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NEW $\check{R}(x)$ MATRICES VIA NEW TRIGONOMETRIC YANG-BAXTERIZATION

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New $\check{R}(x)$ Matrices via New Trigonometric Yang-Baxterization

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ABSTRACT

Several new $\check{R}(x)$ matrices as solutions of Yang-Baxter equations are found via a developed trigonometric Yang-Baxterization for three eigenvalues. The consistency conditions of the new trigonometric Yang-Baxterization are obtained explicitly.

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In recent years, much progress has been made in the study of solutions of Yang-Baxter equation[1,2], which includes a systematic method formulated by Jimbo[3] on the basis of quantum group, and Yang-Baxterization[4,5] approach firstly proposed by Jones[4] and latter developed by other authors[5]. These Yang-Baxterizaton approaches all produce Jimbo type solutions due to they obey Jimbo equation. In a recent systematic study of the author[6] on Yang-Baxterization, several new formulae beyond Jimbo-type case were derived. In present paper , we report new solutions of Yang-Baxter equation, which are obtained from some braid group representations in [7] via the approach developed in [6]. Meanwhile the consistency conditions of new Yang-Baxterization are presented explicitly.

Quantum Yang-Baxter equation with trigonometric parameterization is known as

$$\check{R}_1(x)\check{R}_2(xy)\check{R}_1(y) = \check{R}_2(y)\check{R}_1(xy)\check{R}_2(x) \tag{1}$$

where $\check{R}_1(x) = \check{R}(x) \otimes I$, $\check{R}_2(x) = I \otimes \check{R}(x)$. The S-matrix of a representation of braid group is a solution of Yang-Baxter relation.

$$S_1S_2S_1 = S_2S_1S_2 \tag{2}$$

which can be considered as a spectral parameter independent case of Yang-Baxter equation. Obviously Solutions of algebraic equation (2) is easy to find than that of functional equation (1). One natural question is whether some solutions can be find by means of solutions of (2). This is the idea of Baxterization originated in [4]

Given that the S-matrix of a representation of braid group has three distinct eigenvalues λ_i $(i = 1, 2, 3)$ and obeys a cubic reduction relation

$$(S - \lambda_1)(S - \lambda_2)(S - \lambda_3) = 0 \tag{3}$$

then we have

$$S = \sum_{i=1}^{3} \lambda_i P_i \tag{4a}$$

where P_i is the projector onto subspace of λ_i and is solved uniquely to be

$$P_i = \prod_{j \neq i} \frac{(S - \lambda_j)}{(\lambda_i - \lambda_j)} \tag{4b}$$

If we consider Abelian Yang-Baxterization, we can write the $\check{R}(x)$-matrix as

$$\check{R}(x) = \sum_{i=1}^{3} \Lambda_i(x) P_i, \tag{5}$$

where P_i's are the same projectors as in (4b) and $\lambda_i = \Lambda_i(0)$. It has be shown[6] that $\Lambda_i(x)$ are determined if $\check{R}(x)$ is assumed to obey initial condition

$$\check{R}(1) \propto I \tag{6}$$

A simple example beyond Jimbo-type case is

$$\Lambda_1(x) = (x + \frac{\lambda_1}{\lambda_2})(x^2 + \frac{\lambda_2}{\lambda_3})\lambda_3$$

$$\Lambda_2(x) = (\frac{\lambda_1}{\lambda_2}x + 1)(x^2 + \frac{\lambda_2}{\lambda_3})\lambda_3$$

$$\Lambda_3(x) = (\frac{\lambda_1}{\lambda_2}x + 1)(\frac{\lambda_2}{\lambda_3}x^2 + 1)\lambda_3 \tag{7}$$

Using (7), (3) and (4b), we obtain that

$$\check{R}(u) = L(u)S + M(u)I + N(u)S^{-1} \tag{8a}$$

where

$$L(u) = [x-1][bx-a]a^{-1},$$
$$M(u) = (\lambda_1+\lambda_2)(\lambda_2+\lambda_3)x[\lambda_1 - \lambda_3 x]a^{-1},$$
$$N(u) = x[x-1][a\lambda_3 x + b\lambda_2]\lambda_1 a^{-1},$$
$$a = (\lambda_1 - \lambda_3)\lambda_2, \quad b = (\lambda_1+\lambda_2)\lambda_3. \tag{8b}$$

