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H. YAN

S. M. FEI

J. ZHANG

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The Spin-$\frac{1}{2}$ Heisenberg Chains in Constant Magnetic Field and Coherent States

H. Yan†, S.-M. Fei† and J. Zhang†

† CCAST (World Laboratory), P. O. Box 8730, Beijing 100080, China
and Institute of Theoretical Physics, Academia Sinica
P. O. Box 2735, Beijing 100080, China[1]

Abstract

In the ferromagnetic Heisenberg chains of XXX and XXZ types with the hidden symmetries of double Lie algebra $su(2)$ and quantum bi-algebra $su_q(2)$, we show that at thermodynamic limit the algebra contractions give the boson algebra $h(4)$ and the q-deformed boson algebra $h_q(2)$ as the hidden symmetries respectively. The chains in constant magnetic field are studied and the ground states and lowest excited states are given explicitly with energy spectra. The phonon (or angular momentum) excitations are shown to be bosonic for the isotropic case and q-bosonic for the anisotropic case, and the ground states and lowest excited states of the systems of the chains in field are given explicitly. We give the phonon coherent states in the isotropic Heisenberg chain and the q-coherent states of the anisotropic chain at thermodynamic limit, and the q-coherent states are shown to be a squeezed states of phonon excitations.

§1. Double Lie Algebra $su(2)$

For completeness and to facilitate the forthcoming introduction of the quantum bi-algebra theory [1], we start our discussion at the very beginning of double Lie algebra theory. But we will confine our introduction to the necessary discussions of double Lie algebra $su(2)$ and in section 6, the quantum bi-algebra $su_q(2)$.

Lie algebra $su(2)$ is spanned by $\{J^{\pm}, J^3\}$ and defined by the following commutation relations

$$[J^+, J^-] = 2J^3 \ , \quad [J^3, J^{\pm}] = \pm J^{\pm} \ , \tag{1}$$

[1]mailing address

and for the convenience of the following description, we supplement it with unity 1. The center of this algebra is therefore spanned by 1 and the Casimir element $C = J^+J^- + J^3(J^3 - 1)$.

The algebra structure is built with two maps, the unit map u and the map of multiplication m,

$$\begin{aligned} m: \quad & su(2) \otimes su(2) \longrightarrow su(2) \\ u: \quad & \mathbf{C} \longrightarrow su(2) \end{aligned} \tag{2}$$

and m is associative.

A co-algebra structure can also be defined based on the co-unit map ϵ and the map of diagonalization Δ, which are algebra homomorphisms, i.e., $\Delta(ab) = \Delta(a)\Delta(b)$, $\forall a, b \in su(2)$, and same to ϵ. The maps turn the algebra into its tensor representation as shown below

$$\begin{aligned} \Delta: \quad & su(2) \longrightarrow su(2) \otimes su(2) \\ \epsilon: \quad & su(2) \longrightarrow \mathbf{C} \end{aligned} \tag{3}$$

and the diagonalization is associative, i.e., $su(2) \otimes [\Delta(su(2))] = [\Delta(su(2))] \otimes su(2)$. We give the actions of Δ and ϵ on generators $J^{\pm,3}$ explicitly

$$\begin{aligned} \Delta(J^{\pm,3}) &= J^{\pm,3} \otimes 1 + 1 \otimes J^{\pm,3} \ , \\ \epsilon(J^{\pm,3}) &= 1 \end{aligned} \tag{4}$$

As $su(2)$ is simultaneously an algebra and co-algebra, it (more precisely $su(2) \otimes su(2)$) is a double Lie algebra.

The Casimir element of this double Lie algebra is the diagonalization of C, often called the (1-st rank) co-Casimir, which reads,

$$\begin{aligned} \Delta(C) &= J^+J^- \otimes 1 + 1 \otimes J^+J^- + J^+ \otimes J^- + J^- \otimes J^+ \\ &+ 1 \otimes (J^3)^2 + (J^3)^2 \otimes 1 + 2J^3 \otimes J^3 - 1 \otimes J^3 - J^3 \otimes 1 \end{aligned} \tag{5}$$

