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A q-Deformation of Virasoro Algebra

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Abstract

A q-deformation of Virasoro algebra with center and the Hopf algebra structure has been discussed in this paper. Using higher order difference operators, we can get an infinite family of q-deformation of Virasoro algebra. In particular, when q is a root of unit, the q-deformation is still available.

The Virasoro algebra is a central extension of complex Lie algebra of vector fields on the circle. It plays an important role in mathematical physics. Recently, there are many papers[4-10] deal with the q-deformation of Virasoro algebra. We will give a new one which has 1-dimensional central extension and the Hopf algebra structure. In section 1, we will introduce the elegant results on the central extension given by C. Kassel[4]. Then we will review the Hopf algebra H constructed by H. Hiro-Oka *et. al.*[6] in section2. At last we will consider our q-deformation in section 3.

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1. Algebra D and its q-deformation D_q

Let $\mathbf{C}[x, x^{-1}]$ be the algebra of complex Laurent polynomials and $\frac{d}{dx}$ be its derivation. Consider the associative algebra $D = \mathbf{C}[x, x^{-1}, \frac{d}{dx}]$ which has a basis $\{x^i(\frac{d}{dx})^j \mid i \in \mathbf{Z},\ j \in \mathbf{N}\}$. The Virasoro algebra is a Lie subalgebra of the central extension of D as a Lie algebra . C. Kassel computes the 2-cocycle explicitly. The Virasoro algebra Vir has the generators $L_m = x^{m+1}\frac{d}{dx}$, $m \in \mathbf{Z}$ and the relation

$$[L_m\ ,\ L_n] = (n-m)\, L_{m+n} + \frac{c}{2}\,\frac{m^3-m}{6}\,\delta_{m+n,0} \tag{1}$$

where c is its center element. Replacing $\frac{d}{dx}$ by Jackson's q-differentiation operator ∂_q on $\mathbf{C}[x, x^{-1}]$, we get the q- deformation $D_q = \mathbf{C}[x, x^{-1}, \partial_q]$ of the associative

algebra $D = \mathbf{C}[x, x^{-1}, \frac{d}{dx}]$, where ∂_q is defined by

$$\partial_q(P) = \frac{P(qx) - P(x)}{qx - x}$$

for any polynomial $P \in \mathbf{C}[x, x^{-1}]$. There is the relation

$$\partial_q x - qx\partial_q = 1$$

in D_q. Here D_q has a basis $\{x^i(\partial_q)^j \mid i \in \mathbf{Z},\ j \in \mathbf{N}\}$.

Let τ_q be the algebra automorphism of $\mathbf{C}[x, x^{-1}]$ defined by

$$\tau_q(x) = qx.$$

We have the equation

$$\tau_q = 1 + (q-1)x\partial_q \tag{2}$$

So $\tau_q \in D_q$ and

$$\tau_q x = qx\tau_q \tag{3}$$

in D_q. Formally, we often write $q^{x\frac{d}{dx}}$ into τ_q because they are same as an operator on $\mathbf{C}[x, x^{-1}]$.

The 1- dimensional central extension of the associative algebra D_q

$$0 \longrightarrow \mathbf{C} \longrightarrow \hat{D}_q \longrightarrow D_q \longrightarrow 0$$

has been considered by C. Kassel[4]. These quantum 2-cocycles in explicit formula

are as follows:

$$\begin{aligned}
&\varphi_q(x^i\partial_q^j\ ,\ x^k\partial_q^l) \\
&= (-1)^l (j!)_q (l!)_q \textstyle\sum_{p=j}^{j-i-1} q^{l(l-2k+2p+1)/2} \binom{p}{j}_q \binom{k-p-1}{l}_q \delta_{i+k,j+l} \\
&= \begin{cases} (j!)_q (l!)_q \sum_{p=j}^{j-i-1} \binom{p}{j}_q \binom{i-j+p}{l}_q \delta_{i+k,j+l} & \text{if } i < 0 < k \\ -\varphi_q(x^k\partial_q^l\ ,\ x^i\partial_q^j) & \text{if } k < 0 < i \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
(r)_q &= (q^r - 1)/(q-1) \\
(j!)_q &= (j)_q (j-1)_q \cdots (1)_q \\
\binom{p}{j}_q &= (p!)_q / ((j!)_q ((p-j)!)_q)
\end{aligned}$$

