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BICOVARIANT DIFFERENTIAL CALCULUS ON QUANTUM GROUP $GL_q(n)^+$

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Bicovariant Differential Calculus on Quantum Group $GL_q(n)$ †

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Abstract

The de Rham complex of the quantum group $GL_q(n)$ is presented. And we show that the differential calculus on the quantum group $GL_q(n)$ given in this paper is bicovariant. The noncommutative differentail calculus on the quantum group $SL_q(n)$ is also discussed.

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§0. Introduction

Recently, much attention has been paid to the non-commutative differential calculus on quantum groups[1-5]. In this paper we will describe the exterior differential calculus on the quantum group $GL_q(n)$. Quantum group theories were developed and several different approaches to construct quantum groups were also introduced in papers [6-9]. In the first part of this paper, we will adopt Faddeev's approach to give the construction of the quantum group $GL_q(n)$ and a concrete subalgebra of the dual of the coordinate ring of $GL_q(n)$. The second part is mainly applied to discuss the first order differential calculus on the quantum group $GL_q(n)$, namely the construction of the exterior differential operator d and the first order differential bimodule. In the trird part we will demonstrate in detail that the first order differential calculus given in section two is bicovariant. In the fourth part we will describe how to get the quantum de Rham complex of $GL_q(n)$. A general theory for bicovariant differential calculus on compact matrix pseudogroups was developed by Woronowicz [1]. And the discussions of noncommutative differential calculus on more general quantum groups and quantum spaces can be found in the papers [2]. In the third and fourth sections we mainly adopt Woronowicz's methods and some basic results that are true in Hopf algebras level for general quantum groups. In the last part we shortly remark how the quantum exterior differential calculus on the quantum group $GL_q(n)$ is induced to give the quantum de Rham complex on the quantum group $SL_q(n)$.

This paper is an extension of [10] for more general case $GL_q(n)$, most proofs in [10] are still valid in this paper. In this paper quantum groups are understood as the objects of the inverse category of the Hopf algebras with antipode, which are neither commutative, nor co-commutative. As to Hopf algebras, please see [11]. For simplicity, summation convention is used in the paper.

By the method provided in this paper, we can also give bicovariant differential calculus on quantum groups of B_n, C_n, D_n series and other types[12].

§1. Quantum group $GL_q(n)$

In this section, we will cite some results on the quantum group $GL_q(n)$ without proof, and give some explanation to the symbols applied in this paper.

Let

$$R_q = \sum_{i,j=1}^{n} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + \chi \sum_{\substack{i,j=1 \\ i>j}}^{n} e_{ij} \otimes e_{ji}, \quad q \in \mathbb{C}^*. \tag{1.1}$$

where $\chi = q - q^{-1}$, e_{ij} $(i,j = 1,2,\cdots,n)$ is the element matrix of order n, entries of which are all zeros except that the one on i-th row j-th column is 1, and the symbol $\otimes$ means the tensor product of matrices. One easily checks that the matrix R_q is a solution of the quantum Yang-Baxter equation(QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1.2}$$

with R_{ij} as $n^3 \times n^3$ matrices defined via

$$R_{12} = R_q \otimes E,\ R_{13} = (E \otimes P)R_{12}(E \otimes P),\ R_{23} = E \otimes R_q,$$

where E is the unit matrix of order n and P is the permutation matrix in $\mathbf{C}^n \otimes \mathbf{C}^n$. We can also write the matrix R_q in the form of submatrices, i.e.,

$$R_q = (r_{ij})_{1\le i,j\le n}, \tag{1.3}$$

with

$$r_{ij} = \begin{cases} \chi e_{ji}, & i > j, \\ 0, & i < j, \\ E + (q-1)e_{ii}, & i = j, \end{cases} \tag{1.4}$$

Take n^2 elements t_{ij} $(i,j = 1,2,\cdots,n)$ and arrange them into a matrix $T = (t_{ij})_{1\le i,j\le n}$. Let $\mathbf{C}[T]$ denote the free associative algebra with unit 1 generated by the n^2 elements t_{ij} $(i,j = 1,2,\cdots,n)$, and let $\{R_qT_1T_2 - T_2T_1R_q\}$ be the two-sided ideal of $\mathbf{C}[T]$ generated by the relations $R_qT_1T_2 - T_2T_1R_q$, where $T_1 = T \otimes E$, $T_2 = E \otimes T$. Then the quotient

$$\mathrm{Fun}(M_q(n)) = \mathbf{C}[T]/\{R_qT_1T_2 - T_2T_1R_q\} \tag{1.5}$$

has the structure of a bialgebra with the $\mathbf{C}$-linear structure maps, the comultiplication Δ and the counit ε, fixed by the following values for the generators:

$$\Delta T = T \odot T, \tag{1.6}$$
$$\varepsilon(T) = E, \tag{1.7}$$

where the symbol $\odot$ means $\Delta t_{ij} = t_{ik} \otimes t_{kj}$. Both Δ and ε are algebra homomorphism. And the multiplication m on $\mathrm{Fun}(M_q(n))$ corresponds to the ordinary one of functions, i.e.,

$$m(x \otimes y) = xy,\ \forall x, y \in \mathrm{Fun}(M_q(n)),$$

and the unit map i is defined by

$$\begin{array}{rccl} i: & \mathbf{C} & \longrightarrow & \mathrm{Fun}(M_q(n)), \\ & \lambda & \longmapsto & \lambda \cdot 1. \end{array}$$

When $q = 1$, $\mathrm{Fun}(M_q(n))$ coincides with the commutative algebra $\mathrm{Fun}(M(n))$ of coordinate functions on the matrix algebra $M(n,\mathbf{C})$. So, we can regard $\mathrm{Fun}(M_q(n))$ as the deformation of $\mathrm{Fun}(M(n))$, or the algebra of coordinate functions on the quantum matrix algebra $M_q(n)$ of rank n associated with the matrix R_q.

Write S_n for the symmetric group on n letters and write $l(\sigma)$ for the length of $\sigma \in S_n$. Namely, $l(\sigma)$ is the minimal number of the terms required to express σ as a product of the simple transposition $(i, i+1)$. For the quantum matrix algebra $M_q(n)$, the quantum determinant can be defined as:

$$Det_qT = \sum_{\sigma\in S_n} (-q)^{l(\sigma)} t_{1\sigma_1} t_{2\sigma_2} \cdots t_{n\sigma_n}. \tag{1.8}$$

The quantum determinant has the following properties

$$\Delta(Det_qT) = Det_qT \otimes Det_qT, \tag{1.9}$$
$$\varepsilon(Det_qT) = 1. \tag{1.10}$$

Remark 1. In what follows, we identify the element t_{ij} $(i,j = 1,2,\cdots,n)$ and Det_qT with their corresponding equivalent classes.

Definition 1.1

$$\mathrm{Fun}(GL_q(n)) = \mathrm{Fun}(M_q(n))[t]/\{tDet_qT - (Det_qT)t,\ tDet_qT - 1\}, \tag{1.11}$$

where t is a new generator and $\{tDet_qT - (Det_qT)t,\ tDet_qT - 1\}$ means the two-sided ideal of $\mathrm{Fun}(M_q(n))[t]$ generated by the two relations $tDet_qT - (Det_qT)t$, $tDet_qT - 1$.

At this time, we naturally extend the structure maps m, i, Δ and ε of the bialgebra $\mathrm{Fun}(M_q(n))$ to the quotient $\mathrm{Fun}(GL_q(n))$ and require

$$\Delta(t) = t \otimes t,\ \varepsilon(t) = 1 \tag{1.12}$$

to make it also a bialgebra. Furthermore, the antipode S on $\mathrm{Fun}(GL_q(n))$ can be uniquely determined by the requirement that $TS(T) = E \cdot 1 = S(T)T$, its definition on the generators t_{ij} $(i,j = 1,2,\cdots,n)$ and t is given by

$$S(t_{ij}) = (-q)^{i-j} tDet_qT_{ji},\quad i,j = 1,2,\cdots,n, \tag{1.13}$$
$$S(t) = Det_qT, \tag{1.14}$$

where T_{ij} denote the $(n-1)\times(n-1)$ generic matrix obtained by deleting row i and column j of the generated matrix $T = (t_{ij})_{1\le i,j\le n}$. After introducing the antipode we obtain

Theorem 1.1 $\mathrm{Fun}(GL_q(n))$ is a Hopf algebras with respect to m, i, Δ, ε and S.

$\mathrm{Fun}^*(GL_q(n))$ denotes the dual of $\mathrm{Fun}(GL_q(n))$. We now give two sets of linear functionals $l^{\pm}_{ij}$ $(i,j = 1,2,\cdots,n)$ and arrange them into two $n \times n$ matrices.

$$L^{\pm} = (l^{\pm}_{ij})_{1\le i,j\le n}.$$

To describe $l^{\pm}_{ij}$ $(i,j = 1,2,\cdots,n)$ explicitly. We first define that the values of the linear functionals $l^{\pm}_{ij}$ $(i,j = 1,2,\cdots,n)$ on the generators t_{ij} $(i,j = 1,2,\cdots,n)$ of $\mathrm{Fun}(GL_q(n))$ are given by

$$l^{\pm}_{ij}(T) = \lambda_{\pm}^{\pm 1} r^{\pm}_{ij},\quad 0 \ne \lambda_{\pm} \in \mathbf{C}, \tag{1.15}$$
$$l^{\pm}_{ij}(1) = \delta_{ij},\quad i,j = 1,2,\cdots,n, \tag{1.16}$$

where

$$r_{ij}^{+} = \begin{cases} \chi e_{ji}, & i < j, \\ E + (q-1)e_{ii}, & i = j, \\ 0, & i > j. \end{cases} \tag{1.17}$$

$$r_{ij}^{-} = \begin{cases} 0, & i < j, \\ E + (q^{-1}-1)e_{ii}, & i = j, \\ -\chi e_{ji}, & i > j. \end{cases} \tag{1.18}$$

If denote $R^{+} = (r_{ij}^{+})_{1\le i,j\le n}$, $R^{-} = (r_{ij}^{-})_{1\le i,j\le n}$, then $R^{+} = PR_qP$, $R^{-} = R_q^{-1}$, where P is the permutation matrix, and can be written in the form of submatrices as

$$P = (P_{ij})_{1\le i,j\le n} = (e_{ji})_{1\le i,j\le n}.$$

From the fact that the matrix R_q satisfies QYBE, it follows that

$$R_{12}^{+}R_{13}^{\pm}R_{23}^{\pm} = R_{23}^{\pm}R_{13}^{\pm}R_{12}^{+} \tag{1.19}$$

with $R_{ij}^{\pm}$ as $n^3 \times n^3$ matrices defined by

$$R_{12}^{\pm} = R^{\pm} \otimes E,\ R_{13}^{\pm} = (E \otimes P)R_{12}^{\pm}(E \otimes P),\ R_{23}^{\pm} = E \otimes R^{\pm}.$$

For arbitrary element of $\mathrm{Fun}(GL_q(n))$ the definition of $l_{ij}^{\pm}$ is given by the following induction,

$$l_{ij}^{\pm}(xy) = l_{ik}^{\pm}(x)l_{kj}^{\pm}(y),\ \ \forall x, y \in \mathrm{Fun}(GL_q(n)). \tag{1.20}$$