§2. The Hamiltonian of the 1D Isotropic Heisenberg Model

Consider the co-Casimir for the spin-$\frac{1}{2}$ representation, from (5) we learn that

$$\Delta(C) = \sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+ + \frac{1}{2}\sigma^3 \otimes \sigma^3 + \frac{3}{2} \tag{6}$$

where $\sigma^{\pm,3}$ are Pauli matrices. Therefore the operator

$$E = 2\cdot 1\otimes 1 - \Delta(C) = \begin{pmatrix} 0 & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & 0 \end{pmatrix} \tag{7}$$

is in the center of the double Lie algebra $su(2)\otimes su(2)$. Hence it is true that

$$e_i = 1_1\otimes 1_2\otimes\cdots\otimes 1_{i-1}\otimes(E)_{i,i+1}\otimes 1_{i+2}\cdots\otimes 1_{N+1} \tag{8}$$

(where $i = 1,2,\cdots N$) is $su(2)$-invariant, i.e.,

$$\left[e_i, su(2)^{\otimes N}\right] = 0\ . \tag{9}$$

It is interesting that e_i are elements of the N-dimensional permutation group algebra P_N satisfying

$$\begin{aligned} &e_i^2 = 2e_i \\ &e_i e_{i\pm1} e_i = e_i \\ &[e_i, e_j] = 0\ , \quad |i-j|\geq 2 \end{aligned} \tag{10}$$

which is a special case of Temperley-Lie-Jones (T-L-J) algebra [2]. Therefore the Hamiltonian for spin-$\frac{1}{2}$ Heisenberg chain

$$\begin{aligned} H_{XXX} &= \sum_{i=1}^{N} e_i = \sum_{i=1}^{N}(\Delta(2-C))^{(i,i+1)}\ , \\ &= -\sum_{i=1}^{N}\left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{1}{2}\sigma_i^3\sigma_{i+1}^3\right) + \frac{1}{2}N\ . \end{aligned} \tag{11}$$

is invariant under the action of double Lie algebra $su(2)$.

§3. The Ground and the Lowest Excited States in the Chain

As is shown in the above, the Hamiltonian of the spin-$\frac{1}{2}$ chain is invariant under the action of the $su(2)$ generators, and therefore the solution spaces are completely reduced into irreducible spaces characterized by their highest weight states, and each state can be labelled by the quantum numbers of total spin, the third component of the total spin, and the energy eigenvalues. As a method of book-keeping, we will first give the highest weight states, and the complete solution spaces will be obtained by the application of the lowering operators of the double Lie algebra.

The ground states are irreducible representation spaces with energy eigenvalues 0, and total spin numbers the largest ones, i.e., $(N+1)/2$. We assume the space of the ground states is characterized by the highest weight state of all up spins, i.e.,

$$|0\rangle = |\uparrow_1\uparrow_2\cdots\uparrow_{N+1}\rangle\ . \tag{12}$$

It is clear the energy eigenvalue of this state is 0.

The lowest excitation states are those characterized by the highest weight state with the turning-down of an arbitrary spin,

$$|\downarrow_i\rangle = |\uparrow_1\uparrow_2\cdots\uparrow_{i-1}\downarrow_i\uparrow_{i+1}\cdots\uparrow_{N+1}\rangle\ . \tag{13}$$

Since

$$e_i|\downarrow_j\rangle = \begin{cases} |\downarrow_i\rangle - |\downarrow_{i+1}\rangle\ , & \text{if } i=j\ ; \\ |\downarrow_{i+1}\rangle - |\downarrow_i\rangle\ , & \text{if } i=j-1\ ; \\ 0 & \text{if otherwise}\ , \end{cases} \tag{14}$$

the action of the Hamiltonian on this state yields

$$H_{XXX}|\downarrow_i\rangle = e_i|\downarrow_i\rangle + e_{i-1}|\downarrow_i\rangle = -|\downarrow_{i+1}\rangle + 2|\downarrow_i\rangle - |\downarrow_{i-1}\rangle\ , \tag{15}$$

i.e., the spin deviation propagates along this chain, which is the familiar spin-wave.