Substituting (8a) into (1) and using Yang-Baxter relation (2) and cubic reduction relation, one obtain the following equation

$$\theta_3^+(S_1S_2^{-1}S_1 - S_2S_1^{-1}S_2) + \theta_3^-(S_1^{-1}S_2S_1^{-1} - S_2^{-1}S_1S_2^{-1})$$
$$+\theta_2^+(S_2^{-1}S_1 - S_2S_1^{-1}) + \theta_2^-(S_1^{-1}S_2 - S_1S_2^{-1})$$
$$+\theta_1^+(S_1 - S_2) + \theta_1^-(S_1^{-1} - S_2^{-1}) = 0 \tag{9a}$$

where

$$\theta_3^+ = L(u)N(u+v)L(v), \quad \theta_3^- = N(u)L(u+v)N(v),$$
$$\theta_2^+ = M(u)[N(u+v)L(v) - L(u+v)N(v)],$$
$$\theta_2^- = [N(u)L(u+v) - L(u)N(u+v)]M(v),$$
$$\theta_1^+ = [L(u)M(v) + M(u)L(v) + \sigma_1 L(u)L(v)$$

$$+\sigma_3 \; N(u)N(v)]M(u+v) - M(u)L(u+v)M(v),$$
$$\theta_1^- = [N(u)M(v) + M(u)N(v) + \sigma_3 L(u)L(v)$$
$$+\sigma_2\sigma_3^{-1}N(u)N(v)]M(u+v) - M(u)N(u+v)M(v),$$
$$\sigma_1 = \sum_i^3 \lambda_i, \qquad \sigma_2 = \sum_{i<j}\lambda_i\lambda_j, \qquad \sigma_3 = \lambda_1\lambda_2\lambda_3. \tag{9b}$$

We noticed that[6] Yang-Baxter equation (1) holds for the Yang-Baxterization formulae under consideration at least when $x=0$, $y=0$, $x=1$, $y=1$, and $x = y^{-1}$. This means that the θ's in (9b) must have $xy(x-1)(y-1)(xy-1)$ as a common factor.

Substituting (8b) into the first six equations of (9b), factorizing them and dropping out that common factor, we get seven consistency conditions

$$\varphi_3^{(i)}\Phi_3 + \tilde{\varphi}_3^{(i)}\Phi_3^{-1} + (\lambda_1+\lambda_2)(\lambda_2+\lambda_3)[\varphi_2^{(i)}\Phi_2 + \tilde{\varphi}_2^{(i)}\Phi_2^{-1} + \varphi_1^{(i)}\Phi_1 + \tilde{\varphi}_2^{(i)}\Phi_1^{-1}] = 0$$
$$i = 1, 2, \cdots 7. \tag{10}$$

where

$$\Phi_3 := S_1S_2^{-1}S_1 - S_2S_1^{-1}S_2$$
$$\Phi_2 := S_2^{-1}S_1 - S_2S_1^{-1}$$
$$\Phi_1 := S_1 - S_2 \tag{11}$$

and the seven sets of coefficients are

(i)

$$\begin{aligned} \varphi_3^{(1)} &= b^2, & \tilde{\varphi}_3^{(1)} &= \lambda_1\lambda_3 ab; \\ \varphi_2^{(1)} &= -\lambda_3 b, & \tilde{\varphi}_2^{(1)} &= \lambda_3 b, \\ \varphi_1^{(1)} &= -a\lambda_3/\lambda_2, & \tilde{\varphi}_1^{(1)} &= -\lambda_3(a\sigma_2/\lambda_2 + (\lambda_2+\lambda_3)b); \end{aligned} \tag{12a}$$

(ii)

$$\begin{aligned} \varphi_3^{(2)} &= -ab, & \tilde{\varphi}_3^{(2)} &= \lambda_1\lambda_2 b^2, \\ \varphi_2^{(2)} &= \lambda_3 a, & \tilde{\varphi}_2^{(2)} &= -\lambda_1 b, \\ \varphi_1^{(2)} &= -b, & \tilde{\varphi}_1^{(2)} &= \lambda_2\lambda_3 b; \end{aligned} \tag{12b}$$

(iii)

$$\begin{aligned} \varphi_3^{(3)} &= -ab, & \tilde{\varphi}_3^{(3)} &= \lambda_1\lambda_2 b^2, \\ \varphi_2^{(3)} &= \lambda_1 b, & \tilde{\varphi}_2^{(3)} &= -\lambda_3 a, \\ \varphi_1^{(3)} &= -b, & \tilde{\varphi}_1^{(3)} &= \lambda_2\lambda_3 b; \end{aligned} \tag{12c}$$