Now what we need to do is to give the value of $l_{ij}^{\pm}$ $(i, j = 1, 2, \cdots, n)$ on the generator t of $\mathrm{Fun}(GL_q(n))$. For this we rewrite (1.15), (1.16) and (1.20) in the form of submatrices as follows

$$< L^{\pm}, T >= (l_{ij}^{\pm}(T))_{1\le i,j\le n} = \lambda_{\pm}^{\pm 1}R^{\pm}, \tag{1.21}$$

$$< L^{\pm}, 1 >= E, \tag{1.22}$$

$$< L^{\pm}, xy >= (l_{ij}^{\pm}(xy))_{1\le i,j\le n} =< L^{\pm}, x >< L^{\pm}, y > . \tag{1.23}$$

Then the action of $l_{ij}^{\pm}$ $(i, j = 1, 2, \cdots, n)$ on the generator t is

$$< L^{\pm}, t >= (l_{ij}^{\pm}(t))_{1\le i,j\le n} =< L^{\pm}, Det_qT >^{-1} . \tag{1.24}$$

In fact, we have

$$< L^{+}, t_{ij} >= \lambda_{+}r_{ij},$$

$$< L^{+}, Det_qT >=< L^{+}, \prod_{i=1}^{n} t_{ii} >= \lambda_{+}^{n}qE.$$

Thus

$$< L^{+}, t >= \lambda_{+}^{-n}q^{-1}E \tag{1.25}$$

holds. Similarly, we have

$$< L^{-}, t >= \lambda_{-}^{n}qE. \tag{1.26}$$

We can check that the action of the linear functionals $l_{ij}^{\pm}$ $(i, j = 1, 2, \cdots, n)$ given in above way on the two-sided ideal generated by the relations $R_qT_1T_2 - T_2T_1R_qt$, $Det_qT-(Det_qT)t$, $tDet_qT-1$ is zero. This shows $l_{ij}^{\pm}$ $(i, j = 1, 2, \cdots, n)$ is well defined on the Hopf algebra $\mathrm{Fun}(GL_q(n))$ and then the two sets of functionals $l_{ij}^{\pm}$ $(i, j = 1, 2, \cdots, n)$ belong to $\mathrm{Fun}^{*}(GL_q(n))$ (also see Proposition 1.2 in [10]). Furthermore, with the comultiplication Δ of $\mathrm{Fun}(GL_q(n))$, the multiplication m^{*} among $l_{ij}^{\pm}$ $(i, j = 1, 2, \cdots, n)$ can be introduced. Suppose ξ, η are two polynomials of $l_{ij}^{\pm}$. We define

$$m^{*}(\xi \otimes \eta)(x) = (\xi\eta)(x) = (\xi \otimes \eta)\Delta x, \qquad \forall x \in \mathrm{Fun}(GL_q(n)), \tag{1.27}$$

and introduce two new linear functionals $l^{\pm}$ by the following formulas

$$< l^{\pm}, T >= l^{\pm}(T) = (l_{11}^{\pm}l_{22}^{\pm}\cdots l_{nn}^{\pm}(T))^{-1}, \tag{1.28}$$

$$< l^{\pm}, t >= l^{\pm}(t) = (l_{11}^{\pm}l_{22}^{\pm}\cdots l_{nn}^{\pm}(t))^{-1}, \tag{1.29}$$

$$< l^{\pm}, 1 >= l^{\pm}(1) = 1, \tag{1.30}$$

$$< l^{\pm}, xy >= l^{\pm}(xy) = l^{\pm}(x)l^{\pm}(y),\ \forall x, y \in \mathrm{Fun}(GL_q(n)). \tag{1.31}$$

It is also easy to see that

$$l^{\pm}(\{R_qT_1T_2 - T_2T_1R_q,\ tDet_qT - (Det_qT)t,\ tDet_qT - 1\}) = 0.$$

$\mathrm{Fun}_0^{*}(GL_q(n))$ denotes the associative subalgebra of $\mathrm{Fun}^{*}(GL_q(n))$ generated by $l_{ij}^{\pm}$ $(i, j = 1, 2, \cdots, n)$ and $l^{\pm}$ via the multiplication m^{*} in (1.27). Obviously, the unit of the algebra $\mathrm{Fun}_0^{*}(GL_q(n))$ is ε, i.e. the counit of $\mathrm{Fun}(GL_q(n))$. However, it should be pointed out that the $2(n^2+1)$ elements $l_{ij}^{\pm}$ $(i, j = 1, 2, \cdots, n)$ and $l^{\pm}$ are not free generators, which are subordinate to the communication relations given by the following two propositions proofs of which are due to (1.19) and definition of $L^{\pm}$ (also see Proposition 1.4 in [10]).

Proposition 1.1

$$R^{+}L_1^{\pm}L_2^{\pm} = L_2^{\pm}L_1^{\pm}R^{+}, \tag{1.32}$$

$$R^{+}L_1^{+}L_2^{-} = L_2^{-}L_1^{+}R^{+}, \tag{1.33}$$

where $L_1^{\pm} = L^{\pm} \otimes E$, $L_2^{\pm} = E \otimes L^{\pm}$.

Proposition 1.2

(i) $l^{\pm}\prod_{i=1}^{n} l_{ii}^{\pm} = \varepsilon,$ (1.34)

(ii) $l^{+}L^{\pm} = L^{\pm}l^{+},\ l^{-}L^{\pm} = L^{\pm}l^{-},$ (1.35)

(iii) $l^{+}l^{-} = l^{-}l^{+},$ (1.36)

(iv) $l_{ij}^{+} = 0,\ i > j,\ l_{ij}^{-} = 0,\ i < j.$ (1.37)

The homomorphisms Δ^{*}, ε^{*}, S^{*} on $\mathrm{Fun}_0^{*}(GL_q(n))$ are defined as

$$\Delta^{*}(L^{\pm}) = L^{\pm} \odot L^{\pm}, \qquad \Delta^{*}(l^{\pm}) = l^{\pm} \otimes l^{\pm},$$

$$\varepsilon^{*}(L^{\pm}) = E, \qquad \varepsilon^{*}(l^{\pm}) = 1,$$

$$S^{*}(L^{\pm}) = (-q)^{j-i}l^{\pm}(Det_{q^{-1}}L_{ji}^{\pm})_{1\le i,j\le n}, \qquad S^{*}(l^{\pm}) = l_{11}^{\pm}l_{22}^{\pm}\cdots l_{nn}^{\pm},$$

where $L^{\pm}_{ij}$ is the submatrix of $L^{\pm}$ defined like T_{ij} in (1.13). We can check the compatibility of the maps Δ^*, ε^*, S^* and the relations in Propositions 1.1 and 1.2. Namely, the actions of Δ^*, ε^*, and S^* on the relations (1.32)-(1.37) are all zeros (as Proposition 1.5 and 1.6 of [10]). We can also see

$$S^*(L^{\pm})L^{\pm} = L^{\pm}S^*(L^{\pm}) = \varepsilon \cdot E. \tag{1.38}$$

Finally, we have

Theorem 1.2 $\mathrm{Fun}^*_0(GL_q(n))$ is a Hopf subalgebra of $\mathrm{Fun}^*(GL_q(n))$ with respect to m^*, Δ^*, ε^*, S^*.

§2. The first order differential calculus on $GL_q(n)$

Assume $\mathcal{A}$ is an associative algebra with unit. The first order differential calculus on $\mathcal{A}$, which is denoted by (Γ, δ), consists of a bi-module Γ of $\mathcal{A}$ and a linear operator δ satisfying

(i) Leibnitz rule

$$\delta(xy) = (\delta x)y + x\delta y, \qquad \forall x, y \in \mathcal{A}, \tag{2.1}$$

(ii) for arbitrary element ρ in Γ, there always exist some elements $x_k, y_k \in \mathcal{A}$ $(k = 1, 2, \cdots, N)$ in $\mathcal{A}$ such that

$$\rho = \sum_{k=1}^{N} x_k \delta y_k. \tag{2.2}$$

Now we regard $\mathrm{Fun}(GL_q(n))$ as $\mathcal{A}$, and for simplicity, give it a special symbol Ω^0. To construct the one order differential calculus on quantum group $GL_q(n)$, what one first has to do is to determine a Ω^0-bimodule which is denoted by Ω^1. For this end, we introduce the convolution "$*$" on Ω^0. For $f \in \mathrm{Fun}^*(GL_q(n))$, the convolution "$*$" from Ω^0 to Ω^0 is defined by

$$f * (x) = (id \otimes f)\Delta x, \ x \in \mathrm{Fun}(GL_q(n)), \tag{2.3}$$

where id is the identity operator on Ω^0. Furthermore, we introduce two sets of functionals on Ω^0 as follows:

$$\text{(i)} \quad \nabla_{ij} := \frac{1}{\chi}(S^*(l^-_{ik})l^+_{kj} - \delta_{ij}\varepsilon), \quad i, j = 1, 2, \cdots, n, \tag{2.4}$$

$$\text{(ii)} \quad \theta_{ijkl} := S^*(l^-_{ki})l^+_{jl}, \quad i, j, k, l = 1, 2, \cdots, n. \tag{2.5}$$

For the operators $*$, ∇_{ij}, θ_{ijkl}, we have

Proposition 2.1 For $\forall x, y \in \Omega^0, i, j, k, l = 1, 2, \cdots, n$ the formulas

$$\text{(i)} \quad \nabla_{ij}(1) = 0, \quad \theta_{ijkl}(1) = \delta_{ik}\delta_{jl}, \tag{2.6}$$

$$\text{(ii)} \quad \Delta^*\nabla_{ij} = \nabla_{uv} \otimes \theta_{uvij} + \varepsilon \otimes \nabla_{ij},$$

$$\Delta^*\theta_{ijkl} = \theta_{ijuv} \otimes \theta_{uvkl}, \tag{2.7}$$

$$\text{(iii)} \quad \nabla_{ij} * (xy) = (\nabla_{uv} * x)(\theta_{uvij} * y) + x(\nabla_{ij} * y),$$

$$\theta_{ijkl} * (xy) = (\theta_{ijuv} * x)(\theta_{uvkl} * y) \tag{2.8}$$

hold.

Proof: Now we prove the first equation of (2.7). A directly calculation shows

$$\begin{aligned}
\Delta^*\nabla_{ij} &= \tfrac{1}{\chi}\Delta^*(S^*(l^-_{ik}))\Delta^* l^+_{kj} - \tfrac{1}{\chi}\delta_{ij}\varepsilon \otimes \varepsilon \\
&= \tfrac{1}{\chi}S^*(l^-_{uk})l^+_{kv} \otimes S^*(l^-_{iu})l^+_{vj} - \tfrac{1}{\chi}\delta_{ij}\varepsilon \otimes \varepsilon \\
&= (\nabla_{uv} + \tfrac{1}{\chi}\delta_{uv}\varepsilon) \otimes S^*(l^-_{iu})l^+_{vj} - \tfrac{1}{\chi}\delta_{ij}\varepsilon \otimes \varepsilon \\
&= \nabla_{uv} \otimes \theta_{uvij} + \tfrac{1}{\chi}\varepsilon \otimes (\delta_{uv}S^*(l^-_{iu})l^+_{vj} - \delta_{ij}\varepsilon) \\
&= \nabla_{uv} \otimes \theta_{uvij} + \varepsilon \otimes \nabla_{ij}.
\end{aligned}$$

Next we prove the first equation of (2.8). Let $\Delta x = x_{1,\alpha} \otimes x_{2,\alpha}$, $\Delta y = y_{1,\beta} \otimes y_{2,\beta}$. Since

$$\begin{aligned}
\nabla_{ij} * (x \cdot y) &= (id \otimes \nabla_{ij})\Delta(xy) \\
&= (id \otimes \nabla_{ij})\Delta x \Delta y \\
&= x_{1,\alpha}y_{1,\beta}\nabla_{ij}(x_{2,\alpha}y_{2,\beta}) \\
&= x_{1,\alpha}y_{1,\beta}\Delta^*\nabla_{ij}(x_{2,\alpha} \otimes y_{2,\beta}),
\end{aligned}$$

applying the first equation of (2.7), we have

$$\begin{aligned}
\nabla_{ij} * (xy) &= x_{1,\alpha}y_{1,\beta}(\nabla_{uv}(x_{2,\alpha})\theta_{uvij}(y_{2,\beta}) + \varepsilon(x_{2,\beta})\nabla_{ij}(y_{2,\beta})) \\
&= x_{1,\alpha}\nabla_{uv}(x_{2,\alpha})y_{1,\beta}\theta_{uvij}(y_{2,\beta}) + x_{1,\alpha}\varepsilon(x_{2,\alpha})y_{1,\beta}\nabla_{ij}(y_{2,\beta}) \\
&= (\nabla_{uv} * x)(\theta_{uvij} * y) + x(\nabla_{ij} * y).
\end{aligned}$$

As for the remained formulae, we leave them to readers.