In particular, if q is a root of unit of order $d > 1$ and N is an integer $\neq 0$, then there is an infinite family of quantum cocycle in the following formula:

$$\begin{aligned}
&\varphi_q^{(Nd)}(x^i\partial_q^j\ ,\ x^k\partial_q^l) \\
&= \begin{cases} \displaystyle\sum_{p=j}^{min(k-1,d-1)} \frac{(p!)_q}{((p-j)!)_q} \frac{((i-j+p)!)_q}{((i-j+p-l)!)_q} \delta_{i+k,j+l+Nd} & \text{if } k \geq 1 \\ 0 & \text{if } k = 0 \\ \displaystyle\sum_{max(k,j-d)}^{-1} \frac{(p!)_q}{((p-j)!)_q} \frac{((i-j+p)!)_q}{((i-j+p-l)!)_q} \delta_{i+k,j+l+Nd} & \text{if } k < 0 \end{cases}
\end{aligned} \tag{5}$$

Let $L_m(q) = x^{m+1}\partial_q$, we have

$$L_m(q)L_n(q) - q^{n-m}L_n(q)L_m(q) = (n-m)_q L_{m+n}(q) \tag{6}$$

Therefore, we get Kassel's q-deformation Vir_q of Virasoro algebra which is generated by the generators $L_m(q)$, $m \in \mathbf{Z}$ and the relation (6). The algebra Vir_q is a subalgebra of $\hat{D}_q$ as an associative algebra. Using the formulas (4) and (5) (if q is a root of unit), we can get its central extension. When $q \to 1$, $Vir_q \to Vir$. But we have not got its Hopf algebra structure.

2. Hopf algebra H

H.Hiro-Oka *et. al*[6] construct the Hopf algebra H which is defined by generators τ_q and $\mathcal{L}_{m,n} = x^m \frac{q^{nd/dx+n(m+1)/2} - q^{-nd/dx-n(m+1)/2}}{q^n - q^{-n}}$, $m, n \in \mathbf{Z}$ and the relations

$$\begin{aligned} \tau_q \mathcal{L}_{m,n} &= q^m \mathcal{L}_{m,n} \tau_q \\ [\mathcal{L}_{m,n}, \mathcal{L}_{m',n'}] &= C^{m+m',n+n'}_{m,n,m',n'} \mathcal{L}_{m+m',n+n'} + C^{m+m',n-n'}_{m,n,m',-n'} \mathcal{L}_{m+m',n-n'} \end{aligned} \tag{7}$$

where

$$C^{m'',n''}_{m,n,m',n'} = (q^{(nm'-n'm)/2} - q^{-(nm'-n'm)/2})(q^{n''} - q^{-n''})/(q^n - q^{-n})(q^{n'} - q^{-n'})$$

As we know, $\mathcal{L}_{m,n}$ is also a kind of difference operator on $\mathbf{C}[x, x^{-1}]$. The following definition is more convenient for us:

$$\mathcal{L}_{m,n} = x^m (\tau_q^n q^{n(m+1)/2} - \tau_q^{-n} q^{-n(m+1)/2})/(q^n - q^{-n}) \tag{8}$$

The Hopf algebra structure of algebra H can be given as follows:

$$\begin{aligned} \Delta(\mathcal{L}_{m,n}) &= \mathcal{L}_{m,n} \otimes \tau_q^m + \tau_q^m \otimes \mathcal{L}_{m,n} \\ \Delta(\tau_q) &= \tau_q \otimes \tau_q \\ \varepsilon(\mathcal{L}_{m,n}) &= 0 \\ \varepsilon(\tau_q) &= 1 \\ S(\mathcal{L}_{m,n}) &= -\tau_q^m \mathcal{L}_{m,n} \tau_q^m \\ S(\tau_q) &= \tau_q^{-1} \end{aligned} \tag{9}$$

It is a co-commutative Hopf algebra.