(iv)

$$\varphi_3^{(4)} = \lambda_3 a^3 + \lambda_2 b^3, \quad \tilde{\varphi}_3^{(4)} = \lambda_1\lambda_2^2 b^3 - \lambda_1\lambda_3^3 a^3,$$
$$\varphi_2^{(4)} = \lambda_3(\lambda_3 - \lambda_1)a^2, \quad \tilde{\varphi}_2^{(4)} = \lambda_3(\lambda_3 - \lambda_1)a^2,$$
$$\varphi_1^{(4)} = -(a^2(\lambda_1 - \lambda_3)\lambda_3 + \lambda_2(\lambda_2 + \lambda_3)b^2)/\lambda_2,$$
$$\tilde{\varphi}_1^{(4)} = -(a^3\lambda_3^2 + \lambda_1(\lambda_2 b - \lambda_3 a)b^2)/\lambda_3; \tag{12d}$$

(v)

$$\begin{aligned} \varphi_3^{(5)} &= -\lambda_2 b^2, & \tilde{\varphi}_3^{(5)} &= -\sigma_3 ab, \\ \varphi_2^{(5)} &= \lambda_2\lambda_3 b, & \tilde{\varphi}_2^{(5)} &= \lambda_1\lambda_3 a, \\ \varphi_1^{(5)} &= -\lambda_1 b, & \tilde{\varphi}_1^{(5)} &= -\sigma_3 b; \end{aligned} \tag{12e}$$

(vi)

$$\begin{aligned} \varphi_3^{(6)} &= -\lambda_2 b^2, & \tilde{\varphi}_3^{(6)} &= -\sigma_3 ab, \\ \varphi_2^{(6)} &= -\lambda_1\lambda_3 a, & \tilde{\varphi}_2^{(6)} &= -\lambda_2\lambda_3 b, \\ \varphi_1^{(6)} &= -\lambda_1 b, & \tilde{\varphi}_1^{(6)} &= -\sigma_3 b; \end{aligned} \tag{12f}$$

(vii)

$$\begin{aligned} \varphi_3^{(7)} &= \lambda_2 ab, & \tilde{\varphi}_3^{(7)} &= -\lambda_1\lambda_2^2 b^2, \\ \varphi_2^{(7)} &= -\lambda_1\lambda_2 b, & \tilde{\varphi}_2^{(7)} &= \lambda_1\lambda_2 b, \\ \varphi_1^{(7)} &= \lambda_3(\lambda_1(\lambda_1 + \lambda_2) + \lambda_2(\lambda_2 + \lambda_3)), & \tilde{\varphi}_1^{(7)} &= -\sigma_3 a. \end{aligned} \tag{12g}$$

Therefore if (10) holds for some permutations of eigenvalues of S matrix, then relating to the permutation, the $\check{R}$ matrix obtained from (8) is a solution of Yang-Baxter equation (1). Obviously when $\lambda_1 + \lambda_2 = 0$, the seven consistency conditions reduce to be one simple equation.

$$(S_1 S_2^{-1} S_1 - S_2 S_1^{-1} S_2) - \lambda_1\lambda_3(S_1^{-1} S_2 S_1^{-1} - S_2^{-1} S_1 S_2^{-1}) = 0 \tag{13}$$

In this case the Yang-Baxterization formulae (an overall factor $1 - x$ is omitted) becomes

$$\check{R}(x) = S + \lambda_1\lambda_3 x S^{-1} \tag{14}$$

which is formally the same as the Hacke case[4] (i.e. the case for braid group representation with two eigenvalues) but the S matrix is not a representation of Hacke algebra.

Now we use the obtained formula to baxterize some braid group representations e.g.

$$S = \begin{pmatrix} 1 & 0 & 0 & q^2/p \\ 0 & q & 1 & 0 \\ 0 & 1 & q & 0 \\ p & 0 & 0 & 1 \end{pmatrix} \tag{15}$$

It has three distinct eigenvalues $q + 1$, $q - 1$ and $1 - q$. If we denote them as λ_1, λ_2 and λ_3 respectively, We can find that, for permutations (i) $(\lambda_1, \lambda_2, \lambda_3)$, (ii) $(\lambda_1, \lambda_3, \lambda_2)$, (iii) $(\lambda_3, \lambda_1, \lambda_2)$, (iv) $(\lambda_3, \lambda_2, \lambda_1)$, the consistency conditions are satisfied. Then for case (i) we obtain a new 4×4 solutions of Yang-Baxter equation.