To diagonalize the Hamiltonian, we consider the states

$$\begin{aligned} |(i,i+1)_s\rangle &= (-)^{i+1}(|\uparrow_1\uparrow_2\cdots\uparrow_{i-1}\uparrow_i\downarrow_{i+1}\cdots\uparrow_{N+1}\rangle \\ &\quad -|\uparrow_1\uparrow_2\cdots\uparrow_{i-1}\downarrow_i\uparrow_{i+1}\cdots\uparrow_{N+1}\rangle)\ . \end{aligned} \tag{16}$$

The lowest excited states $|1\rangle$ (which will be supplemented by an subscript k denoting the degeneracy) should be proper compositions of these states with i running from 1 to N,

$$|1\rangle = \sum_{i=1}^{N} a_i|(i,i+1)_s\rangle\ . \tag{17}$$

Because

$$e_i|(i,i+1)_s\rangle = \begin{cases} 2|(i,i+1)_s\rangle\ , & \text{if } i=j\ ; \\ |(i+1,i+2)_s\rangle\ , & \text{if } i=j+1\ ; \\ |(i-1,i)_s\rangle\ , & \text{if } i=j-1\ ; \\ 0 & \text{if otherwise}\ , \end{cases} \tag{18}$$

and we have the reduction equation

$$(2-E)a_i + a_{i+1} + a_{i-1} = 0 \ , \quad a_0 = a_{N+1} = 0 \ , \tag{19}$$

which can be easily solved

$$\begin{aligned} \frac{a_m^k}{a_1} &= (-)^{m-1}\frac{\sin\left(\frac{mk\pi}{N+1}\right)}{\sin\left(\frac{k\pi}{N+1}\right)} \ , \\ E_k &= 2\left(1-\cos\frac{k\pi}{N+1}\right) \ , \quad k = 0,1,2,\cdots,N \ . \end{aligned} \tag{20}$$

For each k, a_m^k gives only the highest weight state characterizing a solution space, and this space is obtained by the actions of the lowering operator of the symmetry algebra $su(2)$, or precisely the lowering operator $\Delta^{(N)}(J^-)$ of the double Lie algebra $su(2)^{\otimes(N+1)}$.

The complete solution space for the ground states is

$$V_0 = \left\{ |j = \frac{N+1}{2}, j^3 = j-m, E=0\rangle = \left[\Delta^{(N)}(J^-)\right]^m |0\rangle \right\} \ , \tag{21}$$

where $0 \le m \le N+1$, the states are not normalized, and the state $|0\rangle$ is the highest weight vector given in (12). The solution spaces for the lowest excited states are

$$V_1^k = \left\{ |j = \frac{N-1}{2}, j^3 = j-m, E_k\rangle = \left[\Delta^{(N)}(J^-)\right]^m |1\rangle_k \right\} \ , \tag{22}$$

where $0 \le m \le N$, and E_k are indicated in (20).

By the term *lowest excited* we mean there is only one spin turning-down in the highest weight states, and we will see that at the thermodynamic limit, the energy eigenvalues of these highest weight states becomes continuous from the energy eigenvalue of the ground states 0. The lowest energy eigenvalues are really contained in these solutions.

§4. The Chain in Constant Magnetic Field

The behaviour of the excitations in the chain in a constant magnetic field is one of the important problem of theoretic research and also one of the experimentally devoted problem in quasi one dimensional samples. The Hamiltonian for the XXX chain in a constant magnetic field $\vec{B} = B\hat{z}$ is

$$H'_{XXX} = H_{XXX} + B\mu_B \sum_{i=1}^{N+1} J_i^3 \ , \tag{23}$$

where μ_B is Bohr magneton. It shows that each spin along the chain has contribution to the energy of the system, and $\sum_{i=1}^{N+1} J_i^3$ is nothing but the element of the double Lie algebra $su(2)^{\otimes(N+1)}$.

Obviously, the existence of the nonzero B induces the breaking of the $su(2)$-symmetry, and the degeneracy of the states in each solution space is lifted.

In space V_0, every state has a different energy eigenvalue,

$$\begin{aligned} H'_{XXX}|j = \tfrac{N+1}{2}, j^3 = m, E=0\rangle &= H'_{XXX}\left[\Delta^{(N)}(J^-)\right]^m |0\rangle \\ &= mB\mu_B |j = \tfrac{N+1}{2}, j^3 = m, E=0\rangle \end{aligned} \tag{24}$$

where $0 \le m \le N+1$. In space V_1^k,

$$\begin{aligned} H'_{XXX}|j = \tfrac{N-1}{2}, j^3 = m, E_k\rangle &= H'_{XXX}\left[\Delta^{(N)}(J^-)\right]^m |1\rangle \\ &= (E_k + mB\mu_B)\,|j = \tfrac{N-1}{2}, j^3 = m, E_k\rangle \end{aligned} \tag{25}$$

where $0 \le m \le N$.

