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Intrinsic Vertex Regularization and Renormalization in ϕ^4 Field Theory

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ABSTRACT

Based upon the intrinsic relation between the divergent lower point functions and the convergent higher point ones in the renormalizable quantum field theories, we propose a new method for regularization and renormalization in QFT. As an example, we renormalize the ϕ^4 theory at the one loop order by means of this method.

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In the last decades, various regularization and renormalization schemes have been developed in the quantum field theory [1]. However, the problem is still one of the most important issues under investigation in the modern QFT.

In this letter we propose a new method for regularization and renormalization in QFT and take the ϕ^4 field theory at one loop order as an example to show the spirit of the method. The main idea is based upon some simple observations in the renormalizable QFT. First, there are always, to a definite loop order, a limited number of lower point correlation functions which are divergent while the higher point functions are convergent and well defined if the number of points is bigger enough. Secondly, there exist certain intrinsic relations between the lower point functions and the higher point ones. Namely, the lower point functions can be reached as limiting cases of the number of the points, or the vertices, in the higher point convergent function which are well defined. This is in fact the crucial property we want to use in our method.

The main points of our method are as follows. In order to regularize a one particle irreducible (1PI) n point divergent function, we first calculate the $n+2q$ point functions $\Gamma^{(n+2q)}(p_1, \cdots, p_n, \underbrace{0, \cdots, 0}_{2q})$ where the momenta of $2q$ external lines are set to zero. When q is large enough, $\Gamma^{(n+2q)}$ is convergent and well defined. It can also be expressed in terms of $p_1, \cdots, p_n$ and q. Then we introduce $\Gamma^{(n)}(p_1, \cdots, p_n; q; \mu) \equiv \mu^{2q}\Gamma^{(n+2q)}(p_1, \cdots, p_n, \underbrace{0, \cdots, 0}_{2q})$ where μ is a constant with the dimension of mass in order that $\Gamma^{(n)}(p_1, \cdots, p_n; q; \mu)$ has the same dimension as the 1PI n point function. After calculating $\Gamma^{(n)}(p_1, \cdots, p_n; q; \mu)$, we make an analytic continuation of q from the integer to the complex number and it is called in our scheme the regularized n point function. The original 1PI n-point function is recovered when $q \to 0$:

$$\Gamma^{(n)}(p_1, \cdots, p_n; \mu) = \lim_{q\to 0} \Gamma^{(n)}(p_1, \cdots, p_n; q; \mu) \tag{1}$$

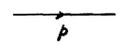
We call this kind of regularization in our scheme the intrinsic vertex regularization. The renormalization of the theory in this method is the same as in the usual approach. Namely, we subtract the divergent part of the vertices of the theory at each loop order by adding the relevant counterterms to the original action. The new action is the renormalized one. The renormalized n point functions are then calculated from the renormalized action. When $q \to 0$, we get a finite result for all the correlation functions.

Let us now concentrate on the divergent diagrams in ϕ^4 theory at the one loop order. We first show that in the ϕ^4 theory the divergent parts of such regularized n point functions at the one loop order behaves as q^{-1} and the convergent parts tend to finite quantities when $q \to 0$.

The action of the theory is

$$S[\phi] = \int d^4x \, (\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4) \tag{2}$$

where λ is the coupling constant. The Feynman rules for the ϕ^4 theory are well known:

Propagator:  $= i(p^2 - m^2)^{-1}$

Vertex: $= -i\lambda$

The superficial degree of divergence for a proper vertex with n external lines (1PI n-point correlation function) is

$$\delta = 4 - n \tag{3}$$

Hence the superficially divergent proper vertices at the one loop order are depicted as the diagrams (a), (b), (c) and (d) in Fig. 1.

In the momentum space, the amplitude of these vertices are

$$\begin{aligned}
(a) &= \tfrac{1}{2}\int \tfrac{d^4l}{(2\pi)^4}\ \tfrac{\lambda}{l^2-m^2} \\
(b) &= \tfrac{1}{2}\int \tfrac{d^4l}{(2\pi)^4}\ \tfrac{\lambda^2}{(l^2-m^2)((p_1+p_2+l)^2-m^2)} \\
(c) &= (b)(p_1+p_2 \to p_1+p_3), \qquad (d) = (b)(p_1+p_2 \to p_1+p_4)
\end{aligned} \tag{4}$$

It is easy to see from (4) that (a) is quadratically divergent and (b), (c) and (d) are logarithmically divergent. We can also see from (3) that the diagrams with $n > 4$ external lines at the one loop order are superficially convergent.

