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A Note on the Equivalence between Canonical and BRST Quantization for Chern-Simons field Theory

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ABSTRACT

The BRST quantization of Chern-Simons field theory is performed. With aid of the nilpotency property of BRST charge, the physical state condition $\hat{Q}_B|phys>=0$ is reduced to the Gauss law constraints associated with the case that Wilson loops are present or not. Further the physical states are shown to satisfy Knizhnik-Zamolodchikov equation and hence the equivalence between canonical and BRST quantization approaches are exhibited.

During the past several years, the quantization of Chern-Simons field theory has been greatly studied in different ways[1-5]. On one hand, Witten performed canonical quantization of Chern-Simons field theory[1], he found that the Hilbert space of the theory can be identical to the space of conformal block of WZW model. On this basis, he evaluated the expectation value of Wilson loop operators and reproduced the celebrated Jones polynomial. On the other hand, Guadagnini *etal* took the quantization of Chern-Simons theory in a alternative way[2]. They performed BRST gauge fixing of the theory and computed expectation value of Wilson loops to second order with aid of perturbation theory, and obtained the same result as Witten's. This fact suggest the problem that whether these two different quantization approaches are equivalent, for just as what Witten did, first impose the constraints, the topology of classical physical phase space may become very complicated, in general geometric quantization must be resorted to deal with this circumstance. Whilist that first quantize the system and then consider the constraints is another completely different story. Hence for a general constrained system, it is very difficult to prove the equivalence between these two quantization schemes. Fortunately for Chern-Simons field theory, Witten has already pointed out that the Hilbert space obtained by first imposing constraints and then quantizing the system can be identical to the space of conformal blocks, this statement was verified by many authors[6][7][8] and make it possible for us to prove the equivalence between these two quantization schemes. The aim of this paper is just give a explicit exhibition of the affirmative answer.

The action for Chern-Simons field theory takes the following form

$$S_{cs} = \frac{k}{4\pi}\int_M (A\wedge dA + \frac{2}{3}A\wedge A\wedge A), \tag{1}$$

where $A=A^a_\mu T^a dx^\mu$ with T^a the generators in some representation of gauge group $\mathcal{G}$, k is chosen to be an interger in order to make the system possess gauge invariance under the action of an arbitrary gauge group. Without loss of generality, we choose $\mathcal{G}=SU(N)$ and there exsits normalization $Tr(T^aT^b)=\frac{1}{2}\delta^{ab}$. Choosing Lorentz gauge condition $\partial_\mu A^\mu=0$ and performing the BRST gauge fixing, we obtain the following effective action

$$\begin{aligned} S_{eff} &= S_{cs} + \int \tfrac{1}{\epsilon} Tr\delta_B[\bar{C}(\partial_\mu A^\mu + B)] \\ &= \int d^3x \tfrac{k}{16\pi}[\epsilon^{\mu\nu\rho}A^a_\mu(\partial_\nu A^a_\rho - \partial_\rho A^a_\nu) + if^{abc}\tfrac{2}{3}A^a_\mu A^b_\nu A^c_\rho \\ &\quad -\tfrac{ik}{8\pi}A^{\mu a}\partial_\mu B^a + \tfrac{ik}{8\pi}B^{a2} - \tfrac{1}{2}\partial_\mu\bar{C}^a D^\mu C^a]. \end{aligned} \tag{2}$$

The BRST transformations for every fields are read as following

$$\delta_B A^a_\mu = \epsilon D_\mu C^a \quad , \quad \delta_B B^a = 0,$$
$$\delta_B C^a = -\epsilon\frac{1}{2}f^{abc}C^bC^c \quad , \quad \delta_B \bar{C}^a = \epsilon\frac{ik}{4\pi}B^a. \tag{3}$$