§5. The Thermodynamic Limit and Coherent States

When $N \to \infty$, if we suppose $k = \alpha(N+1)$, $\alpha \in [0,1]$ then

$$\frac{a_m^k}{a_1} = (-)^{m-1}\frac{\sin(m\alpha\pi)}{\sin(\alpha\pi)} \ , \quad E_k = 4\sin^2\frac{\alpha\pi}{2} \ . \tag{26}$$

Especially, when $\alpha \to 0^+$, we have

$$\frac{a_m^k}{a_1} \to (-)^{m-1}m \ , \quad E_k \to 0 \ ; \tag{27}$$

while when $\alpha \to 1^-$, we have

$$\frac{a_m^k}{a_1} \to m \ , \quad E_k \to 4 \ . \tag{28}$$

When α increases from 0 to 1, the energy eigenvalue E_k increases from 0 to 4. There is no energy gap between the ground states and the lowest excited ones.

Since $N \to \infty$, the total spin numbers for the ground and lowest excited states approach infinity, there is natural algebraic contraction from $su(2)$ to $h(4)$, the linear oscillator algebra. Consider the following linear transformation that transforms the generators $J^{\pm,3}$ to a new set of generators $h^{\pm,3}$

$$\begin{pmatrix} h^\pm \\ h^3 \\ 1 \end{pmatrix} = \begin{pmatrix} c & & \\ & 1 & \frac{1}{2c^2} \\ & & 1 \end{pmatrix} \begin{pmatrix} J^\pm \\ J^3 \\ 1 \end{pmatrix} \tag{29}$$

the algebraic relations are turned into

$$\left[h^3, h^\pm\right] = \pm h^\pm \ , \quad \left[h^+, h^-\right] = 2c^2 h^3 - \mathbf{1} \ . \tag{30}$$

The actions of the new generators in the Dicke space yield

$$\begin{aligned} h^\pm|jm\rangle &= c\sqrt{(j \pm m)(j \mp m + 1)}|j, m \pm 1\rangle \ , \\ h^3|jm\rangle &= c\left(m + \tfrac{1}{2c^2}\right)|j, m\rangle \ , \end{aligned} \tag{31}$$

Therefore when $j \to \infty$, $c^2 = \frac{1}{2j} \to 0$, the transformation (29) becomes singular, but the limit algebraic relation (30) are still well-defined, i.e., the (29) describes an algebraic contraction. The contraction limit is the well known linear oscillator algebra $h(4)$, and the usual generators are $a^\dagger = h^+$, $a = h^-$ and $\hat{N} = h^3 + j$. Denote $\ell = j + m$, we also obtain the Fock space representation of $h(4)$ as the limit of (31)

$$\begin{aligned} a^\dagger|\ell\rangle &= \sqrt{\ell+1}|\ell+1\rangle \ , \\ a|\ell\rangle &= \sqrt{\ell}|\ell-1\rangle \ , \\ \hat{N}|\ell\rangle &= \ell|\ell\rangle \ , \\ a|0\rangle &= 0 \ . \end{aligned} \tag{32}$$

The Hamiltonian for the chain in magnetic field is

$$H'_{XXX} = H_{XXX} + \mu_B B \hat{N} \ . \tag{33}$$

Although the system may be macroscopically large, it is obviously in pure quantum states. The excitations are constrained by the quantization condition, and obey the uncertainty relation. It is interesting to find the minimal uncertainty states, i.e., the coherent states [3], $|z\rangle$ with $z \in \mathbf{C}$ defined as follows

$$|z\rangle = \exp\left(za^\dagger + z^* a\right)|0\rangle = e^{-\frac{|z|^2}{2}} \sum_{\ell=0}^{\infty} \frac{z^\ell}{\sqrt{\ell!}}|\ell\rangle \ . \tag{34}$$

