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On Representation Transformation in Einstein Quantum Cosmology [1]

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Abstract. We consider invertibility and consistency conditions in representation transformation in quantum cosmology, which is regarded as a necessary condition to pick up physical states of the Universe.

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2. Permanent address.

One of approaches to construct a quantum theory of gravity is canonical quantization of Einstein's theory[1]. Up to now a lot of work has been done on the application of canonical quantum gravity to cosmology[2-5]. The Universe is described by a solution to the Wheeler-DeWitt equation. Various Ansätze, so called the boundary condition of the Universe, have been proposed to pick up solutions which are physically meaningful. As we know, the discussions are all done in "coordinate " representation. Therefore, it is natural to ask: what happens in "momentum" representation?

In the present paper we are to discuss the problem of representation transformation. In our opinion physical results should be independent of choice of representation. It gives a necessary condition to pick up physical solutions.

The Hamiltonian constraint for the closed Robertson-Walker universe without cosmological constant is

$$\sigma^2(p^2+\frac{1}{4})=0 \tag{1}$$

where p is the ADM momentum conjugate to σ which is defined in the spacetime metric

$$ds^2=(48\pi^2)^{-1}(-N^2\sigma^3dt^2+\sigma d\Omega_3^2), \tag{2}$$

where $d\Omega_3^2$ is the line element of the 3-sphere with the sectional curvature being $\frac{1}{4}$. In σ-representation, the corresponding quantum constraint gives the Wheeler-DeWitt equation

$$\sigma^2\left(\frac{d^2}{d\sigma^2}-\frac{1}{4}\right)\Psi(\sigma)=0. \tag{3}$$

On the other hand, in p-representation the quantum constraint yields

$$\frac{d^2}{dp^2}(p^2+\frac{1}{4})C(p)=0. \tag{4}$$

Now we consider invertibility condition between representation transformations

$$C(p) = \int_{\sigma_-}^{\sigma_+} e^{-ip\sigma}\Psi(\sigma)d\sigma \tag{5}$$

and

$$\Psi(\sigma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ip\sigma}C(p)dp. \tag{6}$$

Putting (5) into (6), we have

$$\Psi(\sigma) = \frac{1}{2\pi}\int_{\sigma_-}^{\sigma_+} d\sigma'\psi(\sigma')\delta(\sigma-\sigma') = \Psi(\sigma). \tag{7}$$

On the other hand, using (6) in (5) one arrives at

$$C(p) = \int_{-\infty}^{+\infty} dp'C(p')\frac{1}{2\pi i(p'-p)}\left[e^{i(p'-p)\sigma}\right]_{\sigma=\sigma_-}^{\sigma=\sigma_+}. \tag{8}$$

(i) For $[\sigma_-,\sigma_+] = (-\infty,+\infty)$, Eq. (8) becomes

$$C(p) = \int_{-\infty}^{+\infty} dp'C(p')\delta(p'-p) = C(p); \tag{9}$$

(ii) For $[\sigma_-,\sigma_+] = [0,+\infty)$, (8) reduces to

$$C(p) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dp'\frac{C(p')}{p'-p+i\epsilon}; \tag{10}$$

(iii) For $[\sigma_-,\sigma_+] = (-\infty,0]$, we have from (8)

$$C(p) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dp'\frac{C(p')}{p'-p-i\epsilon} \tag{11}$$

It is noticed that invertibility conditions making Eq.(10) (or (11)) valid are the following: (a) $C(p)$ is an analytic function for Im $p < 0$(or Im $p > 0$); (b) for Im $p \leq 0$ (or Im $p \geq 0$), $C(p)$ goes to zero consistently as $|p| \to \infty$.

We now consider consistency conditions for a quantum cosmology. Substituting (5) into (4) and using (3), we obtain[6]

$$\left[\left[\sigma^2\frac{d\Psi(\sigma)}{d\sigma} + (i\sigma^2 p - 2\sigma)\Psi(\sigma)\right]e^{-ip\sigma}\right]_{\sigma=\sigma_-}^{\sigma=\sigma_+} = 0. \tag{12}$$

Conversely, setting (6) in (3) and using (4), one has

$$\left[\left[\left(p^2+\frac{1}{4}\right)\frac{dC(p)}{dp} + (2p - i\sigma p^2 - \frac{i}{4}\sigma)C(p)\right]e^{i\sigma p}\right]_{p=-\infty}^{p=+\infty} = 0 \tag{13}$$

Eqs. (12) and (13) are the consistency conditions for a quantum cosmology.

Next we apply the invertibility conditions and the consistency conditions for solutions to Eqs. (3) and (4). It is easy to find that the solutions to (3) and (4) are

$$\Psi(\sigma) = C_1 e^{-\frac{\sigma}{2}} + C_2 e^{\frac{\sigma}{2}} \tag{14}$$

and

$$C(p) = \frac{D_1}{p-\frac{i}{2}} + \frac{D_2}{p+\frac{i}{2}} \tag{15}$$

respectively, where C_1, C_2, D_1 and D_2 are constants.

In order to make the wave function $\Psi(\sigma)$ in (14) satisfying the consistency condition (12), we should have $[\sigma_-,\sigma_+] = [0,+\infty)$ or $(-\infty,0]$. However in physics we are interested in the region $[0,+\infty)$, and then the wave function satisfying (12) is

$$\Psi(\sigma) = C_1 e^{-\frac{\sigma}{2}} \tag{16}$$

for $[\sigma_-,\sigma_+] = [0,+\infty)$. On the other hand, the following wave function $C(p)$ obtained from (15) for $D_2 = 0$

$$C(p) = \frac{D_1}{p-\frac{i}{2}} \tag{17}$$

satisfies the invertibility conditions (a) and (b) . In short, (16) and (17) satisfy the consistency condition (12) and the representation transformations (5) and (6) for $[\sigma_-,\sigma_+] = [0,+\infty)$. Now we consider the condition (13). As we know, in quantum gravity we cannot introduce the inner product, because of absence of a field-type time variable. Thus we cannot define a

Hermitian operator and then eigenvalues of the momentum operator $\hat{p}$ are not necessarily requested to be real . The following path can be chosen:

$$p = |p|e^{i\epsilon} \quad \text{for} \quad p \to +\infty \tag{18}$$

and

$$p = |p|e^{i(\pi-\epsilon)} \quad \text{for} \quad p \to -\infty \tag{19}$$

where $\epsilon > 0$ is a small constant. Therefore (13) becomes

$$\lim_{\epsilon \to 0} \left\{ \left[(p^2 + \frac{1}{4})C'(p) - i\sigma(p^2 + \frac{1}{4})C(p) + 2pC(p) \right] e^{ip\sigma} \right\}_{|p|e^{i(\pi-\epsilon)}}^{|p|e^{i\epsilon}} = 0. \tag{20}$$

It is easy to see that the wave function $C(p)$ in Eq (17) satisfies the condition (20). In fact, substituting (17) into (6) we have

$$\Psi(\sigma) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{D_1 e^{ip\sigma}}{p - \frac{i}{2}} dp = iD_1 e^{-\frac{\sigma}{2}}. \tag{21}$$

(21) is just (16).

In short, it is pointed out what we discuss the wave function of the Universe only in σ-representation is not enough. We claim that a wave function of the Universe should satisfy the invertibility conditions of the representation transformations and the consistency conditions (12) and (13) for a quantum gravity.

Finally we point out that in a homogeneous and isotropic cosmological model in which the gravitational field minimally-coupled to a massless scalar field we can also pick up solutions satisfying the invertibility conditions and the consistency conditions[7,8].

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