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QUANTUM ALGEBRA AS DEFORMATION SYMMETRY

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JUN 1 1992

ASITP-92-27
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Quantum Algebra as Deformation Symmetry

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Abstract

The deformation maps as well as the general algebraic maps among algebras with three generators are systematically investigated in terms of symplectic geometry and geometric quantization on 2-D manifolds. From which the explicit Hamiltonian of Heisenberg model with $SU_q(2)$ symmetry and arbitrary spin values are given. The deformation symmetries in differential dynamical systems and the q-deformed transformations of $SO(3)$ group in usual $\mathbf{R}^3$ are also discussed.

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I. Introduction

It is well known that the Lie algebra and Lie group play very profound roles in physics. The representations and transformation properties of Lie algebra and Lie group as well as their applications have been throughly investigated. For the case of " quantum algebra " and " quantum group " [1-4], there also have been some significant applications in exactly soluble models in statistical mechanics [5], conformal field theory [6], integrable model field theories [7] etc. together with a great deal of progress in understanding their algebraic structures and geometric properties. However, it has not been fully understood yet if there are more applications of them in physics, especially to the fundamental laws in physics.

Recently some efforts have been made in this direction. In terms of non-commutative geometry approach, it has been tried to " quantize " the usual physics theories such as gauge theories [8], Dirac equation [9], Lorentz group [10] and quantum mechanics [11] etc., though the true physical implications of such a " quantization " are not manifest currently. On the other hand, direct observations of usual physical system with " quantum group " $SU_q(2)$ symmetries are also made in atomic and molecular spectra [12]. The Hamiltonian system with $SU_q(2)$ symmetry are also constructed in the usual commutative differential manifolds, for examples, the q-rotator and q-top [13]. A system of obvious $SU_q(2)$ symmetry is the well-known XXZ Heisenberg spin-$\frac{1}{2}$ chain model with boundary terms [14][15]. Nevertheless, much attention should be paid in order to fully explore the roles played by the " quantum algebras " and " quantum groups " in physics.

We have shown that the " quantum algebra ", the q-deformation of Lie algebra, can be realized both at physically classical and quantum levels with manifest geometric meanings [16][17]. The key points of our observation are that: 1. the "quantum algebras" and " quantum groups" are in fact a kind of deformation symmetries to describe certain deformations of the well-understood systems with ordinary Lie symmetries both at classical and quantum levels. 2. Certain one-to-one correspondences between Lie algebra generators and their counterparts in corresponding "quantum algebra" can be given by a kind of maps, which are induced by quasi-conformal deformations, both at

classical and quantum levels as well. These imply that there should be a lot of applications to be explored for the "quantum algebras" and " quantum groups", which could be at least as many as that of the ordinary Lie symmetries.

In this paper we continue the research made in [16][17] to present a more systematic description for both Lie algebras and their q-deformed counterparts by means of symplectic geometry and geometric quantization, so as to show some mathematical structures of "quantum algebras" and " quantum groups" in the usual commutative manifolds and to explore some physical implications of them. For the sake of simplicity, we mainly concentrate on the Lie algebras with three generators, although the approach may be generalized to arbitrary Lie algebras by means of the symplectic geometry on the coadjoint orbits of Lie groups. We first systematically investigate these deformation maps as well as the general algebraic maps among algebras with three generators on 2-D manifolds. Then, based upon that the "quantum algebras" are a kind of deformation symmetries, we give the explicit Hamiltonian of Heisenberg model with $SU_q(2)$ symmetry and arbitrary spin values. We also study the deformation symmetries in differential dynamical systems and the properties of q-deformed transformations of $SO(3)$ group in usual $\mathbf{R}^3$.

The paper is arranged as follows: In section 2, we give relations between the symplectic geometry of smooth 2-D manifold and the algebras with three generators and their deformed counterparts. The classical (Poisson) algebraic maps among algebras and their q-deformed counterparts are analysed in detail. In section 3, we discuss the general maps among different algebras in Lie commutators, including the case between $SU(2)$ and $SU_q(2)$ algebras presented by Curtright and Zachos [18], by means of geometric quantization method. In section 4, by means of the deformation maps we show the general expression of Hamiltonians of the Heisenberg spin chain models with $SU_q(2)$ symmetries for arbitrary spins. The generalizations to higher dimensional cases are also studied. In section 5, we investigate the deformation symmetries in the solution space of radial Schrödinger equation of free particles. It is shown that this space is of $SU_q(2)$ algebraic symmetries in terms of the algebra maps between $SU(2)$ and $SU_q(2)$. Section 6 is dedicated to survey the q-deformed transformation of $SO(3)$ group in the usual $\mathbf{R}^3$

space. Some conclusion remarks are given in section 7.

II. Symplectic Geometry and Classical Deformation Maps

We first briefly recall some contents of symplectic geometry being used in this paper. The basic object is the symplectic manifold $(M,\ \omega)$ [19] which is a smooth manifold M with a symplectic two form ω on it. ω is closed and non-degenerate, i.e.

$$d\omega = 0, \qquad X = 0 \quad if \quad X\rfloor\omega = 0,$$

where X are vector fields on M and $\rfloor$ denotes the left inner product defined by $(X\rfloor\omega)(Y) = \omega(X,Y)$ for any two vector fields X and Y on M.

The canonical transformations are ω-preserving diffeomorphisms of M onto itself. And a vector filed X on M corresponds to an infinitesimal canonical transformation if and only if the Lie derivative of ω with respect to X vanishes,

$$\mathcal{L}_X\omega = X\rfloor d\omega + d(X\rfloor\omega) = 0\ . \tag{1}$$

A vector field X satisfying (1) is said to be a Hamiltonian vector field. It can be proved that for a smooth function f on M there exits a unique Hamiltonian vector field X_f satisfying

$$X_f\rfloor\omega = -df\ . \tag{2}$$

The Poisson brackets of two smooth functions f and g is defined to be the function $-\omega(X_f, X_g)$ with relations

$$[f,g]_{P.B.} = -\omega(X_f,X_g) = \omega(X_g,X_f) = -X_f g = X_g f\ . \tag{3}$$

In the symplectic approach we find that the algebras with three generators are tightly related to 2-D manifolds [17]. For example, the Poisson algebra for $SU(2)$, also for $SO(3)$, is

$$[S_i,S_j]_{P.B.} = \sum_{k=1}^{3}\epsilon_{ijk}S_k \qquad i,j = 1,2,3. \tag{4}$$

From (3) we may easily write that

$$X_{S_i} = \sum_{jk}^{3}\epsilon_{ijk}S_j\frac{\partial}{\partial S_k}. \tag{5}$$

By writing down a general symplectic two form and using formula (2) we immediately have

$$S_1 dS_1 + S_2 dS_2 + S_3 dS_3 = 0.$$

Hence

$$S_1^2 + S_2^2 + S_3^2 = S_0^2, \tag{6}$$

where S_0 is an integral constant. Therefore the three generators of $SU(2)$ constitute a 2-D manifold, the sphere S^2 with radius S_0 in $\mathbf{R}^3$. In fact, it is well known that the sphere S^2 is a coadjoint orbit of both $SU(2)$ and $SO(3)$. Some of other examples presented in the following are also of this property. But this topic is beyond the scope of this paper.

Similarly for the algebra $SU_q(2)$, and also for $SO_q(3)$,

$$[S_1', S_2']_{P.B.} = \tfrac{1}{2}[2S_3']_q = \frac{\sinh 2\gamma S_3'}{2\sinh\gamma}, \quad [S_2', S_3']_{P.B.} = S_1', \quad [S_3', S_1']_{P.B.} = S_2', \tag{7}$$

where $[x]_q = \dfrac{q^x - q^{-x}}{q - q^{-1}}$ and $q = e^\gamma$ is the deformation parameter, we find that the three elements S_i' of $SU_q(2)$ constitute a 2-D q-deformed sphere S_q^2,

$$S_1'^2 + S_2'^2 + \frac{(\sinh\gamma S_3')^2}{\gamma\sinh\gamma} = S_0'^2, \tag{8}$$

where S_0' is a constant. In fact from the prequantization condition we have [17] $S_0'^2 = \dfrac{(\sinh\gamma S_0)^2}{\gamma\sinh\gamma}$.

