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August, 1992

1992

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Differential Calculi on Quantum Subgroups and Quantum Co-set Spaces*

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Abstract

In this note, quantum subgroups and quantum co-set spaces including corresponding differential calculi are discussed.

*Supported by Natural Science Foundation of Academia Sinica.

1. Quantum subgroups and quantum co-set spaces

More attentions have been paid on quantum groups and their differential calculi by a lot of authors[1-9]. The purpose of this note is to discuss some geometry on quantum groups, i.e. the methods to obtain quantum subgroups, quantum co-set spaces and corresponding differential calculi.

Denote G a quantum group and $Fun(G)$ the algebra of functions on G. The so called quantum subgroup H is defined in term of the algebra of functions on H, which is obtained by the quotient[4]:

$$Fun(H) = Fun(G)/\{I\}, \tag{1.1}$$

where $\{I\}$ is the Hopf ideal generated by I which I is a subset of $Fun(G)$ satisfying:

$$\Delta(I) \subset \{I\} \otimes Fun(G) \cup Fun(G) \otimes \{I\},$$
$$S(I) \subset \{I\}.$$

Therefore (1.1) gives a projection $\mathbf{P}$ from $Fun(G)$ to $Fun(H)$. The coproduct, counit and antipode on H are induced from those operators on G in terms of the projection $\mathbf{P}$. Then the algebra of functions on quantum co-set space G/H is given by following definition.

Definition 1.1 Denote $Fun_L(G/H)$ and $Fun_R(G/H)$ the algebras of functions on left and right quantum co-set space G/H, they are defined as following,

$$Fun_L(G/H) \subset Fun(G), \quad Fun_R(G/H) \subset Fun(G),$$

and for $\forall x \in Fun_L(G/H)$

$$(id \otimes \mathbf{P})\Delta x = x \otimes 1, \tag{1.2}$$

for $\forall x \in Fun_R(G/H)$

$$(\mathbf{P} \otimes id)\Delta x = 1 \otimes x. \tag{1.3}$$

Proposition 1.1 $Fun_L(G/H)$ is a left G-comodule and $Fun_R(G/H)$ is a right G-comodule, i.e.

$$\Delta Fun_L(G/H) \subset Fun(G) \otimes Fun_L(G/H),$$
$$\Delta Fun_R(G/H) \subset Fun_R(G/H) \otimes Fun(G).$$

Proof: Here we only prove the first formula, the proof of the second is similar. Let $x \in Fun_L(G/H)$, since

$$(id \otimes \Delta)\Delta x = (\Delta \otimes id)\Delta x,$$

then by Definition 1.1,

$$(id \otimes id \otimes \mathbf{P})(id \otimes \Delta)\Delta x = (id \otimes id \otimes \mathbf{P})(\Delta \otimes id)\Delta x = (\Delta \otimes id)(id \otimes \mathbf{P})\Delta x = \Delta x \otimes 1. \tag{1.4}$$

If we write $\Delta x = x_{1,\alpha} \otimes x_{2,\alpha}$ where $x_{1,\alpha}$ are linear independent, (1.4) can be written as

$$x_{1,\alpha} \otimes ((id \otimes \mathbf{P})\Delta x_{2,\alpha}) = x_{1,\alpha} \otimes x_{2,\alpha} \otimes 1.$$

By the linear independence of $x_{1,\alpha}$ and Definition 1.1, we know $x_{2,\alpha} \in Fun_L(G/H)$, i.e.

$$\Delta Fun_L(G/H) \subset Fun(G) \otimes Fun_L(G/H).$$

Now we are going to give some examples of quantum subgroups and quantum co-set spaces. Since $Fun_L(G/H)$ and $Fun_R(G/H)$ are similar, in the following part of this note, we only consider $Fun_L(G/H)$. The definitions of quantum groups $GL_q(N)$ and $SL_q(N)$ are given in [1] in which the R-matrix corresponding to quantum groups of A-series of order N is

$$R_q = q^{1/N}\left(\sum_{i,j=1}^{N} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + \chi \sum_{\substack{i,j=1 \\ i>j}}^{N} e_{ij} \otimes e_{ji}\right), \ q \in \mathbb{C}^*, \chi = q - q^{-1}.$$

Example 1 The algebra of functions on subgroup $SL_q(N)$ of $GL_q(N)$ is

$$Fun(SL_q(N)) = Fun(GL_q(N))/\{Det_q T - 1\}$$

Example 2 The algebras of functions on subgroups $GL_q(k, N-k)$ and $SL_q(k, N-k)$ of $GL_q(N)$ and $SL_q(N)$ respectively are

$$\begin{aligned} Fun(GL_q(k, N-k)) &= Fun(GL_q(N))/\{I_1\}, \\ Fun(SL_q(k, N-k)) &= Fun(SL_q(N))/\{I_1\}, \end{aligned}$$

where $I_1 = \{t_{ij} \mid i \leq k, j > k\}$.