Evidently, they are nilpotent: $\delta_B{}^2=0$, Now the classical configuration space is enlarged by the introduction of new fields—ghost field $\bar{C}$, antighost field $\bar{C}$ and multiplier field B, their canonical conjugate momenta defined by $\Pi_\Phi=\frac{\partial\mathcal{L}}{\partial\dot{\Phi}}$, $\Phi = (A_1, B, C, \bar{C})$ can be obtained:

$$\Pi^a_{A_1} = \frac{k}{8\pi}A^a_2, \Pi^a_B = -\frac{ik}{8\pi}A^{0a},$$
$$\Pi^a_{\bar{C}} = -\frac{1}{2}D^0C^a, \Pi^a_C = \frac{1}{2}\dot{\bar{C}}. \tag{4}$$

These fields and their canonical conjugate momenta satisfy Poisson bracket (for bosonic fields) or antibracket (for fermionic ones) relation

$$[\Pi^I_\Phi(\mathbf{x},t), \Phi_J(\mathbf{y},t)]_{\pm P.B} = -i\delta^I_J\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$
$$[\Pi^I_\Phi(\mathbf{x},t), \Pi^J_\Phi(\mathbf{y},t)]_{\pm P.B} = [\Phi_I(\mathbf{x},t), \Phi_J(\mathbf{y},t)]_{\pm P.B} = 0. \tag{5}$$

The BRST charge corresponding to the invariance under BRST transformation eq.(3) can be computed

$$\begin{aligned} Q_B &= \int d^2x[\frac{k}{8\pi}\epsilon^{ij}D_iC^aA^a_j - \frac{1}{4}f^{abc}\dot{\bar{C}}^aC^bC^c - \frac{ik}{8\pi}B^aD^0C^a] \\ &= \int d^2x[-\frac{k}{8\pi}C^aF^a_{12} - \frac{1}{4}f^{abc}\Pi^a_{\bar{C}}C^bC^c + \frac{ik}{4\pi}B^a\Pi^a_{\bar{C}}]. \end{aligned} \tag{6}$$

By a direct calculation, it is easy to prove that

$$\begin{aligned} [Q_B, \Phi]_{\pm P.B} &= \frac{1}{i}\delta_B\Phi, \Phi = (A_1, B, C, \bar{C}), \\ \frac{1}{2}[Q_B, Q_B]_{+,P.B} &= Q^2_B = 0. \end{aligned} \tag{7}$$

The following is performing quantization. According to the general procedure for quantization, the classical observables are replaced by operators and Poisson (anti-)brackets by (anti-)commutation Lie brackets. Especially the polarization must be specified, here we choose $\hat{\Phi}=(\hat{A}_1, \hat{B}, \hat{C}, \hat{\bar{C}})$ to play the role of canonical coordinates and the Hilbert space is composed of square integrable functional in term of $\hat{\Phi}$. Since there exists no anomaly in Chern-Simons field theory, the BRST charge $\hat{Q}_B$ are well defined and the BRST algebra relations

$$\frac{1}{2}[\hat{Q}_B, \hat{Q}_B] = \hat{Q}^2_B = 0 \tag{8}$$

are still satisfied (the problem of operator ordering is ignored here and in what follows). It is well known that the state space at present possesses indefinite metric due to the introduction of non-physical fields. According to the general principle of BRST quantization, the physical states satisfy that

$$\hat{Q}_B|phys>= 0. \tag{9}$$

Notice that the above condition determines physical states up to a zero-norm state due to the nilpotency property of BRST charge operator $\hat{Q}_B$, that is, $|phys> \longrightarrow |phys> + |X>, |X>= \hat{Q}_B|phys>$. Obviously the zero-norm states have the property that they are normal to all physical states including themselves, $<X|phys>=<X_1|X_2>=0$. Hence they make no contribution to the observables and they are in essential unphysical. The genuine physical state space should contains no such states. Kugo and Ojima, making use of the quartet representation of BRST algebra provided by the Hilbert space, found that non-physical states always appear as the zero-norm state combination[9], hence non-physical states are confined and the genuine physical state sector can be reached. Here we adopt the viewpoint of physical operator proposed by Marnelius[10]. Physical operators are defined to be those transform a physical state into another one, according to their action on the physical states, physical operators can be divided into two types: A-type and B-type ones. A A-type physical operator is a genuine physical operator and it transforms a genuine physical operator into the other one, i.e. $\hat{A}|phys>=|phys>'$. A B-type operator take the following form