$$\check{R}(x) = \begin{pmatrix} x-1 & 0 & 0 & -(x+1)q^2/p \\ 0 & -(x+1)q & x-1 & 0 \\ 0 & x-1 & -(x+1)q & 0 \\ -(x+1)p & 0 & 0 & x-1 \end{pmatrix} \tag{16}$$

here and the following we omitted the overall factors in $\check{R}(x)$ matrix obtained from the Yang-Baxterization formula. For the other cases (ii), (iii) or (iv) the results are nothing new but what one obtained from (16) after replacing x by $-x$, $-x^2$, or x^2, respectively. If one introduces a new parameter u such that $x := \exp 2u$, then (16) becomes

$$\check{R}(u) = \begin{pmatrix} \mathrm{sh}u & 0 & 0 & -q^2\mathrm{ch}u/p \\ 0 & -q\mathrm{ch}u & \mathrm{sh}u & 0 \\ 0 & \mathrm{sh}u & -q\mathrm{ch}u & 0 \\ -p\mathrm{ch}u & 0 & 0 & \mathrm{sh}u \end{pmatrix} \tag{17}$$

This may provide an eight vertex model with trigonometric parameterization.

One more example, we consider the following braid group representation

$$S = \begin{pmatrix} 0 & 0 & 0 & q^2/p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

Its three eigenvalues are q, $-q$, and 1. It is easy to find that for the permutations (i) $(\lambda_1, \lambda_2, \lambda_3)$, (ii) $(\lambda_2, \lambda_1, \lambda_3)$, (iii) $(\lambda_3, \lambda_2, \lambda_1)$ and (iv) $(\lambda_2, \lambda_3, \lambda_1)$, the consistency conditions are satisfied. For the first case we obtain the following $\check{R}(x)$ matrix

$$\check{R}(x) = \begin{pmatrix} 0 & 0 & 0 & q^2(q^{-1}x^2-1)/p \\ 0 & qx^2-1 & 0 & 0 \\ 0 & 0 & qx^2-1 & 0 \\ p(q^{-1}x^2-1) & 0 & 0 & 0 \end{pmatrix} \tag{19}$$

which can be written as following form after introducing $x := \exp u$, $q := \exp 2\eta$, $p := \exp\zeta$,

$$\check{R}(u) = \begin{pmatrix} 0 & 0 & 0 & e^{3\eta-\zeta}\mathrm{sh}(u-\eta) \\ 0 & e^{\eta}\mathrm{sh}(u+\eta) & 0 & 0 \\ 0 & 0 & e^{\eta}\mathrm{sh}(u+\eta) & 0 \\ e^{\zeta-\eta}\mathrm{sh}(u-\eta) & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

Similarly for the case (ii), (iii) or (iv) it is just as (19) in which x^2 should be replaced respectively by $-x^2$, x, or $-x$.

Finally we consider a braid group representation without null entries

$$S = \begin{pmatrix} q_3 & q_1 & q_1 & q_1^2q_4 \\ q_2 & q_1q_2q_4 & q_3-2/q_4 & -q_1 \\ q_2 & q_3-2/q_4 & q_1q_2q_4 & -q_1 \\ q_2^2q_4 & -q_2 & -q_2 & q_3 \end{pmatrix} \tag{21}$$

It has three eigenvalues $q_1q_2q_4+q_3$, $-(q_1q_2q_4-q_3+2/q_4)$ and $q_1q_2q_4-q_3+2/q_4$. If we denote them as λ_1, λ_2 and λ_3 respectively, we can find that, for the permutations (i) $(\lambda_1, \lambda_2, \lambda_3)$, (ii) $(\lambda_1, \lambda_3, \lambda_2)$, (iii) $(\lambda_3, \lambda_1, \lambda_2)$ and (iv) $(\lambda_3, \lambda_2, \lambda_1)$, the consistency conditions hold. Then we obtain that

$$\check{R}(x) = \begin{pmatrix} (\frac{2}{q_4}-q_3)x-q_3 & (x-1)q_1 & (x-1)q_1 & (x-1)q_1^2q_4 \\ (x-1)q_2 & (x-1)q_1q_2q_4 & \frac{2}{q_4}-(x+1)q_3 & -(x-1)q_1 \\ (x-1)q_2 & \frac{2}{q_4}-(x+1)q_3 & (x-1)q_1q_2q_4 & -(x-1)q_1 \\ (x-1)q_2^2q_4 & -(x-1)q_2 & -(x-1)q_2 & (\frac{2}{q_4}-q_3)x-q_3 \end{pmatrix} \tag{22}$$

which corresponds to the first permutation, the other permutations (ii), (iii), and (iv) are similar to (22), which can be write out by turning x to be $-x$, $-x^2$, and x^2 respectively.

Acknowledgements

The authors would like to thank Profs H Y Guo, Z M Qiu and K Wu for their hospitality and interesting discussions. The work is supported in part by NSF of China, Zhang is also supported in part by NSF of Xinjiang Provence.

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