Example 3a Let the four generators of $Fun_q(SL_q(2))$ be written in 2×2 matrix as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If we take $I_2 = \{b\}$ and $I_3 = \{b, c\}$, then two subgroups G and H of $SL_q(2)$ are given as

$$Fun(G) = Fun(SL_q(2))/\{I_2\}, \quad Fun(H) = Fun(SL_q(2))/\{I_3\}.$$

In fact H is also a subgroup of G. Since the determinant of G and H are $a \cdot d = 1$, we have $d = a^{-1}$. This implied that $Fun(G)$ is generated by three elements a, a^{-1}, c which can be written in matrix form as

$$T^G = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix},$$

and

$$\begin{array}{lll} \Delta a = a \otimes a, & \Delta a^{-1} = a^{-1} \otimes a^{-1}, & \Delta c = c \otimes a + a^{-1} \otimes c, \\ \varepsilon(a) = 1, & \varepsilon(a^{-1}) = 1, & \varepsilon(c) = 0, \\ S(a) = a^{-1}, & S(a^{-1}) = a, & S(c) = -\frac{1}{q}c. \end{array} \tag{1.5}$$

Here $Fun(G)$ looks similar to the two-dimensional quantum plane with two elements x, y satisfying $yx = qxy$, but they are naturally different from each other since not only

left and right group co-actions of $SL_q(2)$ but also coproduct, counit and antipode are defined on $Fun(G)$. This point has been also noticed by some authers[5], but they did not know why they are different and what the main meaning of $Fun(G)$. In the next example we will try to understand some of the above questions.

$Fun(H)$ is generated by two elements a, a^{-1} which can be written in matrix form as

$$T^G = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

and

$$\begin{array}{ll} \Delta a = a \otimes a, & \Delta a^{-1} = a^{-1} \otimes a^{-1}, \\ \varepsilon(a) = 1, & \varepsilon(a^{-1}) = 1, \\ S(a) = a^{-1}, & S(a^{-1}) = a. \end{array}$$

The definition of projection $\mathbf{P}$ from $Fun(G)$ to $Fun(H)$ is given by

$$\mathbf{P}(a) = a, \ \mathbf{P}(a^{-1}) = a^{-1}, \ \mathbf{P}(c) = 0. \tag{1.6}$$

Now we are going to give $Fun_L(G/H)$. From (1.5) and (1.6) we know that

$$\Delta(ca^{-1}) = ca^{-1} \otimes 1 + a^{-2} \otimes ca^{-1},$$

and

$$(id \otimes \mathbf{P})\Delta(ca^{-1}) = ca^{-1} \otimes 1.$$

Further caculation shows that for all element $x \in Fun(G)$ satisfies (1.2) must be a polynomial of ca^{-1}, so we have

$$Fun_L(G/H) = \mathbb{C}[ca^{-1}, 1].$$

Example 3b Interchange a and c in Example 3a and replace the deformation parameter q by q^{-1}, we have

$$Fun_L(G/H) = \mathbb{C}[1, ac^{-1}].$$

When $q \to 1$, Example 3a and 3b give two local coordinate systems of projection space $\mathbb{C}P^1$, so it can be understand that Example 3a and 3b give two local coordinate functions of quantum projection space $\mathbb{C}P^1_q$.

Example 4a G is a subgroup of $SL_q(3)$, $Fun(G)$ is generated by x_1, x_2, x_3, y_2, y_3, which can be written into a 3×3 matrix as:

$$T^G = \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & y_2 & 0 \\ x_3 & 0 & y_3 \end{pmatrix},$$

and $x_1 y_2 y_3 = 1$. The commutation relations among them are given by

$$RT_1^G T_2^G = T_2^G T_1^G R,$$

where R is the R-matrix of A-series of order 3[1]. H is a subgroup of G, $Fun(H)$ is generated by y_1, y_2, y_3, which can be written into a 3×3 matrix as:

