chain. Obviously

$$[H, J_\alpha] = 0, \quad \alpha = \pm, 3.$$

It can be proven that this T-L algebraic representation (2) with $SU(2)$ symmetry is unique.

The solution of Yang-Baxter equation related to the T-L algebra (2) is

$$\check{R} = (\vec{S} \dot{\otimes} \vec{S})^2 - \frac{5 \pm \sqrt{5}}{2} 1 \otimes 1 = E - \frac{3 \pm \sqrt{5}}{2} 1 \otimes 1.$$

$\check{R}$ satisfies the Yang-Baxter relation

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23},$$

where $\check{R}_{12} = \check{R} \otimes 1$ and $\check{R}_{23} = 1 \otimes \check{R}$.

Now we consider the T-L algebraic structures with q-deformed symmetry. The $SU_q(2)$ algebra is defined by [8]

$$[S'_+, S'_-] = [2S'_3], \quad [S'_3, S'_\pm] = \pm S'_\pm. \tag{5}$$

From the co-multiplication

$$\Delta S'_\pm = q^{S'_3} \otimes S'_\pm + S'_\pm \otimes q^{-S'_3}, \quad \Delta S'_3 = 1 \otimes S'_3 + S'_3 \otimes 1,$$

one gets the total algebra operators of the spin chain

$$\begin{aligned} J'_\pm &= \sum_{i=1}^{N+1} q^{S'_3} \otimes \cdots \otimes q^{S'_3} \otimes S'^{(i)}_\pm \otimes q^{-S'_3} \otimes \cdots \otimes q^{-S'_3}, \\ J'_3 &= \sum_{i=1}^{N+1} 1 \otimes \cdots \otimes 1 \otimes S'^{(i)}_3 \otimes 1 \otimes \cdots \otimes 1 \end{aligned}$$

satisfying

$$[J'_+, J'_-] = [2J'_3], \quad [J'_3, J'_\pm] = \pm J'_\pm.$$

The matrix representation of $S'_{\pm}$ and S'_3 satisfying (5) can be obtained from the algebraic maps between $SU(2)$ and $SU_q(2)$ [9-10]. After a similarity transformation we have

$$S'_{\pm} = \sqrt{\frac{q+q^{-1}}{2}} S_{\pm} , \quad S'_3 = S_3 .$$

We find that the T-L algebraic element with $SU_q(2)$ symmetry is given by

$$e'_i = 1^{(1)} \otimes 1^{(2)} \otimes \cdots \otimes 1^{(i-1)} \otimes E_q \otimes 1^{(i+2)} \otimes \cdots \otimes 1^{(N+1)} , \tag{6}$$

where

$$\begin{aligned} E_q \;=\; & \frac{1}{4q^2}\Big[q^2(S_+^2 \otimes S_-^2 + S_-^2 \otimes S_+^2 + 4S_3^2 \otimes S_3^2 \\ & +2S_3S_+ \otimes S_3S_- + 2S_3S_- \otimes S_3S_+ + S_-S_+ \otimes S_+S_-) \\ & +(q^4 - q^2 + 1)S_+S_- \otimes S_-S_+ \\ & +2q(q^2 - q + 1)(S_+S_3 \otimes S_-S_3 + S_-S_3 \otimes S_+S_3) \\ & +2q^2(q-1)(S_-S_3 \otimes S_+ - S_- \otimes S_+S_3) \\ & +2q(q-1)(S_+S_3 \otimes S_- - S_+ \otimes S_-S_3) \\ & +2(q^2-1)(S_3^2 \otimes S_3 - S_3 \otimes S_3^2) \\ & +2q(q-1)S_+ \otimes S_- - 2q^2(q-1)S_- \otimes S_+ \\ & +2(q^2-1)^2(S_3^2 \otimes 1 + 1 \otimes S_3^2 + 1 \otimes S_3 - S_3 \otimes 1) \\ & -4(q^4 - q^2 + 1)1 \otimes 1)\Big] . \end{aligned}$$

The T-L algebraic element (6) gives rise to the T-L algebra

$$\begin{aligned} & e_i'^2 = (q^2 + 1 + q^{-2})e'_i = [3]e'_i , \qquad e'_i e'_{i\pm1} e'_i = e'_i , \\ & e'_i e'_j = e'_j e'_i , \quad \text{if} \quad |i-j| \geq 2 . \end{aligned} \tag{7}$$

It is straightforward to prove that

$$[e'_i, \; J'_\alpha] = 0 , \quad \alpha = \pm, 3 .$$

Such a T-L algebraic structure is also unique for the case of spin-1 chain models.

The Hamiltonian related to (6) is then

$$H_q = J \sum_{i=1}^{N} e'_i . \tag{8}$$

H_q is obviously of $SU_q(2)$ symmetry

$$[H_q, J'_{\pm,3}] = 0 .$$

Corresponding to the q-deformed spin chain model, there is also a solution of Yang-Baxter equation,

$$\check{R}_q = E_q + C$$

satisfying

$$(\check{R}_q \otimes 1)(1 \otimes \check{R}_q)(\check{R}_q \otimes 1) = (1 \otimes \check{R}_q)(\check{R}_q \otimes 1)(1 \otimes \check{R}_q) ,$$

where

$$C = \frac{-1}{2q^2}\left(q^4 + q^2 + 1 \pm \sqrt{q^8 + 2q^6 - q^4 + 2q^2 + 1}\right) .$$

When q approaches one, the so called quantum algebra $SU_q(2)$ becomes algebra $SU(2)$. And it is direct to find that the T-L algebraic element e'_i and Hamiltonian H_q with $SU_q(2)$ symmetry reduce to e_i and H with $SU(2)$ symmetry when $q \to 1$. So does the related solution of Yang-Baxter equation. Here we would like to indicate that the $SU_q(2)$ invariant Hamiltonian (8) for spin-1 chain model is different to the usual one given in [11]. It is obvious that they are different when q approaches one. In addition, $\check{R}$ and $\check{R}_q$ also give rise to braid group representations and hence to some kinds of link polynomials.

We have investigated the T-L algebraic structures in the spin-1 chain models. Similar to the case of spin-$\frac{1}{2}$, we also find two T-L algebraic constructions with $SU(2)$ and $SU_q(2)$ symmetry respectively. The integrability of the related spin chain models, both for generic q and q being a root of unit, can be studied similarly according to the T-L algebraic properties and $SU(2)$ and $SU_q(2)$ symmetries. From the solutions of Yang-Baxter equation in these models one may also try the Bethe ansatz method on exact solutions of the models.

Acknowledgements The auther would like to thank Prof. H.Y. Guo and Z.Q. Ma for helpful discussions.

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