3. On the q-deformation of Virasoro algebra

Notice that τ_q is not invertible in $\hat{D}_q$. Define the inverse of τ_q by

$$\tau_q^{-1} = \sum_{k=0}^{\infty} (1-q)^k (x\partial_q)^k$$

We have

$$\tau_q^{-1}(x) = q^{-1} x$$

on $\mathbf{C}[x, x^{-1}]$.

Adding τ_q^{-1} to $\hat{D}_q$, we get the extended associative algebra $\hat{D}'_q$. All of relations with respect to τ_q^{-1} can be got from the corresponding equations on τ_q. It is not difficult to confirm that the center element c of $\hat{D}_q$ still is the center of $\hat{D}'_q$.

According to the definition of $\mathcal{L}_{m,n}$ in section 2, the algebra H can be viewed as a subalgebra of the associative algebra $\hat{D}'_q$. So the center c in $\hat{D}'_q$ is also commutative with H.

Let $\hat{L}_m = x^m (\tau_q^{2n} - 1)/(q^{2n} - 1), m \in \mathbf{Z}$, where n is a positive integer and $\hat{L}_0 = 0$. By (2), we have

$$\hat{L}_m = x^m \sum_{k=1}^{2n} \binom{2n}{k} (q-1)^k (x\partial_q)^k / (q^{2n} - 1) \in \hat{D}_q. \tag{10}$$

They are the difference operators of order 2n on $\mathbf{C}[x, x^{-1}]$. we have

$$\begin{aligned} \tau_q \hat{L}_m &= q^m \hat{L}_m \tau_q \\ q^{2nm} \hat{L}_m \hat{L}_{m'} - q^{2nm'} \hat{L}_{m'} \hat{L}_m &= \frac{q^{2nm'} - q^{2nm}}{q^{2n} - 1} \hat{L}_{m+m'} \end{aligned} \tag{11}$$

Denote by $Vir_q^{(n)}$ the associative algebra generated by $\{\tau_q, \hat{L}_m \mid m \in \mathbf{Z}, n \text{ fixed}\}$ and the relations (11). In order to find out its Hopf algebra structure, we at first

want to extend the Hopf algebra H to the new Hopf algebra H' such that H' will contain $Vir_q^{(n)}$ as its Hopf subalgebra.

By means of the definition of $\mathcal{L}_{m,n}$ in section 2, we have the following equations:

$$\begin{aligned} \hat{L}_0 &= q^{-3n/2}\mathcal{L}_{0,n}\tau_q^n - q^{-n}(1+q^n)^{-1} \\ \hat{L}_{-1} &= q^{-n}\mathcal{L}_{-1,n}\tau_q^n \\ \hat{L}_1(1-q^{2n}\tau_q^{2n}) &= (1-q^{2n})\mathcal{L}_{i,n}\tau_q^n\hat{L}_0 \end{aligned} \tag{12}$$

In general,

$$\hat{L}_m(1-q^{n(m+1)}\tau_q^{2n}) = (1-q^{2n})q^{n(m-1)/2}\mathcal{L}_{m,n}\tau_q^n\hat{L}_0, \text{ for } m \in \mathbf{Z} \tag{13}$$