On the other hand we also find that a smooth 2-D manifold uniquely corresponds to a certain algebra [17]. For a general manifold M defined by

$$f(S_1, S_2, S_3) = 0, \tag{9}$$

the corresponding Hamiltonian vector fields of S_i's are

$$X_{S_i} = \sum_{ijk}^{3} \alpha\epsilon_{ijk}\frac{\partial f}{\partial S_j}\frac{\partial}{\partial S_k}, \tag{10}$$

where α is a constant. The symplectic form can also be obtained and the Poisson algebraic relations of S_i's are uniquely given by manifold (9) up to algebraic isomorphisms as α is a constant,

$$[S_i, S_j]_{P.B.} = \alpha\sum_{k}^{3}\epsilon_{ijk}\frac{\partial f}{\partial S_k}. \tag{11}$$

For 2-D sphere S^2 defined by (6), we get the Poisson $SU(2)$ algebra (4) by taking α to be $\frac{1}{2}$, Hamiltonian vector fields (5) and the symplectic form

$$\omega = \frac{-1}{2S_0^2}\sum_{ijk}^{3}\epsilon_{ijk}S_i dS_j \wedge dS_k. \tag{12}$$

For the q-deformed manifold S_q^2 defined by (8), we get the Poisson algebra relations (7) of $SU_q(2)$ and symplectic form

$$\omega' = -\frac{1}{S_0'^2}(S_1' dS_2' \wedge dS_3' + S_2' dS_3' \wedge dS_1' + \frac{\tanh\gamma S_3'}{\gamma} dS_1' \wedge dS_2'). \tag{13}$$

More examples can be readily given. For the case of one sheet hyperboloid

$$S_1^2 + S_2^2 - S_3^2 = constant, \tag{14}$$

from formula (11), taking the constant α to be $\frac{1}{2}$ we immediately have

$$[S_1, S_2]_{P.B.} = -S_3, \quad [S_2, S_3]_{P.B.} = S_1, \quad [S_3, S_1]_{P.B.} = S_2, \tag{15}$$

which is the well known algebra $SU(1,1)$. The only difference between the related manifolds of $SU(2)$ and $SU(1,1)$, defined by equations (6) and (14) respectively, is the sign of the term with S_3. The manifold related to the algebra $SU_q(1,1)$ is the q-deformed one sheet hyperboloid, which is different to the manifold (8) of $SU_q(2)$ by a sign of the third term on the left hand.

For the the case of elliptic paraboloid

$$H_1^2 + H_2^2 - H_3 = \frac{1}{2}, \tag{16}$$

we have

$$[H_1, H_2]_{P.B.} = -\frac{1}{2}, \quad [H_2, H_3]_{P.B.} = H_1, \quad [H_3, H_1]_{P.B.} = H_2. \tag{17}$$

This is just the simple harmonic oscillator algebra $\mathcal{H}(4)$ [20]. It is easy to find that the following q-deformed elliptic paraboloid

$$H_1'^2 + H_2'^2 - \frac{\sinh(2\gamma H_3')}{2\gamma\cosh\gamma} = \frac{\sinh\gamma}{2\gamma\cosh(\gamma)} \tag{18}$$

gives rise to the q-deformed simple harmonic oscillator algebra $\mathcal{H}_q(4)$ [21],

$$[H_1', H_2']_{P.B.} = -\frac{\cosh(2\gamma H_3')}{2\cosh\gamma}, \quad [H_2', H_3']_{P.B.} = H_1', \quad [H_3', H_1']_{P.B.} = H_2' . \tag{19}$$

Another example we would like to mention here is the hidden symmetry in Kepler problem on two dimensional sphere. This symmetry is depicted by the algebra proposed by Higgs [22],

$$[R_1, R_2]_{P.B.} = \frac{1}{4}\left((\lambda - 8E)L + 8\lambda L^3\right) , \quad [R_2, L]_{P.B.} = R_1 , \quad [L, R_1]_{P.B.} = R_2 , \tag{20}$$

where λ is the curvature of the sphere, E is the energy eigenvalue, R_1 and R_2 are the Runge-Lenz vectors, L is the angular momentum of the third direction.

From formula (11) it is easy to find that the 2-D manifold with respect to algebra (20) is

$$R_1^2 + R_2^2 + \frac{1}{4}(\lambda - 8E)L^2 + \lambda L^4 = C_H , \tag{21}$$

where C_H is a constant. C_H is related to S_0 in terms of the prequantization condition,

$$C_H = \frac{1}{4}(\lambda - 8E)S_0^2 + \lambda S_0^4 . \tag{22}$$

It should be noted that the Higgs algebra (20) is in fact isomorphic to the second-approximation of $SU_q(2)$ algebra (7). The invariant S_0' in (8) becomes C_H in (22) when it is expanded to the second order of deformation parameter γ. And the manifold (21) is a second order expansion of S_q^2.

In this way the Poisson algebraic structures for all the smooth 2-D manifolds can be obtained though the algebra may be not familiar to us. For example, the manifold torus

$$(\sqrt{S_1^2 + S_2^2} - a)^2 + S_3^2 = r^2 , \tag{23}$$

where a and r are constants, gives rise to a Poisson algebra

$$[S_1, S_2]_{P.B.} = S_3, \quad [S_2, S_3]_{P.B.} = S_1 - \frac{aS_1}{\sqrt{S_1^2 + S_2^2}}, \quad [S_3, S_1]_{P.B.} = S_2 - \frac{aS_2}{\sqrt{S_1^2 + S_2^2}}. \tag{24}$$

Here what we want to discuss are the maps among different algebras in terms of their related manifolds. As a significant example we consider the algebra $SU(2)$ ($SO(3)$)

and its deformed counterpart $SU_q(2)$ ($SO_q(3)$). The corresponding manifolds are S^2 and S_q^2 defined by equations (6) and (8) respectively and related by a quasi-conformal transformation when q is real [17]. From the discussions above it is clear that, in terms of symplectic geometry, the algebra and manifold have one to one correspondence up to algebraic isomorphisms. Let S_i, i=1,2,3, be the generators of $SU(2)$ and hence satisfy the equation of S^2. Let $S_i' = S_i'(S_1, S_2, S_3)$, i=1,2,3, be functions of S^2. Then we have the following conclusion:

The new " observables " $S_i' = S_i'(S_1, S_2, S_3)$ constitute the $SU_q(2)$ algebra in Poisson brackets if and only if S_i' satisfy the eq. (8) of S_q^2.