It can be checked directly that

$$a|z\rangle = z|z\rangle \ . \tag{35}$$

The adjoint states are

$$\langle z| = (|z\rangle)^\dagger \ , \tag{36}$$

and

$$\langle z|a^\dagger = z^*\langle z| \ . \tag{37}$$

Therefore the coordinate $Q = \dfrac{a^\dagger + a}{\sqrt{2}}$ and momentum $P = i\dfrac{a^\dagger - a}{\sqrt{2}}$ of the bosonic excitation has the following square root differences

$$\begin{aligned} \Delta P &\equiv \sqrt{\langle z|P^2|z\rangle - \langle z|P|z\rangle^2} = \tfrac{1}{\sqrt{2}} \\ \Delta Q &\equiv \sqrt{\langle z|Q^2|z\rangle - \langle z|Q|z\rangle^2} = \tfrac{1}{\sqrt{2}} \end{aligned} \tag{38}$$

That is, the coherent states $|z\rangle$ are minimal uncertainty states, or the quantum states nearest to the classical ones.

In the following, we will generalize the above analysis for isotropic XXX chain to the anisotropic XXZ one, but this generalization is rather indirect, for this sake we have to make use the mathematical tool of quantum bi-algebra theory, or the q-analogue of the double Lie algebra theory.

§6. Quantum Bialgebra $su_q(2)$ and T-L-J Algebra

Quantum bi-algebra [4] $su_q(2)$ is generated by $\mathbf{1}, J_q^{\pm,3}$ satisfying the following relations

$$[J_q^+, J_q^-] = \left[2J_q^3\right]_q \ , \quad [J_q^3, J_q^\pm] = \pm J_q^\pm \ , \tag{39}$$

where $[x]_q = \dfrac{q^x - q^{-x}}{q - q^{-1}}$. Please note that $[x]_q \to x$ if $q \to \pm 1$. The independent central elements are $\mathbf{1}$ and $C_q = J_q^+ J_q^- + \left[J_q^3\right]_q \left[J_q^3 - 1\right]_q$. The algebra structure is still based on two maps: unit map u and associative map of multiplication m. The co-algebra structure is built on the co-unit map ϵ and map of diagonalization Δ, which are algebra homomorphisms and defined as follows

$$\begin{aligned} &\Delta(J_q^3) = J_q^3 \otimes \mathbf{1} + \mathbf{1} \otimes J_q^3 \ , \\ &\Delta(J_q^\pm) = J_q^\pm \otimes q^{J_q^3} + q^{-J_q^3} \otimes J_q^\pm \ , \\ &\epsilon(J_q^{\pm,3}) = 1 \ . \end{aligned} \tag{40}$$

It can be seen that J_q^3 is Lie-like under diagonalization and $q^{\pm J_q^3}$ is group-like, because

$$\Delta\left(q^{\pm J_q^3}\right) = q^{\pm J_q^3} \otimes q^{\pm J_q^3} \ . \tag{41}$$

The parameter q is called the deformation parameter, and when $q \to 1$ the double Lie algebra $su(2)$ is recovered.

When the spin number is $\frac{1}{2}$, the representation of quantum algebra $su_q(2)$ is the same to that of $su(2)$ which is constituted by Pauli matrices, but note that the tensor

representations are deformed. This is obvious from the relations (39) and the definition of diagonalization (40).

Consider the diagonalization of $C'_q = [2]_q - C_q$ for spin-$\frac{1}{2}$ representation, we have the co-Casimir

$$\begin{aligned}\Delta\left(C'_q\right) &= \tfrac{q+q^{-1}}{4}1\otimes 1 - [\sigma^+\otimes\sigma^- + \sigma^-\otimes\sigma^+ + \\ &\quad \tfrac{q+q^{-1}}{4}\sigma^3\otimes\sigma^3 - \tfrac{q-q^{-1}}{4}(\sigma^3\otimes 1 - 1\otimes\sigma^3)] \\ &= \begin{pmatrix} 0 & & & \\ & q & -1 & \\ & -1 & q^{-1} & \\ & & & 0 \end{pmatrix} .\end{aligned} \tag{42}$$

which is just the e matrix in [5] and E matrix in [6]. Therefore the elements of T-L-J algebra [2] can be given

$$e_i = 1_1\otimes 1_2\otimes\cdots\otimes 1_{i-1}\otimes\left[\Delta\left(C'_q\right)\right]_{i,i+1}\otimes 1_{i+2}\cdots\otimes 1_{N+1} \tag{43}$$