The elements $(1-q^{n(m+1)}\tau_q^{2n})$ are not invertible in H. Define their inverse by

$$(1-q^{n(m+1)}\tau_q^{2n})^{-1} = \sum_{k=0}^{\infty} q^{kn(m+1)}\tau_q^{2kn}$$

and add all of them to the algebra H. We get a new associative algebra H'. Futhermore, for these new elements of H', we define

$$\begin{aligned} \Delta(1-q^{n(m+1)}\tau_q^{2n})^{-1} &= \sum_{k=0}^{\infty} q^{kn(m+1)}\Delta(\tau_q^{2kn}) \\ S(1-q^{n(m+1)}\tau_q^{2n})^{-1} &= (S(1-q^{n(m+1)}\tau_q^{2n}))^{-1} \\ \varepsilon(1-q^{n(m+1)}\tau_q^{2n})^{-1} &= (1-q^{n(m+1)})^{-1} \end{aligned} \tag{14}$$

They are compatible with the Hopf algebra structure of H. Therefore, H' is also a Hopf algebra.

The formulas (12) and (13) imply that $Vir_q^{(n)}$ is the Hopf subalgebra of Hopf

algebra H'. So we have

$$\begin{aligned} \Delta(\tau_q) &= \tau_q \otimes \tau_q \\ \Delta(\hat{L}_m) &= q^{n(m-1)/2}(1-q^{2n})\Delta(\mathcal{L}_{m,n})\Delta(\tau_q^n)\Delta(\hat{L}_0)\Delta(1-q^{n(m+1)}\tau_q^{2n})^{-1} \\ &= (\hat{L}_m(1-q^{n(m+1)}\tau_q^{2n})\hat{L}_0^{-1} \otimes \tau_q^{m+n} + \tau_q^{m+n} \otimes \hat{L}_m(1-q^{n(m+1)}\tau_q^{2n})\hat{L}_0^{-1}) \\ &\quad \times \Delta(\hat{L}_0)\Delta(1-q^{n(m+1)}\tau_q^{2n})^{-1} \end{aligned} \tag{15}$$

where $\hat{L}_0^{-1} = (1-q^{2n})(1-\tau_q^{2n})^{-1}$. In particular,

$$\begin{aligned} \Delta(\hat{L}_0) &= (\hat{L}_0 + q^{-n}(1+q^n)^{-1}) \otimes \tau_q^n + \tau_q^n \otimes (\hat{L}_0 + q^{-n}(1+q^n)^{-1}) \\ &\quad -q^{-n}(1+q^n)^{-1}\, 1 \otimes 1 \\ &= (1-q^{2n})^{-1}(1-\tau_q^{2n} \otimes \tau_q^{2n}) \\ \Delta(\hat{L}_{-1}) &= \hat{L}_{-1} \otimes \tau_q^{n-1} + \tau_q^{n-1} \otimes \hat{L}_{-1} \\ \Delta(\hat{L}_1) &= (\hat{L}_1(1-q^{2n}\tau_q^{2n})\hat{L}_0^{-1} \otimes \tau_q^{n+1} + \tau_q^{n+1} \otimes \hat{L}_1(1-q^{2n}\tau^{2n})\hat{L}_0^{-1}) \\ &\quad \times \Delta(\hat{L}_0)\Delta(1-q^{2n}\tau_q^{2n})^{-1} \end{aligned} \tag{16}$$

and

$$\begin{aligned} \varepsilon(\tau_q) &= 1 \\ \varepsilon(\hat{L}_m) &= 0 \\ S(\tau_q) &= \tau_q^{-1} \\ S(\hat{L}_m) &= -S(\tau_q^{n-m}\hat{L}_0(1-q^{n(m+1)}\tau_q^{2n})^{-1})^{-1}\hat{L}_m(\tau_q^{n-m}\hat{L}_0(1-q^{n(m+1)}\tau_q^{2n})^{-1}). \end{aligned} \tag{17}$$

Now we have the Hopf algebra (we still denote it by $Vir_q^{(n)}$) which is generated by the set $\{\tau_q,\ \hat{L}_m,\ (1-q^{2n})/(1-q^{n(m+1)}\tau_q^{2n}) \mid m \in \mathbf{Z},\ n \text{ fixed}\}$ and the relations (11). The Hopf algebra structure are given by the formulas (15) and (17).