$$\hat{B} = [\hat{C}, \hat{Q}_B]_\pm \tag{10}$$

where $\hat{C}$ a nonphysical operator. Evidently A B-type operator transforms a physical state into a zero-norm state and it can be regarded as the generator of a new type of gauge transformation for its action on a physical operator $\hat{\Phi}$

$$[\hat{B}_i, \hat{\Phi}]_\pm = f_{ij}(\hat{\Phi})\hat{B}_j. \tag{11}$$

B-type operators form a ideal in the operator algebra[10], the product of an arbitrary operator $\hat{K}$ (physical or non-physical) with a B-type operator is also the generator of gauge transformation for the fact

$$[(\hat{K}\hat{B})_i, \hat{\Phi}]_\pm = f'_{ij}(\hat{\Phi})(\hat{K}\hat{B})_j \tag{12}$$

Followed BRST charge eq.(6), it can be proved that

$$\begin{aligned} \hat{B}^a_1 &= [\hat{Q}_B, \hat{\Pi}^a_{\bar{C}}] = -\frac{k}{8\pi}\hat{F}^a_{12} - \frac{1}{2}f^{abc}\hat{\Pi}^b_{\bar{C}}\hat{C}^c, \\ \hat{B}^a_2 &= [\hat{Q}_B, \hat{\Pi}^a_B] = \frac{ik}{4\pi}\hat{\Pi}^a_{\bar{C}}, \\ \hat{B}^a_3 &= [\hat{Q}_B, \partial_\mu\hat{A}^{\mu a}] = M_{ab}\hat{C}^b, \end{aligned} \tag{13}$$

where M_{ab}=-$\frac{k}{8\pi}[\hat{F}^a_{12}, \partial_\mu \hat{A}^{\mu b}]$. Note that the matrix $M=(M_{ab})$ is nonsingular for the reason that -$\frac{k}{8\pi}F_{12}$ and $\partial_\mu A^\mu$ constitute a pair of second-class constraint, in classical sense the Poisson bracket between them doesn't vanish. It is worthy to stress the point that during above process we have used on-shell condition. The nonsingularity of M states that $\hat{C}^a=(M^{-1})^{ab}\hat{B}_{3b}$ are also generators of gauge transformation.

From eqs.(6) and (13), one can see that the three parts consisting of BRST charge $\hat{Q}_B$ are all gauge transformation generators, but the second and third terms suit for non-physical fields. As for the first part, one can see that there exists no coupling between the unphysical gauge transformation generators $\hat{C}^a$ and the physical ones $\frac{k}{8\pi}\hat{F}^a_{12}$, so if there exist no Wilson loop operators in the universe, the physical state condition $\hat{Q}_B|Phys>=0$ reduces to that

$$\frac{k}{8\pi}\hat{F}^a_{12}|phys>=0. \tag{14}$$

In the case that Wilson loop operators are present, since they are not local operators, they locate in a finite region of space time, they only produce the excitation from the vacuum to vacuum. If one want consider their effect, one must choose a proper time at which Wlison loop operators excite the vacuum, and the generated state can be illustrately represented by a punctured surface, which is obtained geometrically by the intersection of links, on which Wilson loop operators are defined, with surface Σ determined by some time t. Under above polarization choice, the gauge invariant state functional at some time t takes the following form

$$\begin{aligned}\Psi_{phys} &= \int \mathcal{D}X\ exp(i\int^t_{t_0=-\infty} S_{eff}dt + \int_\Sigma d^2x \sum_{i=1}^2 A^a_i A^{ia})\Pi^N_{n=1} exp\int^{Q_n}_{P_n} A^{(n)}\Psi_0[A],\\ &X=(A,C,\bar{C},B),\end{aligned} \tag{15}$$

where n denotes the different components of links and P_n,Q_n the punctured points created by the intersection of nth components of links with Σ. $\Psi_0[A]$ is the vacuum wave functional at time t=-∞, which is determined by eq.(14) and will be explicitly shown later. In the language of a operator form, eq.(15) can be rewritten as following