$$T^H = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix},$$

and $y_1 y_2 y_3 = 1$. The commutation relations among them are given by

$$RT_1^H T_2^H = T_2^H T_1^H R,$$

where R is the R-matrix of A-series of order 3. Then we have

$$Fun_L(G/H) = \mathbb{C}[1, x_2 x_1^{-1}, x_3 x_1^{-1}].$$

Example 4b Similar to Example 4a, $Fun(G)$ is generated by x_1, x_2, x_3, y_1, y_3 which can be written into a 3×3 matrix as

$$T^G = \begin{pmatrix} y_1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & y_3 \end{pmatrix},$$

and $y_1 x_2 y_3 = 1$. The commutation relations among them are given by

$$RT_1^G T_2^G = T_2^G T_1^G R.$$

$Fun(H)$ is same as in Example 4a and

$$Fun_L(G/H) = \mathbb{C}[x_1 x_2^{-1}, 1, x_3 x_2^{-1}].$$

Example 4c Similar to Example 4a, $Fun(G)$ is generated by x_1, x_2, x_3, y_2, y_3 which can be written into a 3×3 matrix as

$$T^G = \begin{pmatrix} y_1 & 0 & x_1 \\ 0 & y_2 & x_2 \\ 0 & 0 & x_3 \end{pmatrix},$$

and $y_1 y_2 x_3 = 1$. The commutation relations among them are given by

$$RT_1^G T_2^G = T_2^G T_1^G R.$$

$Fun(H)$ is same as in Example 4a and

$$Fun_L(G/H) = \mathbb{C}[x_1 x_3^{-1}, x_2 x_3^{-1}, 1].$$

When $q \to 1$, Example 4a, 4b and 4c give three local coordinate systems of projection space $\mathbb{C}P^2$, so it can be understand that Example 4a, 4b and 4c give three local coordinate function of quantum projection space $\mathbb{C}P_q^2$. Although one can discribe

quantum $\mathbb{C}P^2$ using the deformation of continuous functions on $\mathbb{C}P^2$ as well as $\mathbb{C}P^1$, here we think it is another way to discribe the $\mathbb{C}P^k$ $(k = 1, 2)$ more geometrically.

It is easy to extend above examples of quantum projection spaces to high orders.

2. Differential calculi on quantum subgroups and quantum co-set spaces

Assume that we have had differential calculus on quantum group G, the problem is how to obtain differential calculus on H, a subgroup of G. From (1.1) we know that we can obtain $Fun(H)$ by letting some of the elements of $Fun(G)$ be zero, this will force some of the elements of $\Omega^k(G)$, set of k-forms on G, be zero. Then we can obtain $\Omega^k(H)$, set of k-forms on H by a quotient similar to (1.1).

Taking quantum group G in Example 3a as an example, we know G is a subgroup of $SL_q(2)$, and we have already discussed differential calculus on $SL_q(2)$ in [6,7]. Let us briefly review the way to construct differential calculus on $SL_q(2)$.

$Fun(SL_q(2))$ is generated by a, b, c, d which can be written in to a 2×2 matrix

$$T = (t_{ij})_{i,j=1,2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

First step, we give two sets of linear functionals $l_{ij}^{\pm}$ $(i, j = 1, 2)$ and arrange them into two 2×2 matrices.

$$L^{\pm} = (l_{ij}^{\pm})_{i,j=1,2}.$$

To describe $l_{ij}^{\pm}$ $(i, j = 1, 2)$ explicitly, we first define that the values of the linear functionals $l_{ij}^{\pm}$ $(i, j = 1, 2)$ on the generators $t_{ij}^{\pm}$ $(i, j = 1, 2)$ of $Fun(SL_q(2))$ are given by

$$l_{ij}^{\pm}(T) = \lambda_{\pm}^{\pm 1} r_{ij}^{\pm}, \quad \lambda_{\pm}^2 = 1, \tag{2.1}$$

$$l_{ij}^{\pm}(1) = \delta_{ij}, \quad i, j = 1, 2, \tag{2.2}$$

where

$$r_{ij}^{+} = \begin{cases} (q - q^{-1}) e_{ji}, & i < j, \\ E + (q-1) e_{ii}, & i = j, \\ 0, & i > j. \end{cases} \tag{2.3}$$

$$r_{ij}^{-} = \begin{cases} 0, & i < j, \\ E + (q-1) e_{ii}, & i = j, \\ -(q - q^{-1}) e_{ji}, & i > j. \end{cases} \tag{2.4}$$