From (1.38) and (1.7) it follows that

$$< S^*(L^-)L^-, T > = < \varepsilon \cdot E, T > = (\delta_{ij}\varepsilon(T))_{1 \le i,j \le n} = E_{n^2},$$

where E_{n^2} is the unit matrix of order n^2. On the other hand, due to (1.6) one has

$$< S^*(L^-)L^-, T > = < S^*(L^-), T > < L^-, T > = < S^*(L^-), T > \lambda_-^{-1}R^-.$$

So we obtain

$$< S^*(L^-), T > = \lambda_- R_q,$$

i.e.

$$S^*(l^-_{ij})(T) = \lambda_- r_{ij}. \tag{2.9}$$

Combining (1.4), (1.15), (1.17) with (2.9), and letting $r = \lambda_+\lambda_-$, we can get, if $i = j$,

$$\begin{aligned}(S(l^-_{ik})l^+_{kj})(T) &= S(l^-_{ik})(T)l^+_{kj}(T)\\ &= r\textstyle\sum_{k<i} r_{ik}r^+_{kj} + rr_{ii}r^+_{ii}\\ &= r\chi^2\textstyle\sum_{k<i} e_{kk} + r(E_n + (q^2-1))e_{ii},\end{aligned}$$

and if $i \neq j$,

$$(S(l^-_{ik})l^+_{kj})(T) = r\chi e_{ji}.$$

Thus

$$\nabla_{ij}(T) = \begin{cases} re_{ji}, & i \neq j,\\ \frac{1}{\chi}[(r-1)E_n + r(q^2-1)e_{ij} + r\chi^2\sum_{k<i} e_{kk}], & i = j.\end{cases} \tag{2.10}$$

Similarly, it follows from (1.25) and (1.26) that

$$\nabla_{ij}(t) = \frac{1}{\chi}(\frac{1}{r^nq^2} - 1)\delta_{ij}. \tag{2.11}$$

If we arrange $\nabla_{ij}(t_{kl})$ as a matrix of $n \times n$ blocks,

$$\nabla(T) = (\nabla_{ij}(T))_{1\le i,j\le n},$$

where the submatrices are

$$\nabla_{ij}(T) = (\nabla_{ij}(t_{kl}))_{1\le k,l\le n},\ 1 \le i,j \le n.$$

Then

$$\begin{aligned}\nabla(T) &= \frac{1}{\chi}(S^*(L^-)L^+(T) - \varepsilon\cdot E(T))\\ &= \frac{1}{\chi}(< S^*(L^-), T >< L^+, T > -E_{n^2})\\ &= \frac{1}{\chi}(rR_qR^+ - E_{n^2})\\ &= \frac{1}{\chi}(rR_qPR_qP - E_{n^2}).\end{aligned} \tag{2.12}$$

And

$$(\nabla_{ij}(t))_{1\le i,j\le n} = \frac{1}{\chi}(\frac{1}{r^nq^2} - 1)E. \tag{2.13}$$

Now we apply the matrices $\nabla(T)$ and $\frac{1}{\chi}(\frac{1}{r^nq^2} - 1)E$ to construct another matrix. Let

$$M(\lambda) = (M^{n(k-1)+l}_{n(i-1)+j})_{1\le k,l,i,j\le n} = (\nabla_{ij}(t_{kl}) + \lambda\delta_{kl}\nabla_{ij}(t))_{1\le k,l,i,j\le n}. \tag{2.14}$$

where λ is a complex parameter. Sometimes we also write the matrix $M(\lambda)$ as $M(\lambda) = (M^{kl}_{ij})_{1\le k,l,i,j\le n}$.

Proposition 2.2 If $r^nq^2 \neq 1$, for fixed q and r there always exists λ, s.t. the matrix $M(\lambda)$ is invertible.

Proof: If $i \neq j$, then one has

$$M^{n(k-1)+l}_{n(i-1)+j} = \frac{r}{\chi}\chi\delta_{k-j,i-l} = r\delta_{k-j,i-l}.$$

The above equation shows that there is only one non-zero element r in every row or column of $M(\lambda)$ except the rows $(n(k-1)+k)$ and columns $n(i-1)+i$ $(i,k = 1,2,\cdots,n)$. Hence, to determine whether or not the matrix $M(\lambda)$ is invertible we need only to consider the matrix $N(\lambda)$ of order n,

$$\begin{aligned}N(\lambda) &= (M^{n(k-1)+k}_{n(i-1)+i})_{1\le k,i\le n},\\ M^{n(k-1)+k}_{n(i-1)+i} &= \nabla_{ii}(t_{kk}) + \tfrac{\lambda}{\chi}(\tfrac{1}{r^nq^2} - 1).\end{aligned}$$

In fact, the expression of the matrix $N(\lambda)$ is

$$N(\lambda) = \frac{1}{\chi}\begin{pmatrix} b & a & a & \cdots & a\\ c & b & a & \cdots & a\\ c & c & b & \cdots & a\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ c & c & c & \cdots & b\end{pmatrix},$$

where

$$\begin{aligned}a &= r(\chi^2+1) - 1 + \lambda(\tfrac{1}{r^nq^2} - 1),\\ b &= rq^2 - 1 + \lambda(\tfrac{1}{r^nq^2} - 1),\\ c &= r - 1 + \lambda(\tfrac{1}{r^nq^2} - 1).\end{aligned}$$

Straightforward calculation gives

$$\begin{aligned}DetN(\lambda) &= \frac{(b-a)^{n-1}}{\chi^n}(b\frac{q^{2n}-1}{q^2-1} - q^2(b-a)\frac{q^{2n-2}-1}{q^2-1})\\ &= \frac{r^{n-1}}{\chi q^{n-1}}\left(b\frac{q^{2n}-1}{q^2-1} - (q^{2n-2}-1)\right).\end{aligned}$$

So, when $r^nq^2 \neq 1$, for fixed q and r we have λ such that $DetN(\lambda) \neq 0$.

Now we are going to construct a Ω^0-bimodule Ω^1. Define dt_{ij} and dt the one order differentials of the generators t_{ij} $(i,j = 1,2,\cdots,n)$ and t of Ω^0. And let Ω^1 be the left Ω^0-module generated by the elements ω^{kl} $(k,l = 1,2,\cdots,n)$, satisfying the following conditions:

$$S(t_{im})dt_{mj} = \nabla_{kl}(t_{ij})\omega^{kl}, \tag{2.15}$$

$$Det_qTdt = \nabla_{kl}(t)\omega^{kl}. \tag{2.16}$$

The right multiplication in the left module Ω^1 is defined by

$$\omega^{ij}\cdot x = (\theta_{ijkl} * x)\omega^{kl},\ \forall x \in \Omega^0, \tag{2.17}$$

so that Ω^1 is a bimodule of Ω^0. Due to the argument in the last section that θ_{ijkl} is a functional on Ω^0, the right multiplication is well defined. Namely,

$$\omega^{ij} \cdot x = (id \otimes \theta_{ijkl})\Delta x \omega^{kl}$$

is independent of the choice of the representation element x. Furthermore, by Proposition 2.1 this right multiplication is associable, i.e.

$$\omega^{ij}(xy) = (\omega^{ij}x)y, \quad \forall x, y \in \Omega^0.$$

And it is clear

$$\omega^{ij} \cdot 1 = \omega^{ij}.$$

So Ω^1 is a Ω^0-bimodule.

Definition 2.1 The differential operation d from Ω^0 to Ω^1 is defined by

$$dx = \nabla_{ij} * (x)\omega^{ij}. \tag{2.18}$$

Theorem 2.1 $\{\Omega^1, d\}$ constructed above is the first order differential calculus on Ω^0.

Proof: We need only to check (2.1) and (2.2) hold. Combining (2.8) in Proposition 2.1 with (2.17) one directly verifies d is a differential operator satisfying Leibnitz rule (2.1). To verify $\{\Omega^1, d\}$ satisfies (2.2) we need to prove ω^{ij} $(i, j = 1, 2, \cdots, n)$, the generators of Ω^1 , can be represented by the form $\sum_{k=1}^{N} x_k dy_k$, $x_k, y_k \in \Omega^0$. By (2.18), (2.15) and (2.16), we have

$$\begin{aligned} dt_{kl} &= (id \otimes \nabla_{ij})\Delta t_{kl}\omega^{ij}, \\ dt &= (id \otimes \nabla_{ij})\Delta t\omega^{ij}. \end{aligned}$$

Using the Proposition 2.2 we can find λ such that the matrix $M(\lambda)$ is nonsingular. So from

$$S(t_{km})dt_{ml} + \lambda\delta_{kl}Det_q T dt = (\nabla_{ij}(t_{kl}) + \lambda\delta_{kl}\nabla_{ij}(t))\omega^{ij},$$

one obtains

$$\omega^{ij} = M^{-1}(\lambda)^{n(i-1)+j}_{n(k-1)+l}[S(t_{km})dt_{ml} + \lambda\delta_{kl}Det_q T dt], \tag{2.19}$$

which implies that for $\{\Omega^1, d\}$, (2.2) holds.

It should be point out that the expression of ω^{ij} in (2.19) is independent of the parameter λ. This can be proved by a simple argument in linear algebra.

By (2.17), we also have the cross relation among dt_{ij}, dt and t_{ij}, t as follows,

$$\begin{aligned} dT \cdot z &= (\nabla_{ij} * T)\omega^{ij} z \\ &= (\nabla_{ij} * T)\theta_{ijkl} * (z) M^{-1}(\lambda)^{n(k-1)+l}_{n(\alpha-1)+\beta}(S(t_{\alpha\gamma})dt_{\gamma\beta} + \lambda\delta_{\alpha\beta}(Det_q T)dt), \\ dt \cdot z &= (\nabla_{ij} * t)\omega^{ij} z \\ &= (\nabla_{ij} * t)\theta_{ijkl} * (z) M^{-1}(\lambda)^{n(k-1)+l}_{n(\alpha-1)+\beta}(S(t_{\alpha\gamma})dt_{\gamma\beta} + \lambda\delta_{\alpha\beta}(Det_q T)dt), \end{aligned} \tag{2.20}$$

where $z = t_{uv}$ or t $(u, v = 1, 2, \cdots, n)$.

Remark 2. The case of $r^n q^2 = 1$ will be discussed in §5.