The proof is straightforward. By using eq. (6) we rewrite the symplectic from (12) on S^2 as

$$\omega = -\frac{1}{S_3} dS_1 \wedge dS_2 . \tag{25}$$

The Hamiltonian vector fields of S_i' with respect to (25) are simply

$$X_{S_i'} = S_3\left(\frac{\partial S_i'}{\partial S_2}\frac{\partial}{\partial S_1} - \frac{\partial S_i'}{\partial S_1}\frac{\partial}{\partial S_2}\right) . \tag{26}$$

From formula (3) and that S_i' constitute the $SU_q(2)$ algebra (7) we have

$$\begin{aligned} -S_3\left(\frac{\partial S_1'}{\partial S_2}\frac{\partial S_2'}{\partial S_1} - \frac{\partial S_1'}{\partial S_1}\frac{\partial S_2'}{\partial S_2}\right) &= \frac{\sinh(2\gamma S_3')}{2\sinh\gamma} , \\ -S_3\left(\frac{\partial S_2'}{\partial S_2}\frac{\partial S_3'}{\partial S_1} - \frac{\partial S_2'}{\partial S_1}\frac{\partial S_3'}{\partial S_2}\right) &= S_1' , \\ -S_3\left(\frac{\partial S_3'}{\partial S_2}\frac{\partial S_1'}{\partial S_1} - \frac{\partial S_3'}{\partial S_1}\frac{\partial S_1'}{\partial S_2}\right) &= S_2' , \end{aligned} \tag{27}$$

which give rise to

$$S_1'\frac{\partial S_1'}{\partial S_i} + S_2'\frac{\partial S_2'}{\partial S_i} + \frac{\sinh(2\gamma S_3')}{2\sinh\gamma}\frac{\partial S_3'}{\partial S_i} = 0 , \quad i = 1, 2, 3,$$

i.e.

$$\frac{\partial}{\partial S_i}\left(S_1'^2 + S_2'^2 + \frac{(\sinh\gamma S_3')^2}{\gamma\sinh\gamma}\right) = 0 .$$

Hence S_i' satisfy the eq. (8) of the deformed sphere S_q^2. The inverse conclusion is obvious from formula (11) and discussions above. And similarly if S_i' are elements of $SU_q(2)$, we can prove that the functions $S_i = S_i(S_1', S_2', S_3')$ on S_q^2 constitute the $SU(2)$ algebra if and only if S_i satisfy the eq. (6) of S^2.

In fact, for arbitrary two algebras with generators (S_1, S_2, S_3) and (S'_1, S'_2, S'_3) satisfying equations $f(S_1, S_2, S_3) = 0$ and $g(S'_1, S'_2, S'_3) = 0$ respectively, there may exist many kinds of maps between generators S_i and S'_i. S_i and S'_i can be expressed mutually. Namely, one algebra can be realized through another one. The only requirement is that both the constraint equations be satisfied. In this way the algebraic maps between $SU(2)$ $(SO(3))$ and $SU_q(2)$ $(SO_q(3))$, $\mathcal{H}(4)$ and $\mathcal{H}_q(4)$ as well as other algebra pairs such as $SU(2)$ and $\mathcal{H}(4)$, $SU_q(2)$ and $\mathcal{H}_q(4)$ can be investigated similarly. Here is a kind of algebraic maps between $SU(2)$ $(SO(3))$ and $SU_q(2)$ $(SO_q(3))$,

$$S'_{\pm} = \frac{1}{\sqrt{\gamma \sinh \gamma}} S_{\pm} \frac{\sinh \gamma (S_0 \mp S_3)}{S_0 \mp S_3} , \quad S'_3 = S_3 , \tag{28}$$

where $S'_{\pm} = S'_1 \pm i S'_2$ and $S_{\pm} = S_1 \pm i S_2$. The maps (28) are so selected that, in terms of (6) and (8), when γ approaches zero S'_i becomes S_i apparently. As $S_{\pm,3}$ and $S'_{\pm,3}$ can be expressed by the complex coordinates on S^2 and S^2_q respectively, while the complex coordinate transformations between S^2 and S^2_q are quasi-conformal ones when q is real, hence hidden in all deformation maps between $SU(2)$ $(SO(3))$ and $SU_q(2)$ $(SO_q(3))$ there are quasi-conformal transformations for real q.

The classical maps between $\mathcal{H}(4)$ and $\mathcal{H}_q(4)$ algebras can also be obtained from the related manifolds (16) and (18). For example,

$$H'_+ = \frac{\sinh \gamma (H_3 + \frac{1}{2})}{(H_3 + \frac{1}{2})\gamma} H_+ , \quad H'_- = \frac{\cosh \gamma (H_3 - \frac{1}{2})}{\cosh \gamma} H_- , \quad H'_3 = H_3 , \tag{29}$$

where $H_{\pm} = H_1 \pm i H_2$. And from the manifolds (6) and (21) we have the algebraic maps between $SU(2)$ and the Poisson Higgs algebra (20),

$$\begin{aligned} R_+ &= S_+ \left(\lambda S_3 - \sqrt{2E - \lambda S_0^2 - \frac{\lambda}{4}} \right) , \\ R_- &= S_- \left(\lambda S_3 + \sqrt{2E - \lambda S_0^2 - \frac{\lambda}{4}} \right) , \quad L = S_3 , \end{aligned} \tag{30}$$

where $R_{\pm} = R_1 \pm i R_2$.

Similarly, the deformation maps between $SU(1,1)$ and $SU_q(1,1)$ can also be readily given. And although we only proved that $S'_i = S'_i(S_1, S_2, S_3)$ constitute the $SU_q(2)$ algebra in Poisson brackets if and only if S'_i satisfy the equation of S^2_q for the algebraic

maps between $SU(2)$ and $SU_q(2)$, the similar conclusions can be obtained for all other algebraic maps above.

III. Geometric Quantization and Operator Deformation Maps

The Poisson algebra can be realized in Lie commutators from prequantization and geometric quantization method [19]. The prequantization line bundle and quantum one exist if and only if the symplectic form, up to a dimensional constant factor, defines an integral de Rham cohomology class. For the cases of $SU(2)$ and $SU_q(2)$, this prequantization condition requires that $S_0 = j\beta$ and $S'_0 = \frac{\sinh(\gamma j \beta)}{\sqrt{\gamma \sinh \gamma}}$ respectively [17]. Where β is a dimensional constant and j take values of integers and half integers. The discreteness of S_0 and S'_0 values are due to the topology of the manifolds (6) and (8). Only when β is taken to be the Planck constant $\hbar$, it stands for physically quantization. The prequantization operators of the $SU(2)$ and $SU_q(2)$ elements can be expressed in terms of the complex coordinates and their derivatives on (6) and (8) respectively [17]. When polarization is introduced, these prequantization operators become quantum ones.

For equations (6) and (8) we have their operator versions,

$$\hat{S}_+ \hat{S}_- + \hat{S}_3^2 = j(j+1) \tag{31}$$

and

$$\hat{S}'_+ \hat{S}'_- + \frac{\sinh \gamma \hat{S}'_3 \sinh \gamma (\hat{S}'_3 - 1)}{\gamma \sinh \gamma} = \frac{\sinh(\gamma j) \sinh(\gamma (j+1))}{\gamma \sinh \gamma} , \tag{32'}$$

where $\hat{S}_{\pm} = \hat{S}_1 \pm i \hat{S}_2$, $\hat{S}'_{\pm} = \hat{S}'_1 \pm i \hat{S}'_2$, $\hat{S}_i$ and $\hat{S}'_i$ are operator representations of S_i and S'_i respectively, and for simplicity the constant β has taken to be unit.

The operators with respect to eq. (32′) give rise to an algebra isomorphic to $SU_q(2)$. In order to compare with the usual results we may redefine $\hat{S}'_+$ and $\hat{S}'_-$ by multiplying a factor $\sqrt{\frac{\sinh \gamma}{\gamma}}$. Then eq. (32′) becomes

$$\hat{S}'_+ \hat{S}'_- + [\hat{S}'_3]_q [\hat{S}'_3 - 1]_q = [j]_q [j+1]_q , \tag{32}$$

which corresponds to the $SU_q(2)$ algebraic relations

$$[\hat{S}'_+, \hat{S}'_-] = [2\hat{S}'_3]_q , \quad [\hat{S}'_3, \hat{S}'_{\pm}] = \pm \hat{S}'_{\pm} \tag{33}$$

with co-product operations

$$\begin{aligned}\Delta(\hat{S}'_3) &= \hat{S}'_3 \otimes 1 + 1 \otimes \hat{S}'_3, \\ \Delta(\hat{S}'_\pm) &= q^{\hat{S}'_3} \otimes \hat{S}'_\pm + \hat{S}'_\pm \otimes q^{-\hat{S}'_3} .\end{aligned} \tag{34}$$

The equations (31) and (32) are the necessary conditions for the operators of $SU(2)$ and $SU_q(2)$ respectively. One example is the harmonic oscillator representations of $SU(2)$ and $SU_q(2)$. The direct calculations show that the operators of $SU(2)$ and $SU_q(2)$ in oscillator representations satisfy the eqs. (31) and (32) respectively [17].