If $q \to 1$, then $\tau_q \to 1, \hat{L}_m \to x^{m+1}d/dx$ and $(1-q^{2n})/(1-q^{n(m+1)}\tau_q^{2n}) \to$

$(xd/dx + (m+1)/2)^{-1}$. Hence, we can consider $Vir_q^{(n)}$ as the q-deformation of Virasoro algebra.

Remark. We have got an infinite family of q-deformation of Virasoro algebra when q runs over N. They are not isomorphic. In particular, when q is a root of unit, the q-deformation $Vir_q^{(n)}$ is still available except for $q^{2n} = 1$.

Finally, we will consider the central extension of $Vir_q^{(n)}$. We define

$$\Delta(c) = c \otimes 1 + 1 \otimes c, \ \varepsilon(c) = 0, \ S(c) = -c$$

where c is the center element of $Vir_q^{(n)}$. By (10)

$$\hat{L}_m = x^m \sum_{k=1}^{2n} \binom{2n}{k} (q-1)^k (x\partial_q)^k / (q^{2n}-1)$$

Let $(x\partial_q)^k = a_{k1} x\partial_q + a_{k2} x^2 \partial_q^2 + \cdots + a_{kk} x^k \partial_q^k$, where a_{ki} , $1 \le i \le k$ are some constants. So we have

$$\hat{L}_m = x^m \sum_{k=1}^{2n} \sum_{s=k}^{2n} \binom{2n}{s} (q-1)^s a_{sk} \ x^k \ \partial_q^k \ / \ (q^{2n}-1)$$

Using the formulas (4) and (5) , we get

$$\varphi_q(\hat{L}_m \ , \ \hat{L}_{m'}) = \sum_{k=1}^{2n} \sum_{s=k}^{2n} \sum_{l=1}^{2n} \sum_{t=l}^{2n} \binom{2n}{s} \binom{2n}{t} (1-q)^{s+t} a_{sk} \ a_{tl} \ \varphi_q(x^k \partial_q^k \ , \ x^l \partial_q^l \ )$$

For instance, if $n = 1$, then $\hat{L}_m = x^{m+1}\partial_q + q\frac{q-1}{q+1} x^{m+2}\partial_q^2$. We have

$$\begin{aligned} \varphi_q(\hat{L}_m \ , \ \hat{L}_{m'}) &= \varphi_q(x^{m+1}\partial_q \ , \ x^{m'+1}\partial_q) + q \ \frac{q-1}{q+1} \ \varphi_q(x^{m+1}\partial_q \ , \ x^{m'+2}\partial_q^2 \ ) \\ &+ \ q \ \frac{q-1}{q+1} \ \varphi_q(x^{m+2}\partial_q^2 \ , \ x^{m'+1}\partial_q) \\ &+ \ q^2 \ (\frac{q-1}{q+1})^2 \ \varphi_q(x^{m+2}\partial_q^2 \ , \ x^{m'+2}\partial_q^2 \ ) \end{aligned}$$

If $q \to 1$, the limit of $\varphi_q(\hat{L}_m \ , \ \hat{L}_{m'})$ just is $\frac{m^3-m}{6}\delta_{m+m',0}$.

Before ending this paper, the following eqations are important:

$$\begin{aligned} \hat{L}_m &= x^m \hat{L}_0, \ \ m \in \mathbf{Z} \\ \mathcal{L}_{m,n}\tau_q^n \mathcal{L}_{m',n}\tau_q^n &= q^{-n(m+m'-2)/2} x^{m+m'} (1 - q^{n(m+2m'+1)}\tau_q^{2n}) \\ &\times (1 - q^{n(m'+1)}\tau_q^{2n})(1-q^{2n})^{-2} \end{aligned}$$

We can use them to verify the Hopf algebra structure of $Vir_q^{(n)}$.

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