Proposition 2.3 Let $\mathbf{L} = S^*(L^-)L^+$. We have

$$R\mathbf{L}_1 R^+ \mathbf{L}_2 = \mathbf{L}_2 R \mathbf{L}_1 R^+$$

here $\mathbf{L}_1 = \mathbf{L} \otimes I$, $\mathbf{L}_2 = I \otimes \mathbf{L}$.

Proof: By Proposition 1.1 we easily obtain

$$\begin{aligned} &L_1^+ R^+ S^*(L_2^-) = S^*(L_2^-)R^+ L_1^+, \\ &RS^*(L_1^-)S^*(L_2^-) = S^*(L_2^-)S^*(L_1^-)R, \\ &R^+ L_1^+ L_2^+ = L_2^+ L_1^+ R^+, \\ &S^*(L_1^-)RL_2^+ = L_2^+ RS^*(L_1^-). \end{aligned}$$

Noticing $\mathbf{L}_1 = S^*(L_1^-)L_1^+$ and $\mathbf{L}_2 = S^*(L_2^-)L_2^+$, we have

$$\begin{aligned} R\mathbf{L}_1 R^+ \mathbf{L}_2 &= RS^*(L_1^-)L_1^+ R^+ S^*(L_2^-)L_2^+ \\ &= RS^*(L_1^-)S^*(L_2^-)R^+ L_1^+ L_2^+ \\ &= S^*(L_2^-)S^*(L_1^-)RL_2^+ L_1^+ R^+ \\ &= S^*(L_2^-)L_2^+ RS^*(L_1^-)L_1^+ R^+ \\ &= \mathbf{L}_2 R \mathbf{L}_1 R^+. \end{aligned}$$

Let

$$\begin{aligned} \mathbf{R}^{ij,kl}_{mn,uv} &= < \theta_{ijuv}, T_{mk}S(T_{ln}) >, \\ \mathbf{F}^{ij}_{kl,mn} &= \chi < \nabla_{mn}, T_{ki}S(T_{jl}) > . \end{aligned}$$

By Proposition 2.3, we obtain

Theorem 2.2 For $\mathbf{R}, \mathbf{F}$ defined in the above two formulae, there exist two sets of equations which are equivalent with each other:

$$\begin{aligned} \text{(i)}\quad & \nabla_k\nabla_l - \nabla_i\nabla_j \mathbf{R}^{ij}_{kl} = \nabla_m \mathbf{F}^m_{kl}, \\ & \nabla_k\theta_{ml} = \theta_{mi}\nabla_j \mathbf{R}^{ij}_{kl}, \\ & \mathbf{R}^{ij}_{kl}\theta_{ku}\theta_{lv} = \theta_{im}\theta_{jn}\mathbf{R}^{mn}_{uv}, \\ & \mathbf{F}^i_{jk}\theta_{ju}\theta_{kv} + \theta_{iu}\nabla_v = \nabla_m\theta_{in}\mathbf{R}^{mn}_{uv} + \theta_{in}\mathbf{F}^n_{uv}, \\ \text{(ii)}\quad & \mathbf{F}^w_{uk}\mathbf{F}^v_{wl} - \mathbf{F}^w_{us}\mathbf{F}^v_{wj}\mathbf{R}^{ij}_{kl} = \mathbf{F}^v_{uw}\mathbf{F}^w_{kl}, \\ & \mathbf{F}^w_{uz}\mathbf{R}^{mv}_{wl} = \mathbf{R}^{mw}_{ui}\mathbf{F}^v_{wj}\mathbf{R}^{ij}_{kl}, \\ & \mathbf{R}^{ij}_{kl}\mathbf{R}^{lm}_{uv}\mathbf{R}^{kw}_{ns} = \mathbf{R}^{jm}_{wl}\mathbf{R}^{iw}_{nk}\mathbf{R}^{kl}_{uv}, \\ & \mathbf{R}^{ij}_{kl}\mathbf{F}^k_{uv} + \mathbf{F}^i_{mn}\mathbf{R}^{mj}_{kl}\mathbf{R}^{nk}_{uv} = \mathbf{F}^j_{km}\mathbf{R}^{ik}_{un}\mathbf{R}^{mn}_{lv} + \mathbf{R}^{ij}_{uw}\mathbf{F}^w_{lv}. \end{aligned}$$

Here, for simplicity, we use one index instead of two in above equations, for example i stands for ii', etc.

Proof: Here we only prove the first equations of (i) and (ii), the proofs of remain equations are similar and can be found in [5]. By Proposition 2.3, we have

$$
\begin{aligned}
& \mathrm{L}_{cc'} R_{kc',dl'} \mathrm{L}_{dd'} R^{+}_{d'l',k'b'} = R_{kc,al} \mathrm{L}_{aa'} R^{+}_{a'l,k'b} \mathrm{L}_{bb'} \\
\Longleftrightarrow\ & \mathrm{L}_{cc'} < S^{*}(l^{-}_{kd}), t_{c'l'} > \mathrm{L}_{dd'} < l^{+}_{d'k'}, t_{l'b'} > = < S^{*}(l^{-}_{ka}), t_{cl} > \mathrm{L}_{aa'} < l^{+}_{a'k'}, t_{lb} > \mathrm{L}_{bb'} \\
\Longleftrightarrow\ & \mathrm{L}_{cc'} \mathrm{L}_{dd'} \theta_{dd'kk'}(t_{c'b'}) = \mathrm{L}_{aa'} \mathrm{L}_{bb'} \theta_{aa'kk'}(t_{cb}) \\
\Longleftrightarrow\ & \mathrm{L}_{cc'} \mathrm{L}_{dd'} \theta_{dd'kk'}(t_{c'b'}) \theta_{kk'uu'}(S(t_{b'w})) = \mathrm{L}_{aa'} \mathrm{L}_{bb'} \theta_{aa'kk'}(t_{cb}) \theta_{kk'uu'}(S(t_{b'w})) \\
\Longleftrightarrow\ & \mathrm{L}_{cc'} \mathrm{L}_{dd'} \theta_{dd'uu'}(t_{c'b'} S(t_{b'w})) = \mathrm{L}_{aa'} \mathrm{L}_{bb'} \theta_{aa'uu'}(t_{cb} S(t_{b'w})) \\
\Longleftrightarrow\ & \mathrm{L}_{cc'} \mathrm{L}_{dd'} \delta^{dd'}_{uu'} \delta^{c'}_{w} = \mathrm{L}_{aa'} \mathrm{L}_{bb'} \mathrm{R}^{aa'bb'}_{cwuu'} \\
\Longleftrightarrow\ & \mathrm{L}_{cc'} \mathrm{L}_{dd'} = \mathrm{L}_{aa'} \mathrm{L}_{bb'} \mathrm{R}^{aa'bb'}_{cc'dd'} \\
\Longleftrightarrow\ & (\chi \nabla_{cc'} + \delta_{cc'} \varepsilon)(\chi \nabla_{dd'} + \delta_{dd'} \varepsilon) = (\chi \nabla_{aa'} + \delta_{aa'} \varepsilon)(\chi \nabla_{bb'} + \delta_{bb'} \varepsilon) \mathrm{R}^{aa'bb'}_{cc'dd'}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\delta_{aa'} \mathrm{R}^{aa'bb'}_{cc'dd'} &= \delta_{aa'} \theta_{aa'dd'}(t_{cb} S(t_{b'c'}) \\
&= (\chi \nabla_{dd'} + \delta_{dd'} \varepsilon)(t_{cb} S(t_{b'c'}) \\
&= \mathrm{F}^{bb'}_{cc'dd'} + \delta_{dd'} \delta_{bc} \delta_{b'c'},
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_{bb'} \mathrm{R}^{aa'bb'}_{cc'dd'} &= \delta_{bb'} \theta_{aa'dd'}(t_{cb} S(t_{b'c'}) \\
&= \delta_{ad} \delta_{a'd'} \delta_{cc'},
\end{aligned}
$$

$$
\delta_{aa'} \delta_{bb'} \mathrm{R}^{aa'bb'}_{cc'dd'} = \delta_{cc'} \delta_{dd'},
$$

we have

$$
\nabla_{cc'} \nabla_{dd'} - \nabla_{aa'} \nabla_{bb'} \mathrm{R}^{aa'bb'}_{cc'dd'} = \nabla_{ee'} \mathrm{F}^{ee'}_{cc'dd'}.
$$

Therefore the first equation of (i) is proved, and by applying both sides of it to $t_{uv} S(t_{v'u'})$, we have on the left side

$$
\begin{aligned}
& (\nabla_{cc'} \nabla_{dd'} - \nabla_{aa'} \nabla_{bb'} \mathrm{R}^{aa'bb'}_{cc'dd'})(t_{uv} S(t_{v'u'})) \\
= \ & (\nabla_{cc'} \otimes \nabla_{dd'} - \nabla_{aa'} \otimes \nabla_{bb'} \mathrm{R}^{aa'bb'}_{cc'dd'})(\Delta(t_{uv} S(t_{v'u'}))) \\
= \ & (\nabla_{cc'} \otimes \nabla_{dd'} - \nabla_{aa'} \otimes \nabla_{bb'} \mathrm{R}^{aa'bb'}_{cc'dd'})((t_{uw} S(t_{w'u'})) \otimes (t_{wv} S(t_{v'w'}))) \\
= \ & \mathrm{F}^{ww'}_{uu'cc'} \mathrm{F}^{vv'}_{ww'dd'} - \mathrm{F}^{ww'}_{uu'aa'} \mathrm{F}^{vv'}_{ww'bb'} \mathrm{R}^{aa'bb'}_{cc'dd'},
\end{aligned}
$$

and on the right side

$$
\nabla_{ee'}(t_{uv} S(t_{v'u'})) \mathrm{F}^{ee'}_{cc'dd'} = \mathrm{F}^{vv'}_{uu'ee'} \mathrm{F}^{ee'}_{cc'dd'},
$$

i.e.

$$
\mathrm{F}^{ww'}_{uu'cc'} \mathrm{F}^{vv'}_{ww'dd'} - \mathrm{F}^{ww'}_{uu'aa'} \mathrm{F}^{vv'}_{ww'bb'} \mathrm{R}^{aa'bb'}_{cc'dd'} = \mathrm{F}^{vv'}_{uu'ee'} \mathrm{F}^{ee'}_{cc'dd'}.
$$

Therefore, the first equation of (ii) is true.

In Theorem 2.2, the first set of equations is related to the Lie bracket of the generators of Lie algebra and the second set is a deformation of the Jacobian identity of the structure constants of the classical Lie group $GL(n)$.

§3. Bicovariant differential calculus on $GL_q(n)$

Definition 3.1 Suppose (Γ, δ) is the one order differential calculus on Hopf algebra $\mathcal{A}$. For arbitrary $x_k, y_k \in \mathcal{A}$ $(k = 1, 2, \cdots, N)$ satisfying $x_k \delta y_k = 0$, if $\Delta x_k (id \otimes \delta) \Delta y_k = 0$, then we call (Γ, δ) left-covariant; If $\Delta x_k (\delta \otimes id) \Delta y_k = 0$, then we call (Γ, δ) right-covariant; If (Γ, δ) is not only left-covariant but also right-covariant, then we call (Γ, δ) bicovariant.

Theorem 3.1 The differential calculus (Ω^1, d) on $GL_q(n)$ given in §2 is left-covariant.

Proof: According to the definition of left-covariant, we need only to prove that for arbitrary $x_k, y_k \in \Omega^0$ $(k = 1, 2, \cdots, N)$, if $x_k dy_k = 0$, then $\Delta x_k (id \otimes d) \Delta y_k = 0$.