If we denote $R^+ = (r_{ij}^+)_{1 \le i,j \le n}$, $R^- = (r_{ij}^-)_{i,j=1,2}$, then $R^+ = PR_qP$, $R^- = R_q^{-1}$, where R_q is the Yang-Baxter matrix for quantum group $SL_q(2)$, P is the permutation matrix,

$$P = (e_{ji})_{i,j=1,2}.$$

For arbitrary element of $Fun(SL_q(2))$ the definition of $l^{\pm}_{ij}$ is given by the following induction,

$$l^{\pm}_{ij}(xy) = l^{\pm}_{ik}(x)l^{\pm}_{kj}(y), \quad \forall x, y \in \mathrm{Fun}(SL_q(2)). \tag{2.5}$$

Second step, we introduced the convolution "$*$" on Ω^0, i.e. $Fun(SL_q(2))$. For $f \in Fun^*(SL_q(2))$, the convolution "$*$" from Ω^0 to Ω^0 is defined by

$$f * (x) = (id \otimes f)\Delta x, \quad x \in \mathrm{Fun}(SL_q(2)), \tag{2.6}$$

where id is the identity operator on Ω^0. Then we introduced two sets of functionals on Ω^0 as follows:

$$\nabla_{ij} = \frac{1}{q-q^{-1}}(S(l^-_{ik})l^+_{kj} - \delta_{ij}\varepsilon),$$
$$\theta_{ijkl} = S(l^-_{ki})l^+_{jl},$$

where $i, j, k, l = 1, 2$. These functionals satisfy following proposition.

Proposition 2.1 For $\forall x, y \in \Omega^0, i, j, k, l = 1, 2$,

(i) $\nabla_{ij}(1) = 0, \quad \theta_{ijkl}(1) = \delta_{ik}\delta_{jl},$

(ii) $\Delta\nabla_{ij} = \nabla_{uv} \otimes \theta_{uvij} + \varepsilon \otimes \nabla_{ij},$

$\Delta\theta_{ijkl} = \theta_{ijuv} \otimes \theta_{uvkl},$

(iii) $\nabla_{ij} * (xy) = (\nabla_{uv} * x)(\theta_{uvij} * y) + x(\nabla_{ij} * y),$

$\theta_{ijkl} * (xy) = (\theta_{ijuv} * x)(\theta_{uvkl} * y)$

Let Ω^1 be the left module generated by four generators ω^{ij}, $(i, j = 1, 2)$, and define the right multiplication on Ω^1 by

$$\omega^{ij} \cdot x = (\theta_{ijkl} * x)\omega^{kl}, \quad \forall x \in \Omega^0, \quad i, j, k, l = 1, 2.$$

Therefore, Ω^1 becomes a Ω^0-bimodule. At last we have

$$\delta x = (\nabla_{ij} * x)\omega^{ij}, \quad \forall x \in \Omega^0, i, j = 1, 2. \tag{2.7}$$

And furthermore

$$\begin{aligned}
\omega^{11} &= [q^3(d\delta a - q^{-1}b\delta c) - (q^3 - q - 1)(-qc\delta b + a\delta d)]/(q^3 - 1),\\
\omega^{12} &= q(-qc\delta a + a\delta c),\\
\omega^{21} &= q(d\delta b - q^{-1}b\delta d),\\
\omega^{22} &= [q^2(d\delta a - q^{-1}b\delta c) + q^3(-qc\delta b + a\delta d)]/(q^3 - 1).
\end{aligned} \tag{2.8}$$

Since $Fun(G)$ is obtained by letting $b = 0$ in $Fun(SL_q(2))$, we have $\nabla_{21} = 0$ and $\omega^{21} = 0$ for $Fun(G)$. So we can obtain differential calculus on G by adding condition $\nabla_{21} = 0$ and $\omega^{21} = 0$ to that of $SL_q(2)$.

The commutation relations of ∇_{ij} $(i, j = 1, 2)$ except ∇_{21} are

$$\begin{aligned}
&\nabla_{12}\nabla_{11} - q^2\nabla_{11}\nabla_{12} = q\nabla_{12},\\
&\nabla_{22}\nabla_{11} - \nabla_{11}\nabla_{22} = 0,\\
&\nabla_{22}\nabla_{12} - \nabla_{12}\nabla_{22} + (q^{-2} - 1)\nabla_{11}\nabla_{12} = q^{-1}\nabla_{12}.
\end{aligned}$$

The equivalent relation on $\Omega^1 \otimes \Omega^1$ are

$$\begin{aligned}
&\omega^{11} \wedge \omega^{11} = \omega^{12} \wedge \omega^{12} = \omega^{22} \wedge \omega^{22} = 0,\\
&\omega^{22} \wedge \omega^{12} = -\omega^{12} \wedge \omega^{22}, \quad \omega^{22} \wedge \omega^{11} = -\omega^{11} \wedge \omega^{22},\\
&\omega^{12} \wedge \omega^{11} = -q^{-2}\omega^{11} \wedge \omega^{12} - q^{-2}(q^{-2} - 1)\omega^{12} \wedge \omega^{22}.
\end{aligned}$$