$$|phys>=(\Pi_n Pexp\int^{Q_n}_{P_n\Gamma}\sum_{i=1,2}\hat{A}_i(x_1,x_2)dx^i)|0> \tag{16}$$

with Γ the projection on transverse surface Σ of the links locating in the three-

dimensional space-time region of less time t. Hence we have

$$\begin{aligned}\hat{Q}_B|phys> &= (\int d^2x[-\tfrac{k}{8\pi}\hat{C}^a\hat{F}^a_{12}+\tfrac{k}{4\pi}\hat{B}^a\hat{\Pi}^a_{\hat{C}}+\tfrac{1}{2}f^{abc}\hat{\Pi}^a_{\hat{C}}\hat{C}^b\hat{C}^c]\\ &\quad\times\Pi^N_{n=1}Pexp\int^{Q_n}_{P_n}\sum_{i=1,2}\hat{A}_i dx^i)|0>\\ &= (\int d^2x[-\hat{C}^a[\tfrac{k}{8\pi}\hat{F}^a_{12}-(\sum^N_{n=1}\delta^{(2)}(\mathbf{x}-\mathbf{x}_{P_n})T^a_{(n)}-\delta^{(2)}(\mathbf{x}-\mathbf{x}_{Q_n})]\\ &\quad+\tfrac{k}{4\pi}\hat{B}^a\hat{\Pi}^a_{\hat{C}}-\tfrac{1}{2}f^{abc}\hat{\Pi}^a_{\hat{C}}\hat{C}^b\hat{C}^c])|phys>,\end{aligned} \tag{17}$$

where we have used that

$$\begin{aligned}\partial_{\mathbf{x}_{P_n}}\frac{\delta}{\delta A^a_1(\mathbf{x}_{P_n})}|phys> &= -\tfrac{8\pi}{k}T^a_{(n)}|phys>,\\ \partial_{\mathbf{x}_{Q_n}}\frac{\delta}{\delta A^a_1(\mathbf{x}_{Q_n})}|phys> &= \tfrac{8\pi}{k}T^a_{(n)}\delta(\mathbf{x}-\mathbf{x}_{Q_n})|phys>.\end{aligned} \tag{18}$$

Note that in the locations of punctures, $\hat{Q}_B$ becomes a matrix-valued operator, so do every fields containing in $\hat{Q}_B$, but out the punctures Q_B and the fields containing in it restore to the ordinary ones. Repeating above process, one can reduce the physical state condition $\hat{Q}_B|phys>=0$ to that

$$[\frac{k}{8\pi}F^a_{12}-\sum^N_{n=1}(\delta^{(2)}(\mathbf{x}-\mathbf{x}_{P_n})T^a_n-\delta^{(2)}(\mathbf{x}-\mathbf{x}_{Q_n}])T^a_n]|phys>=0. \tag{19}$$

From Gauss law constraints eqs.(14) and (19) satisfied by physical states, we can show that the physical states indeed can be explained as conformal blocks. Let's first see the case that there exist no Wilson loop operators. As stated above, we choose the polarization that $\hat{A}_1$ plays the role of canonical coordinate and the physical states are represented by the functional with respect to $A_1(x)$, the eigenvalue of operator $\hat{A}_1$. Introducing the new fields $U(\mathbf{x})$ defined by[9]

$$T^aA^a_1(\mathbf{x}) = -iU^{-1}(\mathbf{x})\partial_1 U(\mathbf{x}), \tag{20}$$

or in other words,

$$U(\mathbf{x}) = U(x_1,x_2) = Pexpi\int^{x_1}_{-\infty}dtT^aA^a_1(t,x_2), \tag{21}$$

one can find the solution to the constraint equation eq.(14) takes the following form

$$\begin{aligned}\Psi_0[A] &= exp[-i\tfrac{k}{12\pi}\int_M d^3y\epsilon^{\alpha\beta\gamma}TrU^{-1}\partial_\alpha U^{-1}\partial_\beta U^{-1}\partial_\gamma - i\tfrac{k}{8\pi}\int_\Sigma d^2x tr\partial_i U^{-1}\partial^i U]\\ &= exp - S_{WZW}\end{aligned} \tag{22}$$