Suppose $\Delta y_k = y^{(k)}_{1,\alpha} \otimes y^{(k)}_{2,\alpha}$. Then

$$
\begin{aligned}
\Delta x_k (id \otimes d) \Delta y_k &= \Delta x_k (id \otimes id \otimes \nabla_{ij})(id \otimes \Delta) \Delta y_k \omega^{ij} \\
&= \Delta x_k (id \otimes id \otimes \nabla_{ij})(\Delta \otimes id) \Delta y_k \omega^{ij} \\
&= \sum_{k=1}^{N} \Delta x_k (id \otimes id \otimes \nabla_{ij})((\Delta y^{(k)}_{1,\alpha}) \otimes y^{(k)}_{2,\alpha}) \omega^{ij} \\
&= \sum_{k=1}^{N} \Delta x_k (\Delta y^{(k)}_{1,\alpha} \nabla_{ij} (y^{(k)}_{2,\alpha})) \omega^{ij} \\
&= \sum_{k=1}^{N} \Delta x_k \Delta (y^{(k)}_{1,\alpha} \nabla_{ij} (y^{(k)}_{2,\alpha})) \omega^{ij} \\
&= \Delta x_k \Delta (\nabla_{ij} * y_k) \omega^{ij} \\
&= \Delta (x_k (\nabla_{ij} * y_k)) \omega^{ij}.
\end{aligned}
$$

Since $0 = x_k dy_k = x_k (\nabla_{ij} * y_k) \omega^{ij}$, we have

$$
x_k (\nabla_{ij} * y_k) = 0, \qquad \forall i, j = 1, 2, \cdots, n.
$$

Thus

$$
\Delta x_k (id \otimes d) \Delta y_k = 0.
$$

Therefore, Theorem 3.1 holds.

Now we introduce the concept of ad-invariant. First let two linear mappings $r, s : \Omega^0 \otimes \Omega^0 \longrightarrow \Omega^0 \otimes \Omega^0$ be defined by the following formulas: for $\forall x, y \in \Omega^0$,

$$
r(x \otimes y) = m^{\otimes}((x \otimes 1) \otimes \Delta y), \tag{3.1}
$$

$$
s(x \otimes y) = m^{\otimes}((1 \otimes x) \otimes \Delta y), \tag{3.2}
$$

where $m^{\otimes}$ is the multiplication on $\Omega^0 \otimes \Omega^0$, i.e.

$$
m^{\otimes}((x \otimes y) \otimes (z \otimes w)) = xz \otimes yw.
$$

It can be proved that r, s are bijections, and (see [1])

$$
r^{-1}(x \otimes y) = m^{\otimes}((x \otimes 1) \otimes (S \otimes id) \Delta y). \tag{3.3}
$$

Definition 3.2 We call a linear subspace $\mathcal{B}$ of Ω^0 ad-invariant if

$$ad(\mathcal{B}) \subset \mathcal{B} \otimes \Omega^0$$

where the linear mapping $ad : \Omega^0 \longrightarrow \Omega^0 \otimes \Omega^0$ is defined by

$$ad(x) = s(r^{-1}(1 \otimes x)). \tag{3.4}$$

Proposition 3.1 Let $\mathcal{H} = ker\,\varepsilon \cap (\cap_{i,j=1}^{n} ker \nabla_{ij})$. Then $\mathcal{H}$ is a right ideal of Ω^0.

Proof: Assume $x \in \mathcal{H}$, i.e. the equations

$$\varepsilon(x) = 0, \quad \nabla_{ij}(x) = 0, \quad i,j = 1,2,\cdots,n$$

hold. For $\forall y \in \Omega^0$ using (2.7) of Proposition 2.1, we have

$$\begin{aligned} \nabla_{ij}(xy) &= \Delta^*(\nabla_{ij})(x \otimes y) \\ &= \nabla_{uv}(x)\theta_{uvij}(y) + \varepsilon(x)\nabla_{ij}(y) \\ &= 0 \end{aligned}$$

and

$$\varepsilon(xy) = \varepsilon(x)\varepsilon(y) = 0.$$

So $xy \in \mathcal{H}$.

Now take such a parameter λ that the matrix $M(\lambda)$ is invertible and denote the dual basis of ∇_{kl} by

$$S_{kl} = M^{-1}(\lambda)^{kl}_{ij}(t_{ij} + \lambda\delta_{ij}t),$$

i.e. $\nabla_{ij}(S_{kl}) = \delta_{ik}\delta_{jl}$. For the operator ad we have

Proposition 3.2 The formula

$$\nabla_{uu'}(t_{ij}t_{kl})S_{uu'} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} = ad(\nabla_{uu'}(t_{ab}t_{cd})S_{uu'}) \tag{3.5}$$

holds.

Proof: Due to Proposition 2.1 and the definition of the operator ad, the right side of (3.5) is

$$\begin{aligned} & ad[\nabla_{uu'}(t_{ab}t_{cd})M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} + \lambda\delta_{vv'}t)] \\ = \;& \nabla_{uu'}(t_{ab}t_{cd})M^{-1}(\lambda)^{uu'}_{vv'}(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1) \\ = \;& [\nabla_{ww'}(t_{ab})\theta_{ww'uu'}(t_{cd}) + \delta_{ab}\nabla_{uu'}(t_{cd})]M^{-1}(\lambda)^{uu'}_{vv'} \\ & \cdot(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1) \\ = \;& [\nabla_{ww'}(t_{ab})\theta_{ww'uu'}(t_{cd}) + \delta_{ab}M(\lambda)^{cd}_{uu'} - \lambda\delta_{ab}\delta_{cd}\nabla_{uu'}(t)] \\ & \cdot M^{-1}(\lambda)^{uu'}_{vv'}(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1) \\ = \;& [\nabla_{ww'}(t_{ab})\theta_{ww'uu'}(t_{cd}) - \lambda\delta_{ab}\delta_{cd}\nabla_{uu'}(t)]M^{-1}(\lambda)^{uu'}_{vv'} \\ & \cdot(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1) + \delta_{ab}(t_{kl} \otimes S(t_{ck})t_{ld} + \lambda\delta_{cd}t \otimes 1). \end{aligned}$$

On the other hand, the left side of (3.5) is

$$\begin{aligned} & \nabla_{uu'}(t_{ij}t_{kl})S_{uu'} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} \\ = \;& [\nabla_{ww'}(t_{ij})\theta_{ww'uu'}(t_{kl}) + \delta_{ij}\nabla_{uu'}(t_{kl})]S_{uu'} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} \\ = \;& \nabla_{ww'}(t_{ij})\theta_{ww'uu'}(t_{kl})S_{uu'} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} + \delta_{ab}\nabla_{uu'}(t_{kl})S_{uu'} \otimes S(t_{ck})t_{ld} \\ = \;& I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \nabla_{ww'}(t_{ij})\theta_{ww'uu'}(t_{kl})S_{uu'} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld}, \\ I_2 &= \delta_{ab}\nabla_{uu'}(t_{kl})S_{uu'} \otimes S(t_{ck})t_{ld}. \end{aligned}$$

We also have

$$\begin{aligned} & \nabla_{ww'}(t_{ij})S(t_{ai})t_{jb} \\ = \;& [S^*(l^-_{wv})l^+_{vw'}(t_{ij}) - \delta_{ww'}\varepsilon(t_{ij})]S(t_{ai})t_{jb} \\ = \;& rS(t_{ai})R_{wi,vk}R^+_{vk,w'j}t_{jb} - \delta_{ww'}\delta_{ab}1 \\ = \;& rt_{wi}R_{ia,vk}R^+_{vk,jb}S(t_{jw'}) - \delta_{ww'}\delta_{ab}1 \\ = \;& [S^*(l^-_{iv})l^+_{vj}(t_{ab}) - \delta_{ij}\varepsilon(t_{ab})]t_{wi}S(t_{jw'}) \\ = \;& \nabla_{ij}(t_{ab})t_{wi}S(t_{jw'}), \end{aligned}$$

i.e.

$$\nabla_{ww'}(t_{ij})S(t_{ai})t_{jb} = \nabla_{ij}(t_{ab})t_{wi}S(t_{jw'}). \tag{3.6}$$

Here

$$\begin{aligned} (R_{wi,vk})_{1\le i,k\le n} &= r_{ww}, \\ (R^+_{vk,wj})_{1\le k,j\le n} &= r^+_{vw}. \end{aligned}$$

Applying (3.6) to I_1, we have

$$I_1 = \nabla_{ij}(t_{ab})\theta_{ww'uu'}(t_{kl})S_{uu'} \otimes S(t_{ck})t_{wi}S(t_{jw'})t_{ld}.$$

Since

$$\begin{aligned} & \theta_{ww'uu'}(t_{kl})S(t_{ck})t_{wi}S(t_{jw'})t_{ld} \\ = \;& r(R_{uk,wm}S(t_{ck})t_{wi})(R^+_{w'm,u'l}S(t_{jw'})t_{ld}) \\ = \;& r(\delta_{u\alpha}S(t_{ck})R_{\alpha k,w\beta}t_{wi}\delta_{\beta m})(S(t_{jw'})\delta_{m\alpha}R^+_{w'\alpha,\beta l}\delta_{\beta u'}t_{ld}) \\ = \;& r(\delta_{u\alpha}S(T)r_{\alpha w}t_{wi}E)_{cm}(S(t_{jw'})Er^+_{w'\beta}\delta_{\beta u'}T)_{md} \\ = \;& r(E \otimes S(T) \cdot R_q \cdot T \otimes E)_{uc,im}(S(T) \otimes E \cdot R^+ \cdot E \otimes T)_{jm,u'd} \\ = \;& r(S(T)_2RT_1)_{uc,im}(S(T)_1R^+T_2)_{jm,u'd} \\ = \;& r(T_1R_qS(T)_2)_{uc,im}(T_2R^+S(T)_1)_{jm,u'd} \\ = \;& rR_{kc,iw}t_{uk}S(t_{wm})R^+_{jw',ld}t_{mw'}S(t_{lu'}) \\ = \;& \theta_{ijkl}(t_{cd})t_{uk}S(t_{lu'}), \end{aligned}$$

in which we apply QYBE and

$$\begin{aligned} S(T_1) &= S(T \otimes E) = S(T) \otimes E = S(T)_1, \\ S(T_2) &= S(E \otimes T) = E \otimes S(T) = S(T)_2, \end{aligned}$$

we obtain

$$I_1 = \nabla_{ij}(t_{ab})\theta_{ijkl}(t_{cd})S_{uu'} \otimes t_{uk}S(t_{lu'}). \tag{3.7}$$

By (3.6), we have

$$(\nabla_{ww'}(t_{ij}) + \lambda\delta_{ij}\delta_{ww'}C_0)S(t_{ai})t_{jb} = (\nabla_{ij}(t_{ab}) + \lambda\delta_{ab}\delta_{ij}C_0)t_{wi}S(t_{jw'}),$$

where $\delta_{ij}C_0 = \nabla_{ij}(t)$, i.e.

$$M(\lambda)^{ij}_{ww'}S(t_{ai})t_{jb} = M(\lambda)^{ab}_{ij}t_{wi}S(t_{jw'}).$$

Therefore, one obtains

$$M^{-1}(\lambda)^{kl}_{ab}S(t_{ai})t_{jb} = M^{-1}(\lambda)^{ww'}_{ij}t_{wk}S(t_{lw'}),$$

or

$$M^{-1}(\lambda)^{uu'}_{vv'}t_{uk}S(t_{lu'}) = M^{-1}(\lambda)^{kl}_{ww'}S(t_{wv})t_{v'w'}. \tag{3.8}$$