Regularization and evaluation of Feynman integrals:

In order to make the integral expressions for (a), (b), (c) and (d) to be meaningful, we have to regularize them. The regularization procedure is as follows:

First, we attach to the loop, say in the diag.(a), q vertices or $2q$ extra external lines with the momentum of each external line being zero in order that the diag. (a) turns to the diag.(a') (Fig.2) . Then we introduce a dimensional constant μ with the dimension of mass in order that the dimension of (a) equals to the dimension of $\mu^{2q}(a')$ and when $q = 0$, $\mu^{2q}(a') = (a)$.

The amplitude $\mu^{2q}(a')$ can then be expressed as

$$\mu^{2q}(a') = \mu^{2q}\frac{1}{2}\int \frac{d^4l}{(2\pi)^4}\ \frac{\lambda^{q+1}}{(l^2-m^2)^{q+1}} \tag{5}$$

For q large enough, $\mu^{2q}(a')$ is convergent and is well defined. Furthermore, the amplitude $\mu^{2q}(a')$ can be easily integrated and expressed in terms of the *Gamma* functions of q:

$$\mu^{2q}(a') = \mu^{2q}\frac{i}{2}\frac{\lambda^{q+1}}{(4\pi)^2}\frac{\Gamma(q-1)}{\Gamma(q+1)(-m^2)^{q-1}}. \tag{6}$$

This expression does also make sense when we take the continuation of q from the integer to the complex number. And when $q \to 0$, we have

$$\lim_{q\to 0}\mu^{2q}(a') = \frac{i}{2}\frac{\lambda m^2}{(4\pi)^2}\left[\frac{1}{q} + 1 + \ln\left(-\frac{\lambda\mu^2}{m^2}\right) + o(1)\right]. \tag{7}$$

where $o(1)$ represents those terms that are regular in q.

Obviously, this means that $\mu^{2q}(a')$ can be defined as the regularization of (a) and the procedure of the regularization is completed.

Similarly, we attach to one of the internal lines of diag. (b) q vertices or $2q$ external lines with zero momentum and the diag. (b) turns to the diag. (b') (Fig. 3). The dimension of $\mu^{2q}(b')$ is the same as that of (b) and when $q = 0$, $\mu^{2q}(b') = (b)$.

The amplitude of $\mu^{2q}(b')$ is

$$\mu^{2q}(b') = \frac{\mu^{2q}}{2}\int \frac{d^4l}{(2\pi)^4}\ \frac{\lambda^{2+q}}{(l^2-m^2)^{q+1}((p_1+p_2+l)^2-m^2)} \tag{8}$$

For q large enough, $\mu^{2q}(b')$ is convergent and is well defined.

The integration of $\mu^{2q}(b')$ can be performed by using the Feynman parametrization

$$\begin{aligned}
\mu^{2q}(b') &= \lambda^2\frac{(\mu^2\lambda)^q}{2}\int_0^1 d\alpha\ (q+1)\alpha^q \int \frac{d^4l}{(2\pi)^4}\ \frac{1}{(l^2-m^2+(1-\alpha)[(p_1+p_2)^2-2l\cdot(p_1+p_2)])^{q+2}} \\
&= i\lambda^2\frac{(\mu^2\lambda)^q}{2}\int_0^1 d\alpha\ (q+1)\alpha^q\frac{\Gamma(q)}{(4\pi)^2\Gamma(q+2)}\left(-m^2+\alpha(1-\alpha)(p_1+p_2)^2\right)^{-q} \\
&= \frac{i\lambda^2}{(4\pi)^2}\frac{(\mu^2\lambda)^q}{2q}\int_0^1 d\alpha\ \alpha^q\left(-m^2+\alpha(1-\alpha)(p_1+p_2)^2\right)^{-q}
\end{aligned} \tag{9}$$

From this expression, we make the analytic continuation of q from the integer to the complex number and we regard $\mu^{2q}(b')$ as the regularized form of (b). When $q \to 0$, we find

$$\begin{aligned}
&\lim_{q\to 0}\mu^{2q}(b') \\
&= \frac{i\lambda^2}{(4\pi)^2}\left(\frac{1}{2q} + \frac{7}{3} + \frac{1}{2}\ln\left(\frac{-\lambda\mu^2}{m^2}\right) - \frac{1}{2}\sqrt{1-\frac{(p_1+p_2)^2}{4m^2}}\ln\frac{\sqrt{1-\frac{(p_1+p_2)^2}{4m^2}}+1}{\sqrt{1-\frac{(p_1+p_2)^2}{4m^2}}-1} + o(1)\right).
\end{aligned} \tag{10}$$

We can also get the regularized expressions of the diag. (c) and the diag.(d). It is easy to see that they have the same behavior as the diag.(b) except that $p_1 + p_2$ substituted by $p_1 + p_3$ and $p_1 + p_4$ respectively.