Hence one see that physical vacuum state corresponds to a special conformal block of WZW model — the scattering amplitude from vacuum to vacuum

$$< 0\ out|0\ in >= \int \mathcal{D}U\ exp - iS_{WZW} \tag{23}$$

If there exist Wilson loops in the universe, in order to exhibit the explicit identification of conformal block with physical state, we adopt complex coordinate description with the convention $z=\frac{1}{\sqrt{2}}(x_1+ix_2)$, $\bar{z}=\frac{1}{\sqrt{2}}(x_1-ix_2)$; $\hat{A}_z=\frac{1}{\sqrt{2}}(\hat{A}_1-i\hat{A}_2)$, $\hat{A}_{\bar{z}}=\frac{1}{\sqrt{2}}(\hat{A}_1+i\hat{A}_2)$. It can be easily proved that the $\hat{A}_{\bar{z}}$ polarization in term of complex language is consistent with the real $\hat{A}_1$ polarization

$$\begin{aligned} [\hat{A}_z, \hat{A}_w] &= [\hat{A}_{\bar{z}}, \hat{A}_{\bar{w}}] = 0 \\ [\hat{A}^a_{\bar{z}}, \hat{A}^b_{\bar{w}}] &= -i\frac{8\pi}{k}\delta^{(2)}(z-w)\delta^{ab} \end{aligned} \tag{24}$$

Correspondingly, the constraint equation eq.(20) takes the following form

$$\frac{k}{8\pi}F^a_{z\bar{z}}|phys>=\sum_{n=1}^{N}[\delta^{(2)}(z-z_{P_n})T^a_{(n)}-\delta^{(2)}(z-z_{Q_n})T^a_{(n)}]|phys> \tag{25}$$

Starting from this, one can verify that

$$\begin{aligned} \frac{\partial}{\partial z_{P_n}}\Psi_{phys}[A_z] &= \frac{2}{k+C_V}(\sum_{n\neq m}^{N}\frac{T^a_{(n)}T^a_{(m)}}{z_{P_n}-z_{P_m}}-\sum_{m=1}^{N}\frac{T^a_{(n)}T^a_{(m)}}{z_{P_m}-z_{Q_n}})\Psi_{phys}[A_z], \\ \frac{\partial}{\partial z_{Q_n}}\Psi_{phys}[A_z] &= \frac{2}{k+C_V}(\sum_{m\neq n}^{N}\frac{T^a_m T^a_n}{z_{Q_n}-z_{Q_m}}-\sum_{m=1}^{N}\frac{T^a_{(m)}T^a_{(n)}}{z_{Q_n}-z_{P_m}})\Psi_{phys}[A_z], \end{aligned} \tag{26}$$

that is,

$$\frac{\partial}{\partial z_k}\Psi_{phys}[A_z]=\frac{2}{k+C_V}\sum_{k\neq l}^{2N}\frac{T^a_{(k)}T^a_{(l)}}{z_k-z_l}\Psi_{phys}[A_z], k=1,2,...2N. \tag{27}$$

They are exactly the Knizhnik-Zamolodchikov equations satisfied by the conformal block for $2N$ primary fields

$$\frac{\partial}{\partial z_k}<\Phi_1(z_1)...\Phi_{2N}(z_{2N})>=\frac{2}{k+C_V}\sum_{k\neq l}^{2N}\frac{T^a_{(k)}T^a_{(l)}}{z_k-z_l}<\Phi_1(z_1)...\Phi_{2N}(z_{2N})> \tag{28}$$

Hence from eqs.(23) and (27), we come to the conclusion that physical states can be explained as the conformal blocks of WZW model.

In summary, we perform BRST quantization for Chern-Simons field theory and reduce the physical state condition $\hat{Q}_B$=0 to the Gauss law constraints associated

with the case that Wilson loops are present or not, then we exhibit the identification of constraint equation for physical states with Knizhnik-Zamolodchikov equation satisfied by conformal blocks, this is consistent with the facts coming from Witten's canonical quantization approach. In this sense, we state that for Chern-Simons field theory, its BRST and canonical quantization are equivalent.

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