Thus the equation (3.8) is applied to rewrite I_1 as

$$\begin{aligned} I_1 &= \nabla_{ij}(t_{ab})\theta_{ijkl}(t_{cd})M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} + \lambda\delta_{vv'}t) \otimes t_{uk}S(t_{lu'}) \\ &= \nabla_{ij}(t_{ab})\theta_{ijkl}(t_{cd})M^{-1}(\lambda)^{kl}_{ww'}(t_{vv'} + \lambda\delta_{vv'}t) \otimes S(t_{wv})t_{v'w'} \\ &= \nabla_{ww'}(t_{ab})\theta_{ww'uu'}(t_{cd})M^{-1}(\lambda)^{uu'}_{vv'}(t_{kl} + \lambda\delta_{kl}t) \otimes S(t_{vk})t_{lv'} \\ &= \nabla_{ww'}(t_{ab})\theta_{ww'uu'}(t_{cd})M^{-1}(\lambda)^{uu'}_{vv'}(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1). \end{aligned}$$

Similarly, I_2 can be rewritten as

$$\begin{aligned} I_2 &= \delta_{ab}(\nabla_{uu'}(t_{kl}) + \lambda\delta_{kl}\nabla_{uu'}(t) - \lambda\delta_{kl}\nabla_{uu'}(t)) \\ &\quad \cdot M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} + \lambda\delta_{vv'}t) \otimes S(t_{ck})t_{ld} \\ &= \delta_{ab}(t_{kl} + \lambda\delta_{kl}t) \otimes S(t_{ck})t_{ld} \\ &\quad -\lambda\delta_{ab}\delta_{kl}\nabla_{uu'}(t)M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} + \lambda\delta_{vv'}t) \otimes S(t_{ck})t_{ld} \\ &= \delta_{ab}(t_{kl} \otimes S(t_{ck})t_{ld} + \lambda\delta_{cd}t \otimes 1) \\ &\quad -\lambda\delta_{ab}\delta_{cd}\nabla_{uu'}(t)M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} + \lambda\delta_{vv'}t) \otimes 1. \end{aligned}$$

By (3.8), we have

$$\begin{aligned} & \nabla_{uu'}(t)M^{-1}(\lambda)^{uu'}_{vv'}t_{vv'} \otimes 1 \\ = \; & \delta_{uu'}C_0M^{-1}(\lambda)^{uu'}_{vv'}t_{vv'} \otimes 1 \\ = \; & \delta_{kl}C_0M^{-1}(\lambda)^{uu'}_{vv'}t_{vv'} \otimes t_{uk}S(t_{lu'}) \\ = \; & \nabla_{kl}(t)M^{-1}(\lambda)^{kl}_{ww'}t_{vv'} \otimes S(t_{wv})t_{v'w'} \\ = \; & \nabla_{uu'}(t)M^{-1}(\lambda)^{uu'}_{vv'}t_{kl} \otimes S(t_{vk})t_{lv'}. \end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned} I_2 = \; & \delta_{ab}(t_{kl} \otimes S(t_{ck})t_{ld} + \lambda\delta_{cd}t \otimes 1) \\ & -\lambda\delta_{ab}\delta_{cd}\nabla_{uu'}(t)M^{-1}(\lambda)^{uu'}_{vv'}(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1). \end{aligned}$$

We complete the proof of the Proposition 3.2.

Theorem 3.2 The differential calculus (Ω^1, d) on $GL_q(n)$ given in §2 is right-covariant.

In Theorem 1.8 of [1], S.L. Woronowicz provided a theorem to decide whether a differential calculus is bicovariant. The theorem can also be said as "*Let (Γ, δ) be a left-covariant first order differential calculus. Then (Γ, d) is bicovariant if and only if $\mathcal{H}$ is ad-invariant.*" Based on this theorem, in order to prove the differential calculus on Ω^0 we have provided is bicovariant, it is sufficient to verify $\mathcal{H}$ is ad-invariant.

Proposition 3.3 Let $\mathcal{H} = ker\,\varepsilon \cap (\cap_{i,j=1}^{n} ker\nabla_{ij})$. Then $\mathcal{H}$ is ad-invariant.

Proof: It is easy to check that

$$\begin{aligned} & t_{ij}t_{kl} - \nabla_{uv}(t_{ij}t_{kl})S_{uv} - C_{ijkl}1 \in \mathcal{H}, \\ & t - \nabla_{uv}(t)S_{uv} - \hat{C}1 \in \mathcal{H}, \end{aligned}$$

where

$$\begin{aligned} C_{ijkl} &= \varepsilon(t_{ij}t_{kl} - \nabla_{uv}(t_{ij}t_{kl})S_{uv}), \\ \hat{C} &= \varepsilon(t - \nabla_{uv}(t)S_{uv}). \end{aligned}$$

Denote by $\tilde{\mathcal{H}}$ the right ideal generated by $t_{ij}t_{kl} - \nabla_{uv}(t_{ij}t_{kl})S_{uv} - C_{ijkl}1$ $(i, j, k, l = 1, 2, \cdots, n)$ and $t - \nabla_{uv}(t)S_{uv}\hat{C}1$, and denote the set of the generators by Λ. Obviously, $\tilde{\mathcal{H}} \subseteq \mathcal{H}$.

Now we define an equivalent relation in Ω^0, for $\xi, \eta \in \Omega^0$, we say ξ and η are equivalent or $\xi \sim \eta$ if $\xi - \eta \in \tilde{\mathcal{H}}$.

An arbitrary element ρ of $\mathcal{H}$ can be represented by a polynomial of t_{ij} $(i, j = 1, 2, \cdots, n)$ and t. By the definitions of the generators in Λ, we known that any two order polynomial of t_{ij} $(i, j = 1, 2, \cdots, n)$ and one order polynomial of t is equivalent to a one order polynomial of S_{kl} $(k, l = 1, 2, \cdots, n)$, so ρ is equivalent to $a_{kl}S_{kl}$, $a_{kl} \in \mathbb{C}$, i.e.,

$$\begin{aligned} 0 &= \varepsilon(\rho) = \varepsilon(a_{kl}S_{kl}), \\ 0 &= \nabla_{ij}(\rho) = \nabla_{ij}(a_{kl}S_{kl}), \quad i, j = 1, 2, \cdots, n. \end{aligned}$$

Since S_{kl} $(k, l = 1, 2, \cdots, n)$ is the dual basis of ∇_{ij} $(i, j = 1, 2, \cdots, n)$, $a_{kl} = 0$, i.e. $\rho \in \tilde{\mathcal{H}}$. So we have proved $\mathcal{H} \subseteq \tilde{\mathcal{H}}$, therefore $\mathcal{H} = \tilde{\mathcal{H}}$.

Now we prove Λ is ad-invariant, i.e. $ad\Lambda \subset \Lambda \otimes \Omega^0$. By the definition of ad,

$$ad(t_{ij}) = t_{kl} \otimes S(t_{ik})t_{lj}, \tag{3.10}$$

$$ad(t_{ab}t_{cd}) = t_{ij}t_{kl} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld}, \tag{3.11}$$

$$ad(t) = t \otimes 1. \tag{3.12}$$

By the Proposition 3.2, we have

$$\begin{aligned} & C_{ijkl}1 \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} \\ = \; & \varepsilon(t_{ij}t_{kl} - \nabla_{uv}(t_{ij}t_{kl})S_{uv})1 \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} \\ = \; & \delta_{ij}\delta_{kl}1 \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} - (\varepsilon \otimes id)ad(\nabla_{uu'}(t_{ab}t_{cd})S_{uu'}) \\ = \; & \delta_{ab}\delta_{cd}1 \otimes 1 - \varepsilon(\nabla_{uu'}(t_{ab}t_{cd})S_{uu'})1 \otimes 1 \\ = \; & C_{abcd}1 \otimes 1\,. \end{aligned} \tag{3.13}$$

Therefore, by the Proposition 3.2 and (3.11),

$$\begin{aligned} & ad(t_{ab}t_{cd} - \nabla_{uu'}(t_{ab}t_{cd})S_{uu'} - C_{abcd}1) \\ = \; & t_{ij}t_{kl} \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} - ad(\nabla_{uu'}(t_{ab}t_{cd})S_{uu'}) - C_{abcd}1 \otimes 1 \\ = \; & (t_{ij}t_{kl} - \nabla_{uu'}(t_{ij}t_{kl})S_{uu'} - C_{ijkl}1) \otimes S(t_{ck})S(t_{ai})t_{jb}t_{ld} \\ \in \; & \Lambda \otimes \Omega^0 \end{aligned}$$

holds. By (3.9) we have

$$\begin{aligned} & ad(t - \nabla_{uu'}(t)S_{uu'} - \hat{C}1) \\ = \; & ad(t - \delta_{uu'}C_0M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} + \lambda\delta_{vv'}t) - \hat{C}1) \\ = \; & t \otimes 1 - \delta_{uu'}C_0M^{-1}(\lambda)^{uu'}_{vv'}(t_{kl} \otimes S(t_{vk})t_{lv'} + \lambda\delta_{vv'}t \otimes 1) - \hat{C}1 \otimes 1 \\ = \; & t \otimes 1 - \nabla_{uu'}(t)C_0M^{-1}(\lambda)^{uu'}_{vv'}(t_{vv'} \otimes 1 + \lambda\delta_{vv'}t \otimes 1) - \hat{C}1 \otimes 1 \\ = \; & (t - \nabla_{uu'}(t)S_{uu'} - \hat{C}1) \otimes 1. \end{aligned}$$

Therefore we have proved

$$ad\Lambda \subset \Lambda \otimes \Omega^0.$$

Let ξ_i $(i = 1, 2, \cdots, n^4+1)$ be the n^4+1 generators in Λ. Take $\xi \in \Lambda$. For $\forall\eta \in \Omega^0$, one has

$$\Delta\xi = \xi_{1,\alpha} \otimes \xi_{2,\alpha}, \quad \Delta\eta = \eta_{1,\beta} \otimes \eta_{2,\beta},$$
$$\Delta(\xi\eta) = (\xi\eta)_{1,r} \otimes (\xi\eta)_{2,r} = \xi_{1,\alpha}\eta_{1,\beta} \otimes \xi_{2,\alpha}\eta_{2,\beta}.$$

Then

$$\begin{aligned} ad(\xi) &= s(r^{-1}(1 \otimes \xi)) \\ &= s(m^{\otimes}((1 \otimes 1) \otimes (S \otimes id)\Delta\xi)) \\ &= s(S(\xi_{1,\alpha}) \otimes \xi_{2,\alpha}) \\ &= m^{\otimes}(1 \otimes S(\xi_{1,\alpha}) \otimes \Delta\xi_{2,\alpha}) \\ &= (1 \otimes S(\xi_{1,\alpha}))\Delta\xi_{2,\alpha}. \end{aligned}$$

Similarly,

$$\begin{aligned} ad(\xi\eta) &= [1 \otimes S((\xi\eta)_{1,r})]\Delta((\xi\eta)_{2,r}) \\ &= [1 \otimes S(\xi_{1,\alpha}\eta_{1,\beta})]\Delta(\xi_{2,\alpha}\eta_{2,\beta}) \\ &= [1 \otimes S(\eta_{1,\beta})S(\xi_{1,\alpha})]\Delta(\xi_{2,\alpha})\Delta(\eta_{2,\beta}) \\ &= (1 \otimes S(\eta_{1,\beta}))ad(\xi)\Delta(\eta_{2,\beta}). \end{aligned}$$

Since $ad\xi \in \Lambda \otimes \Omega^0$, one obtains

$$ad(\xi) = \sum_{i=1}^{n^4+1} \xi_i \otimes x_i, \qquad x_i \in \Omega^0, \; i = 1, 2, \cdots, n^4+1 \; .$$

Thus

$$\begin{aligned} ad(\xi\eta) &= (1 \otimes S(\eta_{1,\beta}))(\xi_i \otimes x_i)\Delta(\eta_{2,\beta}) \\ &= (\xi_i \otimes S(\eta_{1,\beta})x_i)\Delta(\eta_{2,\beta}). \end{aligned}$$

Therefore $ad(\xi\eta) \in \mathcal{H} \otimes \Omega^0$. By the linear property of ad, we know

$$ad(\mathcal{H}) \subset \mathcal{H} \otimes \Omega^0,$$

i.e. $\mathcal{H}$ is ad-invariant.