The eqs. (31) and (32) are hence the guiding equations for the operator maps between $SU(2)$ and $SU_q(2)$. For example, an operator map related to (28) found by Curtright and Zachos [18] is

$$\begin{aligned}\hat{S}'_+ &= \hat{S}_+ \frac{[j-\hat{S}_3]_q}{j-\hat{S}_3} = \frac{1}{\sinh\gamma}\hat{S}_+ \frac{\sinh\gamma(j-\hat{S}_3)}{j-\hat{S}_3} , \\ \hat{S}'_- &= \hat{S}_- \frac{[j+\hat{S}_3]_q}{j+\hat{S}_3}, \quad \hat{S}'_3 = \hat{S}_3 ,\end{aligned} \tag{35}$$

where the terms with operators in denominator do not exist in fact because of the sinh function in numerator. Hence operators $\hat{S}'_{\pm,3}$ in eq. (35) are well defined. In fact j may be an operator and $S'_\pm$, S'_3 still constitute an $SU_q(2)$ algebra as long as j is an invariant of $SU(2)$.

Similarly, by prequantizing and quantizing the corresponding manifolds, the guiding equations of maps among $SU(1,1)$ and $SU_q(1,1)$, $\mathcal{H}(4)$ and $\mathcal{H}_q(4)$ can also be obtained. Here we discuss the maps between $\mathcal{H}(4)$ and $\mathcal{H}_q(4)$ algebras. The operator version of equations (16) and (18) are

$$\hat{H}_+\hat{H}_- - \frac{1}{2}(2\hat{H}_3 - 1) = 1 \tag{36}$$

and

$$\hat{H}'_+\hat{H}'_- - \frac{1}{2\cosh\gamma}[2\hat{H}'_3 - 1]_q = 1 \tag{37}$$

respectively. Where $\hat{H}_\pm = \hat{H}_1 \pm i\hat{H}_2$, $\hat{H}'_\pm = \hat{H}'_1 + i\hat{H}'_2$ and $\hat{H}'_\pm$ have been redefined by multiplying a constant factor $\frac{\gamma}{\sinh\gamma}$. $\hat{H}_{\pm,3}$ and $\hat{H}'_{\pm,3}$ constitute the $\mathcal{H}(4)$ and $\mathcal{H}_q(4)$

algebras in Lie brackets

$$[\hat{H}_+, \hat{H}_-] = -1 , \quad [\hat{H}_3, \hat{H}_\pm] = \pm\hat{H}_\pm . \tag{38}$$

$$[\hat{H}'_+, \hat{H}'_-] = -\frac{\cosh(2\gamma\hat{H}'_3)}{\cosh\gamma} , \quad [\hat{H}'_3, \hat{H}'_\pm] = \pm\hat{H}'_\pm . \tag{39}$$

Let $\hat{H}'_3 = \hat{H}_3$. From (37) we have

$$\hat{H}'_+\hat{H}'_- = \frac{2\sinh\gamma(\hat{H}_3 + \frac{1}{2})\cosh\gamma(\hat{H}_3 - \frac{3}{2})}{\sinh 2\gamma}. \tag{40}$$

From (36) that $\hat{H}_+\hat{H}_- = \hat{H}_3 + \frac{1}{2}$, we can rewrite eq. (40) as

$$\hat{H}'_+\hat{H}'_- = \frac{\sinh\gamma(\hat{H}_3 + \frac{1}{2})}{\sinh\gamma\cdot(\hat{H}_3 + \frac{1}{2})}\hat{H}_+\hat{H}_- \frac{\cosh\gamma(\hat{H}_3 - \frac{3}{2})}{\cosh\gamma}. \tag{41}$$

Therefore we get a kind of maps between $\mathcal{H}(4)$ and $\mathcal{H}_q(4)$ algebras,

$$\hat{H}'_+ = \frac{\sinh\gamma(\hat{H}_3 + \frac{1}{2})}{(\hat{H}_3 + \frac{1}{2})\sinh\gamma}\hat{H}_+ , \quad \hat{H}'_- = \frac{\cosh\gamma(\hat{H}_3 - \frac{1}{2})}{\cosh\gamma}\hat{H}_- , \quad \hat{H}'_3 = \hat{H}_3 , \tag{42}$$

which are related to the classical deformation map (29). It is easy to check that $\hat{H}'_{\pm,3}$ satisfy relations (39) in terms of (38). And when q approaches one $\hat{H}'_\pm$ become $\hat{H}_\pm$.

Similarly, in terms of the symplectic form on (21) and the prequantization condition, we find the invariant C_H related to Higgs algebra (20) takes discrete values,

$$C_H = \frac{1}{4}(\lambda - 8E)j^2 + \lambda j^4 ,$$

where j is an integer or half integer. This is why we say that classically C_H has the form of (22). The manifold (21) in operator version becomes

$$\hat{R}_+\hat{R}_- + \frac{1}{4}(\lambda - 8E)\hat{L}(\hat{L}-1) + \lambda\hat{L}^2(\hat{L}-1)^2 = \frac{1}{4}(\lambda - 8E)j(j+1) + \lambda j^2(j+1)^2 \tag{43}$$

with the Higgs algebra in Lie commutators,

$$\begin{aligned}[\hat{L}, \hat{R}_\pm] &= \pm\hat{R}_\pm , \\ [\hat{R}_+, \hat{R}_-] &= \frac{1}{2}[(\lambda - 8E)\hat{L} + 8\lambda\hat{L}^3] ,\end{aligned} \tag{44}$$

where $\hat{R}_\pm = \hat{R}_1 \pm i\hat{R}_2$. The multiplication of this algebra is obviously non-commutative. However it is not a " quantum algebra " unless non-commutative co-multiplication and antipode can be constructed in its universal enveloping algebra.

Equation (43) is the guiding equation for algebraic maps. For instance, maps between Higgs algebra (20) and $SU(2)$ algebra can be obtained from equations (31) and (43):

$$\hat{R}_+ = \hat{S}_+ \left[\sqrt{\lambda}\hat{S}_3 - \left(\sqrt{2E - \lambda j(j+1)} - \frac{\sqrt{\lambda}}{2}\right)\right] ,$$
$$\hat{R}_- = \hat{S}_- \left[\sqrt{\lambda}\hat{S}_3 + \left(\sqrt{2E - \lambda j(j+1)} - \frac{\sqrt{\lambda}}{2}\right)\right] , \quad \hat{L} = \hat{S}_3 .$$

The $\hat{R}_{\pm,3}$ in above map satisfy the equation (43) in terms of the relation (31).

IV. Heisenberg Spin Model with $SU_q(2)$ Symmetry

In this section we discuss the Heisenberg spin model by using the maps (35). The spin-$\frac{1}{2}$ Heisenberg XXZ chain model was studied in terms of the Bethe ansatz in 1966 [23]. It is found that this model with a boundary term is of $SU_q(2)$ symmetry [15]. Accounting to this symmetry this model can be analytically solved [24][26]. The spin-1 Heisenberg XXZ chain model with $SU_q(2)$ symmetry was also constructed [27]. And in principle the $SU_q(2)$ invariant Hamiltonian of this model for higher spin cases can be obtained from fusion rules. Nevertheless it is not easy to get the general expressions. Here we are going to give an $SU_q(2)$ invariant Hamiltonian for arbitrary spin values.