(where $i = 1, 2, \cdots N$), and it is a simple calculation to check that

$$\begin{aligned} &e_i^2 = [2]_q e_i \\ &e_i e_{i\pm 1} e_i = e_i \\ &[e_i, e_j] = 0 , \quad |i-j| \geq 2 \end{aligned} \tag{44}$$

which are just the defining relations of the N-dimensional T-L-J algebra. From (42), we see

$$[e_i, su_q(2)] = 0 . \tag{45}$$

Apparently, when $q \to 1$, the T-L-J algebra coincides with the permutation group algebra. It should be stressed that the T-L-J algebra is a special representation of braid algebra [7] which is an important tool in studying many body systems. The anisotropic Heisenberg chain presented in the next section is a well known example to illustrate this.

§7. Anisotropic Chain and Lowest Excited States

Consider the Hamiltonian for anisotropic Heisenberg chain [5][8][9][10]

$$H_{XXZ} = -\sum_{i=1}^{N}\left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+ + \frac{[2]_q}{4}\sigma_i^3\sigma_{i+1}^3\right) + \frac{[2]_q}{4}N - \frac{q-q^{-1}}{4}\left(\sigma_1^3 - \sigma_{N+1}^3\right) , \tag{46}$$

it is clear

$$H_{XXZ} = \sum_{i=1}^{N} e_i . \tag{47}$$

From (45) we learn that

$$[H_{XXZ}, su_q(2)] = 0 , \tag{48}$$

i.e., that system is $su_q(2)$-invariant.

The ground states are the state

$$|0\rangle = |\uparrow_1\uparrow_2\cdots\uparrow_{N+1}\rangle . \tag{49}$$

and its descendent states from it by the action of the lowering operator $\Delta^{(N)}(J_q^-)$ of the quantum bi-algebra $su_q(2)^{\otimes(N+1)}$. The complete solution space for the ground states is

$${}_qV_0 = \left\{|j = \frac{N+1}{2}, j^3 = j - m, E = 0\rangle_q = \left[\Delta^{(N)}(J_q^-)\right]^m |0\rangle\right\} , \tag{50}$$

where $0 \leq m \leq N+2$.

When there is only one spin deviation, such as $|\uparrow_1\uparrow_2\cdots\downarrow_i\cdots\uparrow_{N+1}\rangle_q$, the solutions are similar to the isotropic case. Let

$$\begin{aligned}|(i,i+1)_s\rangle_q &= (-)^{i+1}(q|\uparrow_1\uparrow_2\cdots\uparrow_{i-1}\uparrow_i\downarrow_{i+1}\cdots\uparrow_{N+1}\rangle_q \\ &\quad -|\uparrow_1\uparrow_2\cdots\uparrow_{i-1}\downarrow_i\uparrow_{i+1}\cdots\uparrow_{N+1}\rangle_q) .\end{aligned} \tag{51}$$

The states $|(1,2)_s\rangle_q$ etc. are q-analogous spin-singlets.

We will confine ourselves to the analysis of the first excited states $|1\rangle_q$ which are proper compositions of these states with i running from 1 to N [11][6], i.e.,

$$|1\rangle_q = \sum_{i=1}^{N} a_i |(i,i+1)_s\rangle_q . \tag{52}$$

Because

$$e_i|(i,i+1)_s\rangle_q = \begin{cases} [2]_q|(i,i+1)_s\rangle_q , & \text{if } i = j ; \\ |(i+1,i+2)_s\rangle_q , & \text{if } i = j+1 ; \\ |(i-1,i)_s\rangle_q , & \text{if } i = j-1 ; \\ 0 & \text{if otherwise} , \end{cases} \tag{53}$$

and we have the reduction equation

$$([2]_q - E)a_i + a_{i+1} + a_{i-1} = 0 , \quad a_0 = a_{N+1} = 0 , \tag{54}$$

which is solved in analogue of the isotropic case,

$$\begin{aligned} \frac{a_m^k}{a_1} &= (-)^{m-1}\frac{\sin\left(\frac{mk\pi}{N+1}\right)}{\sin\left(\frac{k\pi}{N+1}\right)}, \\ E_k &= 2\left(\frac{q+q^{-1}}{2} - \cos\frac{k\pi}{N+1}\right), \quad k = 0,1,2,\cdots,N . \end{aligned} \tag{55}$$