Therefore, by Theorem 1.8 in [1], we have proved the differential calculus given in §2 is bicovariant.

§4. Quantum de Rham complex on $GL_q(n)$

Let Γ be a $\mathcal{A}$-bimodule consisting of all differential forms of one order on an associative algebra $\mathcal{A}$ with unit. Let $\Gamma^{\otimes n}$ be the n-fold tensor product of Γ. If $\mathcal{A}$ is commutative, for example the algebra consisting of all $\mathbf{C}^\infty$ functions on a smooth manifold, then the de Rham complex on $\mathcal{A}$ can be defined as follows.

$$\Gamma^\wedge = \Gamma^\otimes/N, \; \Gamma^\otimes = \oplus_{i=0}^{\infty}\Gamma^{\otimes i} \; , \tag{4.1}$$

where $\Gamma^0 = \mathcal{A}$, $\Gamma^{\otimes 1} = \Gamma$, and N is the two-sided ideal of $\Gamma^\otimes$ generated by the kernel of $1 - \sigma$, in which 1 is the identity operator on $\Gamma \otimes_{\mathcal{A}} \Gamma$ and σ is the automorphism given by the permutation on $\Gamma^{\otimes 2}$. As done in commutative geometry, in order to construct the high order differential calculus on the quantum group $GL_q(n)$, we should first decide a bimodule automorphism σ of $\Gamma \otimes_{\mathcal{A}} \Gamma$. For that reason we first introduce the concept of left-invariant and right-invariant 1-form.

Definition 4.1 Let $\Delta_\Omega : \Omega^1 \longrightarrow \Omega^0 \otimes \Omega^1$ be a linear mapping satisfying

(i) $\forall x \in \Omega^0, \omega \in \Omega^1$,

$$\Delta_\Omega(x\omega) = \Delta(x)\Delta_\Omega(\omega),$$
$$\Delta_\Omega(\omega x) = \Delta_\Omega(\omega)\Delta(x),$$

(ii) $(\Delta \otimes id)\Delta_\Omega = (id \otimes \Delta_\Omega)\Delta_\Omega$,

(iii) $(\varepsilon \otimes id)\Delta_\Omega = id$.

Then we call Δ_Ω the left action on Ω^1. If an element $\omega \in \Omega^1$ satisfying $\Delta_\Omega(\omega) = 1 \otimes \omega$, then we call ω the left-invariant differential 1-form.

Definition 4.2 Let ${}_\Omega\Delta : \Omega^1 \longrightarrow \Omega^1 \otimes \Omega^0$ be a linear mapping satisfying:

(i) $\forall x \in \Omega^0, \omega \in \Omega^1$,

$${}_\Omega\Delta(x\omega) = \Delta(x){}_\Omega\Delta(\omega),$$
$${}_\Omega\Delta(\omega x) = {}_\Omega\Delta(\omega)\Delta(x),$$

(ii) $(id \otimes \Delta){}_\Omega\Delta = ({}_\Omega\Delta \otimes id){}_\Omega\Delta$,

(iii) $(id \otimes \varepsilon){}_\Omega\Delta = id$.

Then we call ${}_\Omega\Delta$ the right action on Ω^1. If an element $\omega \in \Omega^1$ satisfying ${}_\Omega\Delta(\omega) = \omega \otimes 1$, then we call ω the right-invariant differential 1-form.

In general, the differential calculus on a Hopf algebra which only satisfies the conditions (2.1) and (2.2) can not always be provided with a left(right) action. But if the differential calculus is left(right)-covariant, the left(right) action on differential forms can be defined.

Proposition 4.1 For $\forall\omega \in \Omega^1$, $\omega = x_k dy_k$, the left action on ω is defined as

$$\Delta_\Omega(x_k dy_k) = \Delta x_k (id \otimes d)\Delta y_k.$$

and the right action on ω as

$$_\Omega\Delta(x_k dy_k) = \Delta x_k (d \otimes id)\Delta y_k.$$

For the proof of this proposition, please see the Proposition 1.2 and 1.3 in [1].

By this proposition, we have

$$\Delta_\Omega(Det_q T dt) = \Delta(Det_q T)(id \otimes d)\Delta t,$$
$$_\Omega\Delta(Det_q T dt) = \Delta(Det_q T)(d \otimes id)\Delta t.$$

Since

$$\Delta Det_q T = Det_q T \otimes Det_q T,\ \Delta t = t \otimes t,$$

we have

$$\Delta_\Omega(Det_q T dt) = 1 \otimes (Det_q T)dt, \tag{4.2}$$
$$_\Omega\Delta(Det_q T dt) = (Det_q T)dt \otimes 1. \tag{4.3}$$

Noticing

$$\Delta S(t_{ik}) = S(t_{mi}) \otimes S(t_{km}),\ i,k = 1,2,\cdots,n, \tag{4.4}$$

we have

$$\Delta_\Omega(S(t_{ik})dt_{kj}) = 1 \otimes S(t_{ik})dt_{kj}, \tag{4.5}$$
$$_\Omega\Delta(S^{-1}(t_{ki})dt_{jk}) = S^{-1}(t_{ki})dt_{jk} \otimes 1. \tag{4.6}$$

Combining (4.2), (4.3), (4.5) with (4.6), we obtain the following proposition.

Proposition 4.2

(i) $Det_q T dt$ is left-invariant and right-invariant 1-form,

(ii) $S(t_{ik})dt_{kj}$, $(i,j = 1,2,\cdots,n)$ are left-invariant 1-forms,

(iii) $S^{-1}(t_{ki})dt_{jk}$, $(i,j = 1,2,\cdots,n)$ are right-invariant 1-forms.

By the proposition,

$$\omega^{ij} = M^{-1}(\lambda)^{ij}_{kl}[S(t_{km})dt_{ml} + \lambda(\frac{1}{r^n q^2} - 1)\delta_{kl} Det_q T dt] \tag{4.7}$$

is left-invariant. It is easy to see

$$t_{mk} S^{-1}(t_{lm}) = \delta_{kl} 1. \tag{4.8}$$

By (4.8), we can rewrite (4.7) as

$$\omega^{ij} = M^{-1}(\lambda)^{ij}_{kl} S(t_{ku}) t_{vl}[S^{-1}(t_{uv})dt_{uv} + \lambda(\frac{1}{r^n q^2} - 1)\delta_{uv} Det_q T dt], \tag{4.9}$$

Denote

$$\eta_{uv} = S^{-1}(t_{uv})dt_{uv} + \lambda(\frac{1}{r^n q^2} - 1)\delta_{uv} Det_q T dt. \tag{4.10}$$

We have

$$\omega^{ij} = M^{-1}(\lambda)^{ij}_{kl} S(t_{ku}) t_{vl} \eta_{uv}. \tag{4.11}$$

By Proposition 4.2, η_{uv} is right-invariant 1-form, and (4.10) shows η_{uv} $(u,v = 1,2,\cdots,n)$ are also a group of right-invariant generators of Ω^1.

Now we define the bi-module automorphism $\sigma : \Omega^1 \otimes_{\Omega^0} \Omega^1 \longrightarrow \Omega^1 \otimes_{\Omega^0} \Omega^1$ by

$$\sigma(X^{mn}_{uv}\omega^{uv} \otimes \eta_{mn}) = X^{mn}_{uv}\eta_{mn} \otimes \omega^{uv}, \tag{4.12}$$

here $X^{mn}_{uv} \in \Omega^0$. It is easy to check σ satisfies the braid relation,

$$(id \otimes \sigma)(\sigma \otimes id)(id \otimes \sigma) = (\sigma \otimes id)(id \otimes \sigma)(\sigma \otimes id).$$

Obviously, $\omega^{ij} \otimes \omega^{uv}$ $(i,j,k,l = 1,2,\cdots,n)$ is a group of generators of $\Omega^1 \otimes_{\Omega^0} \Omega^1$. By (4.10),

$$\begin{aligned}\sigma(\omega^{ij} \otimes \omega^{kl}) &= \sigma(\omega^{ij} \otimes M^{-1}(\lambda)^{kl}_{\alpha\beta} S(t_{\alpha u}) t_{v\beta}\eta_{uv}) \\ &= M^{-1}(\lambda)^{kl}_{\alpha\beta}(\theta_{ijuv} * (S(t_{\alpha\gamma})t_{\tau\beta}))\eta_{\gamma\tau} \otimes \omega^{uv}\end{aligned} \tag{4.13}$$

Applying (4.10) to (4.13), we obtain

$$\begin{aligned}\sigma(\omega^{ij} \otimes \omega^{kl}) &= M^{-1}(\lambda)^{kl}_{\alpha\beta}\theta_{ijuv}(S(t_{\alpha\mu})t_{w\beta})S(t_{\mu\gamma})t_{\tau w} \\ &\quad \cdot[S^{-1}(t_{m\tau})dt_{\gamma m} + \lambda(\frac{1}{r^n q^2} - 1)\delta_{\tau\gamma} Det_q T dt] \otimes \omega^{uv} \\ &= M^{-1}(\lambda)^{kl}_{\alpha\beta}\theta_{ijuv}(S(t_{\alpha\mu})t_{w\beta})[S(t_{\mu\gamma})dt_{\gamma w} \\ &\quad + \lambda(\frac{1}{r^n q^2} - 1)\delta_{\mu w} Det_q T dt] \otimes \omega^{uv} \\ &= M^{-1}(\lambda)^{kl}_{\alpha\beta}\theta_{ijuv}(S(t_{\alpha\mu})t_{w\beta})M(\lambda)^{\mu w}_{mn}\omega^{mn} \otimes \omega^{uv}.\end{aligned} \tag{4.14}$$

Let

$$\sigma(\omega^{ij} \otimes \omega^{kl}) = \tilde{\mathbf{R}}^{ijkl}_{mnuv}\omega^{mn} \otimes \omega^{uv},$$
$$\mathbf{R}^{ijkl}_{mnuv} = \theta_{ijuv}(t_{mk}S(t_{ln})).$$

Proposition 4.3 $\mathbf{R}^{ijkl}_{mnuv} = \tilde{\mathbf{R}}^{ijkl}_{mnuv}$.