Let I_q denote the left hand side of eq. (32),

$$\mathrm{I}_q = \hat{S}'_+\hat{S}'_- + [\hat{S}'_3]_q[\hat{S}'_3 - 1]_q .$$

From eq. (32) we know that I_q is an invariant under actions of $SU_q(2)$. Hence the co-product,

$$\begin{aligned} \mathrm{I}_q^2 &= \Delta(C - \mathrm{I}_q) = C\mathbf{1}\otimes\mathbf{1} \\ &- \Big(q^{2\hat{S}'_3}\otimes\hat{S}'_+\hat{S}'_- + q^{\hat{S}'_3}\hat{S}'_-\otimes\hat{S}'_+q^{-\hat{S}'_3} + \hat{S}'_+q^{\hat{S}'_3}\otimes q^{-\hat{S}'_3}\hat{S}'_- + \hat{S}'_+\hat{S}'_-\otimes q^{-2\hat{S}'_3} \\ &+ \frac{1}{(q-q^{-1})^2}\left(q^{-1}q^{2\hat{S}'_3}\otimes q^{2\hat{S}'_3} + qq^{-2\hat{S}'_3}\otimes q^{-2\hat{S}'_3}\right) - \frac{q+q^{-1}}{(q-q^{-1})^2}\mathbf{1}\otimes\mathbf{1}\Big) \end{aligned} \tag{45}$$

also commutes with the co-products (34) of $SU_q(2)$ Hopf algebra elements, where C is an arbitrary constant.

Now we consider a chain with $N+1$ sites. Let

$$e_i = \mathbf{1}^{(1)}\otimes\cdots\otimes\mathbf{1}^{(i-1)}\otimes\mathrm{I}_q^2\otimes\mathbf{1}^{(i+2)}\otimes\cdots\otimes\mathbf{1}^{(N+1)} . \tag{46}$$

Correspondingly we have the $SU_q(2)$ elements in N-fold co-products,

$$\begin{aligned} \hat{S}'^{tot}_\pm &= \Delta^{(N)}\left(\hat{S}'_\pm\right) = \sum_{i=1}^{N+1} q^{\hat{S}'_3}\otimes\cdots\otimes q^{\hat{S}'_3}\otimes\hat{S}'^{(i)}_\pm\otimes q^{-\hat{S}'_3}\otimes\cdots\otimes q^{-\hat{S}'_3} , \\ \hat{S}'^{tot}_3 &= \Delta^{(N)}\left(\hat{S}'_3\right) = \sum_{i=1}^{N+1} \mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes\hat{S}'^{(i)}_3\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1} . \end{aligned} \tag{47}$$

It is easy to prove that

$$[e_i, \hat{S}'^{tot}_a] = 0 , \qquad a = 3, + \ and \ - . \tag{48}$$

Therefore the Hamiltonian defined by

$$H_n = \sum_{i=1}^{N} e_i \tag{49}$$

is of $SU_q(2)$ symmetry. Where the term e_i stands for the nearest interactions between spins at sites i and $i+1$. However $\hat{S}'_{\pm,3}$ in Hamiltonian are not the usual spin operators since they constitute $SU_q(2)$ algebra rather than $SU(2)$. To make (49) a Heisenberg Hamiltonian $\hat{S}'_{\pm,3}$ should be expressed in terms of the $SU(2)$ generators $\hat{S}_{\pm,3}$ by using the "quantum" deformation maps (35). In this way we get a general expression of 1-D Heisenberg Hamiltonian with $SU_q(2)$ symmetry, arbitrary spin values and nearest interactions.

The $SU_q(2)$ invariant Heisenberg Hamiltonian with non-nearest interactions can be

similarly obtained by using co-multiplications [25]. Let

$$
\begin{aligned}
\mathrm{I}_q^3 &= \Delta^{(2)}(C' - \mathrm{I}_q) = (\Delta \otimes \mathrm{id})\Delta(C' - \mathrm{I}_q) \\
&= \hat{S}'_+\hat{S}'_- \otimes q^{-2\hat{S}'_3} \otimes q^{-2\hat{S}'_3} + \hat{S}'_+ q^{\hat{S}'_3} \otimes q^{-\hat{S}'_3}\hat{S}'_- \otimes q^{-2\hat{S}'_3} + q^{\hat{S}'_3}\hat{S}'_- \otimes \hat{S}'_+ q^{-\hat{S}'_3} \otimes q^{-2\hat{S}'_3} \\
&\quad + q^{2\hat{S}'_3} \otimes \hat{S}'_+\hat{S}'_- \otimes q^{-2\hat{S}'_3} + \hat{S}'_+ q^{\hat{S}'_3} \otimes 1 \otimes q^{-\hat{S}'_3}\hat{S}'_- + q^{2\hat{S}'_3} \otimes \hat{S}'_+ q^{\hat{S}'_3} \otimes q^{-\hat{S}'_3}\hat{S}'_- \\
&\quad + q^{\hat{S}'_3}\hat{S}'_- \otimes 1 \otimes \hat{S}'_+ q^{-\hat{S}'_3} + q^{2\hat{S}'_3} \otimes q^{\hat{S}'_3}\hat{S}'_- \otimes \hat{S}'_+ q^{-\hat{S}'_3} + q^{2\hat{S}'_3} \otimes q^{2\hat{S}'_3} \otimes \hat{S}'_+\hat{S}'_- \\
&\quad + (q^{-1} q^{2\hat{S}'_3} \otimes q^{2\hat{S}'_3} \otimes q^{2\hat{S}'_3} + q\, q^{-2\hat{S}'_3} \otimes q^{-2\hat{S}'_3} \otimes q^{-2\hat{S}'_3})(q - q^{-1})^{-2} \\
&\quad - (q + q^{-1})(q - q^{-1})^{-2} 1 \otimes 1 \otimes 1 \,,
\end{aligned}
\tag{50}
$$

where C' is an arbitrary constant. Then

$$
H_{nn} = \sum_{i=1}^{N-1} w_i \tag{51}
$$

is an expression of $SU_q(2)$ invariant Hersenberg Hamiltonian with non-nearest neighbour interactions for general spin values in terms of maps (35). Where

$$
w_i = 1^{(1)} \otimes \cdots \otimes 1^{(i-1)} \otimes \mathrm{I}_q^3 \otimes 1^{(i+3)} \otimes \cdots \otimes 1^{(N+1)} \tag{52}
$$

represents three body interactions of spins at sites i, $i+1$ and $i+2$ and the interactions between spin pairs at sites $\{i, i+1\}$, $\{i+1, i+2\}$ and $\{i, i+2\}$.

For the case of spin-$\frac{1}{2}$, the generators of $SU(2)$ can be expressed as Pauli matrices,

$$
\hat{S}_3 = \frac{1}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \hat{S}_+ = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \hat{S}_- = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{53}
$$

Which is also a trivial expression of $SU_q(2)$ from maps (35), i.e,

$$
\hat{S}'_\pm = \hat{S}_\pm \,, \qquad \hat{S}'_3 = \hat{S}_3 \,. \tag{54}
$$

However the co-product operators (34) and (47) are not trivial ones. They really give rise to an $SU_q(2)$ Hopf algebra. In order to compare with the usual results, we take C in (45) to be $[q + q^{-1}]$. From (53), (54) and (45) we have

$$
\begin{aligned}
\mathrm{I}_q^2(j = \tfrac{1}{2}) &= \frac{q + q^{-1}}{4} 1 \otimes 1 - \Big[\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+ + \frac{q + q^{-1}}{4}\sigma_3 \otimes \sigma_3 \\
&\quad - \frac{q - q^{-1}}{4}\left(\sigma^3 \otimes 1 - 1 \otimes \sigma^3\right)\Big] \,.
\end{aligned}
\tag{55}
$$

This is the usual result and e_i defined by (46) constitute the Temperley-Lieb-Jones algebra. In terms of these algebraic properties this Hamiltonian system, with Hamiltonian given by formula (49), can be exactly solved [24][26].