Hence we get the divergent proper vertices at the one loop order:

$$\Gamma^{(2)}(p) = i(p^2 - m^2) + \frac{i}{2}\frac{\lambda m^2}{(4\pi)^2}\left[\frac{1}{q} + 1 + \ln\left(-\frac{\lambda\mu^2}{m^2}\right) + o(q)\right]$$

$$\begin{aligned}
&\Gamma^{(4)}(p_1,\ p_2,\ p_3,\ p_4) \\
&= -i\lambda + \frac{i\lambda^2}{(4\pi)^2}\left(\frac{3}{2q} + \frac{21}{6} + \frac{3}{2}\ln\left(\frac{-\lambda\mu^2}{m^2}\right) - A(p_1+p_2) - A(p_1+p_3) - A(p_1+p_4)\right) + o(q)
\end{aligned} \tag{11}$$

where

$$A(p_1+p_2)=\frac{1}{2}\sqrt{1-\frac{(p_1+p_2)^2}{4m^2}}\ln\frac{\sqrt{1-\frac{(p_1+p_2)^2}{4m^2}}+1}{\sqrt{1-\frac{(p_1+p_2)^2}{4m^2}}-1}.$$

We find that infiniteness of these divergent proper vertices resulted in poles in q. Furthermore, the finite part of these proper vertices are arbitrary depending in our scheme on the mass parameter μ.

Renormalization at the one loop order:

In order to eliminate these divergencies in q at the one loop order, we may add some counterterms to the original action. These counterterms, in general, can be written as

$$\frac{1}{2}(Z_\phi-1)\partial_\mu\phi\partial^\mu\phi+\frac{1}{2}\delta m^2\phi^2+\frac{1}{4!}\lambda(Z_\lambda-1)\phi^4 \tag{12}$$

where Z_ϕ, δm^2 and Z_λ are some constants depending on q, m, λ and μ.

Hence the renormalized action $S_r(\phi)$ is:

$$\begin{aligned}S_r[\phi]=\int d^4x\,\Big(&\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi+\tfrac{1}{2}m^2\phi^2+\tfrac{\lambda}{4!}\phi^4\\&+\tfrac{1}{2}(Z_\phi-1)\partial_\mu\phi\partial^\mu\phi+\tfrac{1}{2}\delta m^2\phi^2+\tfrac{1}{4!}\lambda(Z_\lambda-1)\phi^4\Big)\end{aligned} \tag{13}$$

If we choose Z_ϕ, δm^2 and Z_λ to be

$$\begin{aligned}Z_\phi&=1,\\ \delta m^2&=\tfrac{\lambda m^2}{2(4\pi)^2}\tfrac{1}{q}+o(1),\\ Z_\lambda&=1+\tfrac{3\lambda}{2(4\pi)^2}\tfrac{1}{q}+o(1),\end{aligned} \tag{14}$$

where $o(1)$ represents those terms that are regular in q and can be determined by the renormalization conditions, we find that all the correlation functions of ϕ^4 theory have a finite small q limit at the one loop order.

If we redefine

$$\phi_0=\sqrt{Z_\phi}\phi,\quad m_0^2=Z_\phi^{-1}(m^2+\delta m^2),\quad \lambda_0=Z_\phi^{-2}Z_\lambda\lambda,$$

the renormalized action (13) can be rewritten as

$$S_r(\phi_0)=\int d^4x\,\Big(\frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0+\frac{1}{2}m_0^2\phi_0^2+\frac{\lambda_0}{4!}\phi_0^4\Big), \tag{15}$$

where ϕ_0, m_0 and λ_0 are the so-called bare quantities. The physical quantities ϕ, m and λ can be expressed in terms of these bare ones[2].