Proof: From (4.14), it follows that

$$\begin{aligned}\tilde{\mathbf{R}}^{ijkl}_{mnuv} &= M^{-1}(\lambda)^{kl}_{\alpha\beta}\theta_{ijuv}(S(t_{\alpha\mu})t_{w\beta})M(\lambda)^{\mu w}_{mn} \\ &= \theta_{ijuv}(M^{-1}(\lambda)^{kl}_{\alpha\beta}S(t_{\alpha\mu})t_{w\beta}M(\lambda)^{\mu w}_{mn}).\end{aligned}$$

Applying (3.8) to above equation, one obtains

$$\begin{aligned}\tilde{\mathbf{R}}^{ijkl}_{mnuv} &= \theta_{ijuv}(M^{-1}(\lambda)^{\alpha\beta}_{uw}S(t_{\alpha k})t_{l\beta}M(\lambda)^{\mu w}_{mn}) \\ &= \theta_{ijuv}(t_{mk}S(t_{ln})) \\ &= \mathbf{R}^{ijkl}_{mnuv}.\end{aligned} \tag{4.15}$$

Proposition 4.4 Let $\mathbf{R} = (\mathbf{R}^{ijkl}_{\alpha\beta\mu\lambda}) = (r^a_b)_{1\le a,b\le n^4}$, $a = n^3(i-1) + n^2(j-1) + n(k-1) + l$, $b = n^3(\alpha-1) + n^2(\beta-1) + n(\mu-1) + \lambda$. Then the minimal polynomial of $\mathbf{R}$ is $(\xi-1)(\xi+q^2)(\xi+q^{-2})$.

Proof: By (2.8) in Proposition 2.1,

$$\begin{aligned}
\mathbf{R}^{ijkl}_{\alpha\beta\mu\lambda} &= \theta_{ij\mu\lambda}(t_{\alpha k}S(t_{l\beta})) \\
&= \Delta^*\theta_{ij\mu\lambda}(t_{\alpha k}\otimes S(t_{l\beta})) \\
&= \theta_{ijab}(t_{\alpha k})\theta_{ab\mu\lambda}(S(t_{l\beta})) \\
&= S^*(l^-_{ai})\otimes l^+_{jb}(\Delta t_{\alpha k})S^*(l^-_{\mu a})\otimes l^+_{b\lambda}(\Delta S(t_{l\beta})) \\
&= S^*(l^-_{ai})(t_{\alpha c})l^+_{jb}(t_{ck})S^*(l^-_{\mu a})(S(t_{d\beta}))l^+_{b\lambda}(S(t_{ld})).
\end{aligned}$$

By definition of $l^\pm_{ij}$ $(i,j = 1,2,\cdots,n)$,

$$\begin{aligned}
S^*(l^-_{ai})(t_{\alpha c}) &= \lambda_- R_{a\alpha,ic}, & l^+_{jb}(t_{ck}) &= \lambda_+ R^+_{jc,bk}, \\
S^*(l^-_{\mu a})(S(t_{d\beta})) &= \lambda_-^{-1} R^{-1}_{\mu d,a\beta}, & l^+_{b\lambda}(S(t_{ld})) &= \lambda_+^{-1}(R^+)^{-1}_{bl,\lambda d}.
\end{aligned} \tag{4.16}$$

These formulae give

$$\begin{aligned}
\mathbf{R}^{ijkl}_{\alpha\beta\mu\lambda} &= R^+_{jc,bk}R_{a\alpha,ic}(R^+)^{-1}_{bl,\lambda d}(R_q^{-1})_{\mu d,a\beta} \\
&= R_{cj,kb}(R^t_q)_{ic,a\alpha}(PR_q^{-1}P)_{bl,\lambda d}(R^{t_1}_q)^{-1}_{ad,\mu\beta} \\
&= (PR^{t_1}_q)_{jk,cb}(R^t_qP)_{ic,\alpha a}(PR_q^{-1})_{bl,d\lambda}(PR^{t_1}_q)^{-1}_{ad,\beta\mu},
\end{aligned} \tag{4.17}$$

where t is the transposition of the matrix, and t_1 is the permutation of the first and the third indexes. Write

$$\begin{aligned}
(PR^{t_1}_q)_{jk,cb} &= (PR^{t_1}_q)^{jk}_{cb}, & (R^t_qP)_{ic,\alpha a} &= (R^t_qP)^{ic}_{\alpha a}, \\
(PR_q^{-1})_{lb,\lambda d} &= (PR_q^{-1})^{lb}_{\lambda d}, & (PR^{t_1}_q)^{-1}_{ad,\beta\mu} &= ((PR^{t_1}_q)^{-1})^{ad}_{\beta\mu}.
\end{aligned}$$

Then (4.17) can be rewritten as

$$\begin{aligned}
\mathbf{R}^{ijkl}_{\alpha\beta\mu\lambda} &= (\delta^i_a(PR^{t_1}_q)^{jk}_{bc}\delta^l_d)((R^t_qP)^{ab}_{a'b'}\delta^c_{c'}\delta^d_{d'}) \\
&\quad (\delta^{a'}_m\delta^{b'}_n(PR_q^{-1})^{c'd'}_{uv})(\delta^m_\alpha((PR^{t_1}_q)^{-1})^{nu}_{\beta\mu}\delta^v_\lambda) \\
&= [(E\otimes PR^{t_1}_q\otimes E)\cdot(R^t_qP\otimes E\otimes E) \\
&\quad \cdot(E\otimes E\otimes PR_q^{-1})\cdot(E\otimes PR^{t_1}_q\otimes E)^{-1}]^{ijkl}_{\alpha\beta\mu\lambda}.
\end{aligned} \tag{4.18}$$

Let $M_1 = R^t_qP\otimes E\otimes E$, $M_2 = E\otimes E\otimes PR_q^{-1}$. Obviously, $\mathbf{R}$ and M_1M_2 have the same minimal polynomial

$$(\xi-1)(\xi+q^2)(\xi+q^{-2}).$$

Therefore, we obtain

$$(E_{n^4}-\mathbf{R})(\mathbf{R}+q^2E_{n^4})(\mathbf{R}+q^{-2}E_{n^4}) = 0,$$

where E_{n^4} is the unit matrix of order n^4.

Now we introduce the quantum de Rham complex on quantum group $GL_q(n)$. Denote

$$\Omega^\otimes = \oplus_{i=0}^\infty\Omega^{\otimes i},\ \Omega^{\otimes 0} = \Omega^0,\ \Omega^{\otimes 1} = \Omega^1.$$

Definition 4.3 The quantum de Rham complex on $GL_q(n)$ is defined as

$$\Omega^\wedge = \Omega^\otimes/\{\ker(1-\sigma)\}, \tag{4.19}$$

where $\{\ker(1-\sigma)\}$ is the two-sided ideal on $\Omega^\otimes$ generated by $\ker(1-\sigma)$, and $\ker(1-\sigma) = \{[(\mathbf{R}+q^2E_{n^4})(\mathbf{R}+q^{-2}E_{n^4})]^{\alpha\beta\mu\lambda}_{ijkl}\omega^{ij}\otimes\omega^{kl}\ \alpha,\beta,\mu,\lambda = 1,2,\cdots,n\}$. The production in $\Omega^\wedge$ is denoted by $\wedge$.

Theorem 4.1 There exists a unique linear mapping

$$d:\Omega^\wedge\longrightarrow\Omega^\wedge,$$

so that

(i) d is the derivation of order one, i.e. it maps differential forms of order n to ones of order $n+1$,

(ii) The definition of d on Ω^0 is given by (2.18),

(iii) $d(\xi\wedge\eta) = d\xi\wedge\eta + (-1)^{\deg\xi}\xi\wedge d\eta$, where $\deg\xi = n$ if ξ is a differential form of order n,

(iv) $d^2 = 0$.

The proof of Theorem 4.1 is similar to that of Theorem 4.1 in [1].

In fact, we can write (4.19) as

$$\Omega^\wedge = \mathrm{Fun}(GL_q(n))[\omega^{ij},1]/\{I_1,I_2\}, \tag{4.20}$$

where relation I_1 is given by

$$\omega^{ij}x - (\theta_{ijkl}*x)\omega^{kl},\ x = t_{mn},t \tag{4.21}$$

and the relation I_2 is given by

$$[(\mathbf{R}+q^2E_{n^4})(\mathbf{R}+q^{-2}E_{n^4})]^{\alpha\beta\mu\lambda}_{ijkl}\omega^{ij}\wedge\omega^{kl},\ \alpha,\beta,\mu,\lambda = 1,2,\cdots,n. \tag{4.22}$$

Additionally, we can also obtain the Maurer-Cartan equation by Theorem 4.1 and (2.19).

§5. Noncommutative differential calculus on quantum group $SL_q(n)$

Quantum group $SL_q(n)$ can be obtained by taking the quotient algebra

$$\mathrm{Fun}(M(n))/\{Det_qT-1\}.$$

In fact, as for the Hopf algebra $\mathrm{Fun}(GL_q(n))$, its generator t now equivalents to the unit 1, i.e.

$$t \equiv 1. \tag{5.1}$$

Thus, the algebra of coordinate functions $\mathrm{Fun}(SL_q(n))$ is equivalent to $\mathrm{Fun}(GL_q(n))$ $/\{Det_qT - 1\}$. Namely,

$$\mathrm{Fun}(SL_q(n)) = \mathrm{Fun}(M(n))/\{Det_qT - 1\} = \mathrm{Fun}(GL_q(n))/\{Det_qT - 1\}\ . \tag{5.2}$$

To insure the linear functionals $l^{\pm}_{ij}$ $(i, j = 1, \cdots n)$ and $l^{\pm}$ given in §1 are well defined on $\mathrm{Fun}(SL_q(n))$, from (1.25) and (1.26), we know that the following conditions must be satisfied,

$$\lambda^n_+ q = \lambda^n_- q = 1. \tag{5.3}$$

Therefore,

$$\nabla_{ij}(t) = \nabla_{ij}(1) = 0, \tag{5.4}$$

and by (1.15) (1.17) and (1.18) we have $l^+ = l^- = \varepsilon$. Or we can say, after the condition (5.2) is introduced, all of the equations in §1 still hold, and those related to t and $l^{\pm}$ become trivial. Obviously, $\mathrm{Fun}(SL_q(n))$ and the corresponding algebra $\mathrm{Fun}^*_0(SL_q(n))$ are Hopf algebras.

Now we discuss how to obtain the differential calculus on $SL_q(n)$ and its quantum de Rham complex from that of $GL_q(n)$.

Matrix $M(\lambda)$ plays a very important role in the discussions of the differential calculus on $GL_q(n)$. For quantum group $SL_q(n)$, we have only two extra conditions $Det_qT = t = 1$ and $\lambda^n_+ q = \lambda^n_- q = 1$. Thus,

$$M(\lambda) = (M^{n(k-1)+l}_{n(i-1)+j})_{1\le k,l,i,j\le n} = (\nabla_{ij}(t_{kl}))_{1\le k,l,i,j\le n}. \tag{5.5}$$

And the determinant of $N(\lambda)$ is

$$DetN(\lambda) = \frac{(1-r^n)^{n-2}}{\chi^n r^{n^2-n}}(1 - r^{n-1} - r^n + r^{2n} + r^{n^2+n-1} - r^{n^2+n}), \tag{5.6}$$

where $r^n q^2 = 1$. Therefore, except for finite isolated values of q, the matrix $M(\lambda)$ is invertible. When $M(\lambda)$ is invertible, we can add the conditions $Det_qT \equiv 1$ and $\lambda^n_+ q = \lambda^n_- q = 1$ to the differential calculus of $GL_q(n)$ to obtain that of $SL_q(n)$. The values of q that $M(\lambda)$ is not invertible are the 6th unit roots when $n = 2$, when $n \ge 3$, the discussions will be a bit more complicated, we will discuss the differential calculus of $SL_q(n)$ at the extra values of q elsewhere.

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