When q approaches one, the Hamiltonian becomes the one with $SU(2)$ symmetry,

$$
H_n(j = \tfrac{1}{2}) = \frac{1}{2} - 2\sum_{i=1}^{N} \hat{\mathbf{S}}^i \cdot \hat{\mathbf{S}}^{i+1} \,, \tag{56}
$$

where $\hat{\mathbf{S}} = (\hat{S}_1, \hat{S}_2, \hat{S}_3)$ and $\hat{S}_i$ are given by eq. (53).

For spin-$\frac{1}{2}$ case formula (50) becomes

$$
\begin{aligned}
\mathrm{I}_q^3(j = \tfrac{1}{2}) &= C' 1 \otimes 1 \otimes 1 - \big[2q(q^6 - 3q^4 + 3q^2 - 1)\sigma_+ \otimes \sigma_- \otimes \sigma_3 \\
&\quad + 2q(q^6 - 3q^4 + 3q^2 - 1)\sigma_- \otimes \sigma_+ \otimes \sigma_3 + 2q(-q^6 + 3q^4 - 3q^2 + 1)\sigma_3 \otimes \sigma_+ \otimes \sigma_- \\
&\quad + 2q(-q^6 + 3q^4 - 3q^2 + 1)\sigma_3 \otimes \sigma_- \otimes \sigma_+ + 2q(-q^6 + q^4 + q^2 - 1)\sigma_+ \otimes \sigma_- \otimes 1 \\
&\quad + 2q(-q^6 + q^4 + q^2 - 1)\sigma_- \otimes \sigma_+ \otimes 1 + (-q^8 + 2q^6 - 2q^4 + 2q^2 - 1)\sigma_3 \otimes \sigma_3 \otimes 1 \\
&\quad + 4q^2(-q^4 + 2q^2 - 1)\sigma_+ \otimes 1 \otimes \sigma_- + 4q^2(-q^4 + 2q^2 - 1)\sigma_- \otimes 1 \otimes \sigma_+ \\
&\quad + 2q^2(-q^4 + 2q^2 - 1)\sigma_3 \otimes 1 \otimes \sigma_3 + 2q(-q^6 + q^4 + q^2 - 1)1 \otimes \sigma_+ \otimes \sigma_- \\
&\quad + 2q(-q^6 + q^4 + q^2 - 1)1 \otimes \sigma_- \otimes \sigma_+ + (-q^8 + 2q^6 - 2q^4 + 2q^2 - 1)1 \otimes \sigma_3 \otimes \sigma_3 \\
&\quad + (-q^8 + 2q^6 - 2q^2 + 1)\sigma_3 \otimes 1 \otimes 1 + (q^8 - 2q^6 + 2q^2 - 1)1 \otimes 1 \otimes \sigma_3 \\
&\quad + 2(-q^8 - q^6 + 2q^5 + 2q^3 - q^2 - 1)1 \otimes 1 \otimes 1\big](4q^2(q^4 - 2q^2 + 1))^{-1}
\end{aligned}
\tag{57}
$$

The related Hamiltonian given by eq. (51) is of three spin interactions as well as two nearest spin interactions and next-to-nearest neighbour interactions.

It is notable that when q approaches one, (57) becomes

$$
\begin{aligned}
\mathrm{I}_{q=1}^3(j = \tfrac{1}{2}) &= \frac{3}{2} 1 \otimes 1 \otimes 1 - [\sigma_+ \otimes \sigma_- \otimes 1 + \sigma_- \otimes \sigma_+ \otimes 1 \\
&\quad + \sigma_+ \otimes 1 \otimes \sigma_- + \sigma_- \otimes 1 \otimes \sigma_+ + 1 \otimes \sigma_+ \otimes \sigma_- + 1 \otimes \sigma_- \otimes \sigma_+ \\
&\quad + \tfrac{1}{2}(\sigma_3 \otimes \sigma_3 \otimes 1 + \sigma_3 \otimes 1 \otimes \sigma_3 + 1 \otimes \sigma_3 \otimes \sigma_3)\big] \,,
\end{aligned}
\tag{58}
$$

where C' in (57) has taken to be $\frac{15}{4}$. The terms representing three-spin actions in (57) dispear. The corresponding Hamiltonian (51) becomes the one with $SU(2)$ symmetry and nearest as well as next-to-nearest neighbour interactions.

In fact for $q = 1$ the interactions among spins at i, $i+1$ and $i+2$ are triangular ones. Therefore the Hamiltonian H_{nn} with next-to-nearest neighbour interactions in one dimension is in fact equivalent to the $SU(2)$ invariant Hamiltonian with nearest triangular lattice interactions of a two dimensional band. And generally for the case of $q = 1$ the lower dimensional Heisenberg spin model with higher order non-nearest interactions corresponds to higher dimensional one with lower order non-nearest interactions [25].

For spin-1, the matrix representations are

$$\hat{S}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \hat{S}_+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{S}_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{59}$$

From the maps given by (35) we have

$$\hat{S}'_3 = \hat{S}_3, \quad \hat{S}'_+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{q+q^{-1}}{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{S}'_- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ \frac{q+q^{-1}}{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{60}$$

They can be simply related to $\hat{S}_{3,\pm}$ by similarity transformations,

$$\hat{S}'_{3,\pm} \rightarrow A^{-1}\hat{S}'_{3,\pm}A,$$

where A is the transformation matrix

$$A = \begin{pmatrix} \sqrt{\frac{2}{q+q^{-1}}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{2}{q+q^{-1}}} \end{pmatrix}.$$

After this transformation we have another kind of expressions for $\hat{S}'_\pm$,

$$\hat{S}'_\pm = \sqrt{\frac{q+q^{-1}}{2}}\hat{S}_\pm, \quad \hat{S}'_3 = \hat{S}_3. \tag{61}$$

It is easy to check that $\hat{S}'_{3,\pm}$ in both equations (60) and (61) satisfy the $SU_q(2)$ algebraic relations (33).

Substituting expressions (61) into formula (45) we get

$$\begin{aligned}
\mathrm{I}^2_q(j=1) \;=\; & \frac{1}{4q^3}\Big[(q^6-q^4-q^2+1)\hat{S}_3^2\otimes\hat{S}_3^2-(q^6+3q^4+3q^2+1)\hat{S}_3\otimes\hat{S}_3 \\
& +(-q^6-q^4+q^2+1)\hat{S}_3^2\otimes\hat{S}_3+(q^6+q^4-q^2-1)\hat{S}_3\otimes\hat{S}_3^2 \\
& +2q(q^4-2q^3+2q^2-2q+1)(\hat{S}_+\hat{S}_3\otimes\hat{S}_-\hat{S}_3+\hat{S}_-\hat{S}_3\otimes\hat{S}_+\hat{S}_3) \\
& +2q(-q^3+q^2-q+1)\hat{S}_-\hat{S}_3\otimes\hat{S}_++2q^2(q^3-q^2+q-1)\hat{S}_+\otimes\hat{S}_-\hat{S}_3 \\
& +2q(q^3-q^2+q-1)\hat{S}_-\otimes\hat{S}_+\hat{S}_3+2q^2(-q^3+q^2-q+1)\hat{S}_+\hat{S}_3\otimes\hat{S}_- \\
& -2q(q^2+1)\hat{S}_-\otimes\hat{S}_+-2q^3(q^2+1)\hat{S}_+\otimes\hat{S}_- \\
& +2(-q^6+q^4+q^2-1)(\hat{S}_3^2\otimes 1+1\otimes\hat{S}_3^2) \\
& +2(-q^6-q^4+q^2+1)\hat{S}_3\otimes 1+2(q^6+q^4-q^2-1)1\otimes\hat{S}_3 \\
& +4q^2(-2q^2+qC-2))1\otimes 1\Big].
\end{aligned} \tag{62}$$