The quantum Maurer-Cartan formulae are

$$\begin{aligned}
&\delta\omega^1 = 0, \qquad \delta\omega^4 = 0,\\
&\delta\omega^2 = q^{-1}\omega^1 \wedge \omega^2 + q^{-3}\omega^2 \wedge \omega^4.
\end{aligned}$$

In what follows we will discuss how to give differetial calculi on quantum co-set spaces. Assume H is a subgroup of quantum group G, and we have a bicovariant differential calculus on G. Since $Fun(G/H) \subset Fun(G)$, we can directly apply the derivative δ corresponding to $Fun(G)$ on $Fun_L(G/H)$ or $Fun_R(G/H)$ to obtain $\Omega^1_L(G/H)$ or $\Omega^1_R(G/H)$, $\Omega^2_L(G/H)$ or $\Omega^2_R(G/H)$ and so on.

Take G, H and G/H in Example 3a as an example. In the previous section we have obtained the differential calculus on G, and we have

$$Fun_L(G/H) = \mathbb{C}[ca^{-1}, 1].$$

Let $z = ca^{-1}$, It is sufficient to calculate δz and commutation relation between δz and z. Based on the differential calculus on G and by applying Proposition 2.1, we have

$$\begin{aligned}
\delta z &= \nabla_{ij} * z\omega^{ij}\\
&= (id \otimes \nabla_{ij})\Delta(ca^{-1})\omega^{ij}\\
&= (id \otimes \nabla_{ij})(ca^{-1} \otimes 1 + a^{-2} \otimes ca^{-1})\omega^{ij}\\
&= \tfrac{1}{q}a^{-2}\omega^{12},
\end{aligned}$$

by (2.8) we have

$$\delta z = \delta(ca^{-1}) = -qa^{-2}c\delta a + a^{-1}\delta c = a^{-1}(\delta c - ca^{-1}\delta a), \tag{2.9}$$

and

$$\begin{aligned}
\delta z \cdot z &= \tfrac{1}{q}a^{-2}\omega^{12}ca^{-1}\\
&= q^{-3}ca^{-1}a^{-2}\omega^{12}\\
&= q^{-2}z\delta z,
\end{aligned}$$

$$\delta z \wedge \delta z = q^{-2}a^{-4}\omega^{12} \wedge \omega^{12} = 0.$$

Applying above procedure to Example 3b and let $z^{-1} = ac^{-1}$, we have

$$\delta(z^{-1}) = c^{-1}(\delta a - ac^{-1}\delta c) = -q^2z^{-2}\delta z,$$

$$\delta(z^{-1})z^{-1} = q^2z^{-1}\delta(z^{-1}),$$

and

$$\delta(z^{-1}) \wedge \delta(z^{-1}) = 0.$$

The next question is about the relation between those two different calculi on $\mathbf{C}P^1_q$, i.e. how to introduce the connection to connect the two differential caculi, in other word to introduce the idea of gauge field. The answer will appear in a forthcoming paper.

References

1. N. Yu. Reshetikhin, L. A. Takhtadzhyan & L. D. Faddeev, Quantization of Lie groups and Lie algebras, *Leningrad Math. J.* Vol. 1, 1(1990), 193-224.
2. J. Wess, & B. Zumino, Covariant differential calculus on the hyperplanes, Preprint CERN-5697, April 1990.
3. S. L. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), *Commun. Math. Phys.* **122**(1989) 125-170.
4. B. Parshall & J.-P. Wang, Quantum linear groups, AMS Memoirs, Number 439, Published by the American Mathematical Society, 1991.
5. S. Majid Inter. J. of Mod. Phys. A, Vol. 5, No.1(1990)1-91.
6. K. Wu & R. J. Zhang, *Commun. Theor. Phys.* **17**(1992) 331.
7. X. D. Sun & S. K. Wang, Bicovariant differential calculus on the two-parameter quantum group $GL_{p,q}(2)$, to appear in *Journal of Math. Phys.*.
8. X. D. Sun & S. K. Wang, Preprint CCAST-92-04, ASIAM-92-07, ASITP-92-12.
9. X. D. Sun, S. K. Wang & K. Wu, Preprint CCAST-92-14, ASITP-92-18, ASIAM-92-11.