The bare 1PI n point function $\Gamma_0^{(n)}(p_1,\cdots,p_n;\lambda_0,m_0,q)$, relate with the renormalized function $\Gamma_r^{(n)}(p_1,\cdots,p_n;\lambda,m,\mu,q)$ through

$$\Gamma_0^{(n)}(p_1,\cdots,p_n;\lambda_0,m_0,q)=Z_\phi^{-\frac{n}{2}}\Gamma_r^{(n)}(p_1,\cdots,p_n;\lambda,m,\mu,q) \tag{16}$$

We note that the left hand side of eq. (16) is independent of μ. Based upon this observation, it is easy to get the renormalization group equation:

$$[\mu\frac{\partial}{\partial\mu}+\beta(\lambda,\frac{m}{\mu},q)\frac{\partial}{\partial\lambda}+\gamma_m(\lambda,\frac{m}{\mu},q)\frac{\partial}{\partial m}-\frac{n}{2}\gamma_d(\lambda,\frac{m}{\mu},q)]\Gamma_r^{(n)}(p_1,\cdots,p_n;\lambda,m,\mu,q)=0 \tag{17}$$

where

$$\beta(\lambda,\frac{m}{\mu},q)=\mu\frac{\partial\lambda}{\partial\mu},\quad \gamma_m(\lambda,\frac{m}{\mu},q)=\frac{\mu}{2}\frac{\partial\ln m^2}{\partial\mu},\quad \gamma_d(\lambda,\frac{m}{\mu},q)=\frac{\mu}{2}\frac{\partial\ln Z_\phi}{\partial\mu}.$$

Further discussions on the renormalization group equation are the same as that in the dimensional regularization method.

The new method of regularization and renormalization proposed here is very simple and useful. It works not only for the ϕ^4 theory but also for the QED, chiral fermionic theory, QCD and other field theory models. The relevant results will be presented elsewhere [3].

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1 See, for example, C. Itzykson and J.-B. Zuber, "Quantum Field Theory", McGraw-Hill Inc. and references therein.

2 P. Ramond, "Field Theory: A Modern Primer", 2nd edition, Addison-Wesley Publishing Company, Inc.

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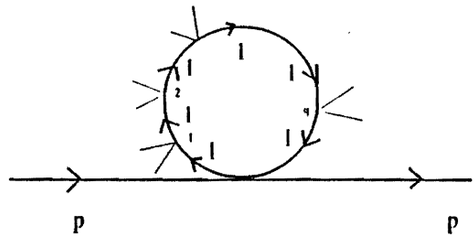
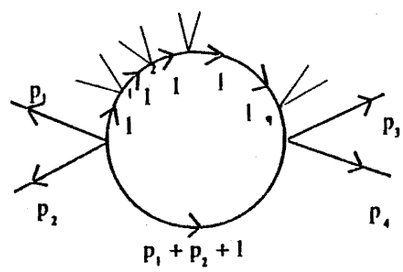


(a')

Fig. 2: The regularized diagram of (a).



(b')

Fig. 3: The regularized diagram of (b).

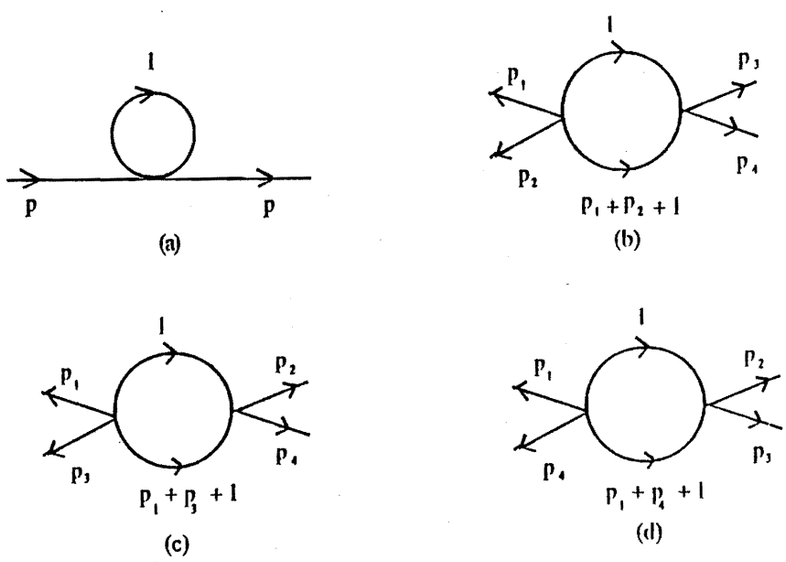


Fig. 1 : Divergent diagrams at the one loop order.