In this way we obtain the Heisenberg spin-1 chain Hamiltonian of $SU_q(2)$ symmetry with nearest neighbour interactions,

$$H_n(j=1) = \sum_{i=1}^{N} 1^{(1)}\otimes\cdots\otimes 1^{(i-1)}\otimes \mathrm{I}^2_q(j=1)\otimes 1^{(i+2)}\otimes\cdots\otimes 1^{(N+1)}, \tag{63}$$

which is different to the Hamiltonian presented in [27]. And when q approaches one we have

$$\mathrm{I}^2_{q=1}(j=1) = 1\otimes 1-(\hat{S}_+\otimes\hat{S}_-+\hat{S}_-\otimes\hat{S}_++2\hat{S}_3\otimes\hat{S}_3). \tag{64}$$

The Hamiltonian (63), corresponding to (56) for the case of spin-$\frac{1}{2}$, simply becomes

$$H_n(j=1) = 1-2\sum_{i=1}^{N}\hat{\mathrm{S}}^{(i)}\cdot\hat{\mathrm{S}}^{(i+1)}, \tag{65}$$

where the constant C has taken to be 3, $\hat{\mathrm{S}} = (\hat{S}_1, \hat{S}_2, \hat{S}_3)$ and $\hat{S}_i$ are given by eqs. (59). The Hamiltonian (65) is obviously of $SU(2)$ symmetry as it should be.

The Heisenberg spin-1 chain Hamiltonian $H_{nn}(j=1)$ with three spin interactions and $SU_q(2)$ symmetry can be similarly obtained by substituting (61) into formula (50). The equivalence between the lower dimensional Heisenberg spin model with higher order

non-nearest interactions and the higher dimensional one with lower order non-nearest interactions can also be similarly studied.

V. Deformation Symmetries in Differential Dynamical Systems

Now we discuss the "quantum group" symmetries in dynamical systems. To this end, we first recall the Lie symmetry cases.

It is well known that for a linear differential equation

$$\Delta\Psi(x_i) = 0 \ , \tag{66}$$

where Δ is a linear differential operator and x_i are independent variables, the operator S is said to be a Lie symmetry operator for the eq. (66) provided [28]

$$[S, \Delta] = R(x_i)\Delta \ , \tag{67}$$

where $R(x_i)$ are complex-valued functions of x_is or differential operators.

Let $\mathcal{F}$ denote the solution space of differential equation (66). It is easy to see that a symmetry operator S maps solutions of (66) into solutions, i.e., if $\Psi \in \mathcal{F}$, then $S\Psi \in \mathcal{F}$. The symmetries of the solution space $\mathcal{F}$ of eq. (66) are hence depicted by the independent bases, the generators of symmetry operators S. A direct result is that if S are first order differential operators, then $[S, S] \subset S$ constitute a Lie algebra. As an example we only analyse the symmetries of the radial Schröginger equation of free particle in this section. But the procedure can be easily applied to other differential dynamical systems.

Let t be the time variable and x the radial coordinate. Then the radial Schröginger equation of free particle is

$$\left(\partial_\tau - \partial_{xx} + \frac{a}{x^2}\right)\Psi(\tau, x) = 0 \ , \tag{68}$$

where $\tau = i\,t$ and a is a constant.

Let S be a general first order differential operator,

$$S = f\partial_x + g\partial_\tau + h \ . \tag{69}$$

f, g, and h are complex-valued functions of x, τ. From eq. (67) we have

$$\frac{x}{2}\partial_\tau R + \tau\partial h = 0 \ , \quad -\partial_\tau h + \partial_{xx} h = 0 \ .$$

Which give rise to three generators [28]

$$\begin{aligned} \hat{S}_3 &= \frac{x}{2}\partial_x + \tau\partial_\tau + \frac{1}{4} \ , \\ \hat{S}_+ &= \tau^2\partial_\tau + \tau x\partial_x + \frac{\tau}{2} + \frac{x^2}{4} \ , \\ \hat{S}_- &= -\partial_\tau \ , \end{aligned} \tag{70}$$

satisfying

$$[\hat{S}_3, \Delta] = -\Delta, \quad [\hat{S}_+, \Delta] = -2\tau\Delta, \quad [\hat{S}_-, \Delta] = 0$$

and

$$[\hat{S}_3, \hat{S}_\pm] = \pm\hat{S}_\pm, \quad [\hat{S}_+, \hat{S}_-] = 2\hat{S}_3 \ .$$

That is, $\hat{S}_3$ and $\hat{S}_\pm$ constitute the $SU(2)$ algebra. Therefore the solution space of this radial Schrödinger equation (68) is of $SU(2)$ symmetry. The solutions of this equation can be throughly investigated from separation of variables and the symmetry operators $\hat{S}_3$ and $\hat{S}_\pm$ in formulae (70).

As the $SU(2)$ operators map solutions into solutions, the universal enveloping algebra of $SU(2)$ also maps $\mathcal{F}$ into $\mathcal{F}$. Therefore from (35) we have

$$\hat{S}_3' = \hat{S}_3 \ , \quad \hat{S}_\pm' = \hat{S}_\pm \frac{[j \mp \hat{S}_3]_q}{[j \mp \hat{S}_3]_q} \ , \tag{71}$$

where $\hat{S}_\pm$ and $\hat{S}_3$ are given by formula (70), j is a constant with respect to the solution space $\mathcal{F}$. Obviously $\hat{S}_\pm'$ and $\hat{S}_3'$ in (71) constitute the $SU_q(2)$ algebra and map solutions into solutions of (68). Therefore the solution space $\mathcal{F}$ also possesses $SU_q(2)$ symmetries. This fact may be understood in this way. The evolutions of a dynamical system are guided not only by the dynamical equation, but also by the initial and boundary conditions. Therefore which kind map of the solutions gives rise to the real description of the dynamical system depends on the detailed physical systems. Of course, it would be interesting to find such a kind of real processes with the deformation symmetries. This subject is under investigation.

VI. q-Deformed Finite Transformation of $SO(3)$ Group

From section I we see that the algebras (including deformed algebras) with three generators can be systematically analysed in symplectic geometry approach. Nevertheless, the finite transformations related to the deformed algebras are not so manifest, though the infinitesimal canonical transformation deduced by S_is (S_i's) in (6) and (8) are clear and S^2 (S_q^2) is the invariant under these transformations respectively. Here we would like to give some considerations on finite transformation properties related to the deformed algebras.

In view of symplectic geometry the only difference between algebra $SO(3)$ ($SU(2)$) and " quantum algebra" $SO_q(3)$ ($SU_q(2)$) is that the former is related to sphere S^2 and the later to a q-deformed sphere S_q^2. While S^2, the coset space of $SU(2)/U(1)$, is invariant under group transformations of $SO(3)$. That is, the $SO(3)$ group transformation keeps

$$S_0^2 = x_1^2 + x_2^2 + x_3^2 = x_+x_- + x_3^2 \tag{72}$$

invariant for coordinates x_i in $\mathbf{R}^3$, where $x_\pm = x_1 \pm ix_2$. Therefore reasonably we would like to know what kind of transformation, let us call it the q-deformed transformation of $SO(3)$ correspondingly, keeps the deformed sphere

$$S_0'^2 = y_1^2 + y_2^2 + \frac{(\sinh\gamma y_3)^2}{\gamma\sinh\gamma} = y_+y_- + \frac{(\sinh\gamma y_3)^2}{\gamma\sinh\gamma} \tag{73}$$

invariant. Where $S_0'^2 = \dfrac{(\sinh\gamma S_0)^2}{\gamma\sinh\gamma}$, $\quad y_\pm = y_1 \pm iy_2$.

Let R be the usual rotation transformation of $SO(3)$ group, R_q the deformed finite transformation of $SO(3)$, Q and Q' the deformation maps related to the quasi-conformal transformations between S^2 and S_q^2 for real q [17], $Q, Q' \ : S^2 \to S_q^2$. Then we have following diagram:

$$\begin{array}{ccc} S^2 & \xrightarrow{R} & S^2 \\ Q\downarrow & & \downarrow Q' \\ S_q^2 & \xrightarrow{R_q} & S_q^2 \end{array} \tag{74}$$

The q-deformed finite transformation of $SO(3)$, $R_q : \ S_q^2 \to S_q^2$ can be deduced from the maps Q, Q' and $SO(3)$ group transformation R in above diagram, i.e.,

$$R_q = Q' \, R \, Q^{-1} \ . \tag{75}$$

Where Q^{-1} maps S_q^2 to S^2. The actions of R_q on $\mathbf{R}^3$ result in all the coordinate transformations that keeps S_q^2 invariant. Let $y_{\pm,3}$, $y'_{\pm,3}$ be the coordinates of S_q^2 related by transformation R_q and $x_{\pm,3}$, $x'_{\pm,3}$ be the coordinates of S^2 related by the rotation matrix $R = (R_{ij})$. From (28) we have the deformation map Q' related to (72) and (73),

$$y_3' = x_3' \ , \quad y_\pm' = \frac{1}{\sqrt{\gamma\sinh\gamma}} x_\pm' \frac{\sinh\gamma(S_0 \mp x_3')}{S_0 \mp x_3'} \ . \tag{76}$$

Where

$$\begin{aligned} x_3' &= \tfrac{1}{2}(R_{31} - iR_{32})x_+ + \tfrac{1}{2}(R_{31} + iR_{32})x_- + R_{33}x_3 \ , \\ x_+' &= \tfrac{1}{2}(R_{11} + R_{22} + i(R_{21} - R_{12}))x_+ \\ &\quad + \tfrac{1}{2}(R_{11} - R_{22} + i(R_{12} + R_{21}))x_- + (R_{13} + iR_{23})x_3 \ , \\ x_-' &= \tfrac{1}{2}(R_{11} - R_{22} - i(R_{21} + R_{12}))x_+ \\ &\quad + \tfrac{1}{2}(R_{11} + R_{22} + i(R_{12} - R_{21}))x_- + (R_{13} - iR_{23})x_3 \ . \end{aligned} \tag{77}$$

From (72), (73) and (76) we have the map of $Q^{-1} = Q'^{-1}$,

$$x_3 = y_3 \ , \quad x_\pm = y_\pm \frac{(S_0 \mp y_3)\sqrt{\gamma\sinh\gamma}}{\sinh\gamma(S_0 \mp y_3)} \ . \tag{78}$$

Equations (76), (77) and (78) give a kind of deformed finite coordinate transformations of $SO(3)$ that keep S_q^2 invariant.

Let $R_3(\phi)$ be the rotation matrix along the third direction with an angle ϕ,

$$R_3(\phi) = \begin{pmatrix} cos\phi & -sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{79}$$

Then equations (76), (77) and (78) give rise to

$$y_+' = (\cos\phi + i\sin\phi)y_+ \ , \quad y_-' = (\cos\phi - i\sin\phi)y_- \ , \quad y_3' = y_3 .$$

Which is the usual rotation around the third direction. And all of these transformations constitute a group $U(1)$. This is because that the deformed sphere S_q^2 is symmetric along the third axis.

However for a rotation along the second direction

$$R_2(\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}, \tag{80}$$

the coordinate transformations are rather complicated. For instance,

$$y_3' = \cos\psi\ y_3 - \sqrt{\gamma\sinh\gamma}\left[\frac{y_3+S_0}{\sinh\gamma(y_3+S_0)}y_- + \frac{y_3-S_0}{\sinh\gamma(y_3-S_0)}y_+\right]\sin\psi\ . \tag{81}$$

It is notable that the $SO(3)$ group transformation R can be written in form of matrices that their elements are independent of coordinates. And the product of two rotation matrices is still a rotation matrix belonging to $SO(3)$. Here from equation (81) it is clear that the deformed finite $SO(3)$ transformations R_q can not be simply expressed by matrices. That is, the coordinate transformations that keep S_q^2 invariant are not linear ones. Although two continuous deformed $SO(3)$ transformations, $R_q^3 = R_q^2 R_q^1 = Q_2' R_2 Q_2^{-1} Q_1' R_1 Q_1^{-1}$ is still a transformation of R_q, the associativity of group actions is no longer preserved. Therefore these transformations do not constitute a usual group.

VII. Conclusions

In the symplectic geometry approach we have found the classical deformation maps among algebras with three generators, especially, the ones between the Lie algebras and their q-deformed counterparts as well as the relations between the general algebras and 2-D smooth manifolds. The prequantization and polarization in geometric quantization method give rise to the operator representations of the classical Poisson algebras, which result in the operator deformation maps among algebras. In fact, the algebras with more generators can also be similarly investigated by means of the symplectic structure on the coadjoint orbits in group theory [29].

The method presented in section IV can be used to construct various Hamiltonian systems with various algebraic symmetries such as $SU_q(n)$, $\mathcal{H}_q(4)$ etc. by using their invariants, the Casimirs or the co-products of their Casimirs.

In addition, the XXZ Heisenberg spin-$\frac{1}{2}$ chain Hamiltonian with $SU_q(2)$ symmetry and nearest interaction is the sum of e_i which is also the representation of Temperley-Lieb-Jones algebra. When $q = 1$ the Hamiltonian with next-to-nearest neighbour interactions is the sum of w_i given by formulae (52) and (58). While calculation shows that

$$w_i = e_i + e_{i+1} + e_{i,i+2}\ ,$$

where $e_{i,i+2}$ represents the interaction between spins at site i and $i+2$,

$$e_{i,i+2} = e_i + e_{i+1} - e_i e_{i+1} - e_{i+1} e_i\ .$$

Therefore w_i can be simply expressed in terms of the Temperley-Lieb elements e_i's and constitute no new algebras. However when q is not a unit, w_i can not be expressed by e_i's. Hence it is interesting to investigate what kind algebra the element w_i, given by formulae (52) and (57), in the XXZ Hamiltonian with next-to-nearest neighbour interactions constitute. The same interesting problems exist for spin-1 case as well.

The survey of the q-deformation symmetries in dynamical systems is a way to give "quantum algebras" some physical interpretations and applications in physics. In fact it is well known that S^2 is an equal potential surface for the central symmetric forces such as Coulomb force. It may be naturally asked what will turn out if the equal potential surface is taken to be S_q^2. All these problems are worthy investigating further.

Finally, it should be mentioned that we have mainly considered in this paper the properties of the " quantum algebras ", almost nothing to do with the "quantum groups"at all. It is well known, however, that the elements of the q-deformed universal enveloping algebra can be expressed in terms of the quantum Borel subalgebras $L^{\pm}$, which are related to the "quantum group" element T through the duality $< L^{\pm}, T >= R^{\pm}$ [1-3]. Therefore, the approach presented in [16][17] and here may also be generalized to the entire structure of the "quantum groups" as long as the duality can be realized explicitly. This subject is also under investigation.

Acknowledgements One of the authers (SMF) is thankful to CCAST (World Laboratory) for hospitality and support. The work is undertaken during the CCAST Workshop on Quantum Groups and Field Theory in Lower Dimensions (March 12 - April 10, 1992) co-sponsored by ITP and NSFC. This work is supported in part by the National Natural Science Foundation of China.

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