For different k, a_m^k give the highest weight states characterizing different solution spaces, and the other states in the spaces are obtained by the actions of the lowering operator $\Delta^{(N)}(J_q^-)$. The solution spaces for the lowest excited states are

$$_qV_1^k = \left\{|j = \frac{N-1}{2}, j^3 = j-m, E_k\rangle_q = \left[\Delta^{(N)}(J_q^-)\right]^m |1\rangle_k^q\right\}, \tag{56}$$

where $0 \le m \le N$, and E_k is indicated in (55).

§8. The Thermodynamic Limit and Chain in Constant Magnetic Field

When $N \to \infty$, if we suppose $k = \alpha(N+1)$, $\alpha \in [0,1]$ then for the lowest excited states we have

$$\frac{a_m^k}{a_1} = (-)^{m-1}\frac{\sin(m\alpha\pi)}{\sin(\alpha\pi)}, \quad E_k = 4\sin^2\frac{\alpha\pi}{2} . \tag{57}$$

When $\alpha \to 0^+$ and $\alpha \to 1^-$, we have respectively

$$\frac{a_m^k}{a_1} \to (-)^{m-1}m, \quad E_k \to [2]_q - 2 ; \tag{58}$$

and

$$\frac{a_m^k}{a_1} \to m, \quad E_k \to [2]_2 + 2 . \tag{59}$$

As α increases from 0 to 1, the energy eigenvalue E_k increases from $[2]_q - 2$ to $[2]_q + 2$. There is an energy gap between the ground states and the lowest excited ones.

Since $N \to \infty$, the total spin numbers for the ground and lowest excited states approach infinity, there is an algebraic contraction from $su_q(2)$ to $h_q(4)$, the q-oscillator algebra. Consider the following linear transformation that transforms the generators $J_q^{\pm,3}$ to a new set of generators $h_q^{\pm,3}$

$$\begin{pmatrix} h_q^{\pm} \\ h_q^3 \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} c & & \\ & 1 & \\ & & \frac{1}{2c^2} \\ & & 1 \end{pmatrix} \begin{pmatrix} J_q^{\pm} \\ J_q^3 \\ \mathbf{1} \end{pmatrix} \tag{60}$$

the algebraic relations are turned into

$$\left[h_q^3, h_q^{\pm}\right] = \pm h_q^{\pm}, \quad \left[h_q^+, h_q^-\right] = c^2\left[2h_q^3 - \frac{\mathbf{1}}{c^2}\right]_q . \tag{61}$$

The actions of the new generators in the Dicke space yield

$$\begin{aligned} h_q^{\pm}|jm\rangle_q &= c\sqrt{[j \pm m]_q[j \mp m + 1]_q}|j, m \pm 1\rangle_q , \\ h_q^3|jm\rangle_q &= \left(m + \tfrac{1}{2c^2}\right)|j,m\rangle_q , \end{aligned} \tag{62}$$

Therefore when $j \to \infty$, $c^2 = \frac{1}{[2j]_q} \to 0$, the transformation (60) becomes singular, but the limit algebraic relation (61) are still well-defined, i.e., the (60) describes an algebraic contraction. The contraction limit is an analogue of linear oscillator algebra $h_q(4)$, and the usual generators are $a_q^\dagger = h_q^+$, $a_q = h_q^-$ and $\hat{N} = h^3 + j$. Denote $\ell = j + m$, we also obtain the q-Fock space representation of $h_q(4)$ as the limit of (62)

$$\begin{aligned} a_q^\dagger|\ell\rangle_q &= \sqrt{[\ell+1]_q}|\ell+1\rangle_q , \\ a_q|\ell\rangle_q &= \sqrt{[\ell]_q}|\ell-1\rangle_q , \\ \hat{N}|\ell\rangle_q &= \ell|\ell\rangle_q , \\ a_q|0\rangle_q &= 0 . \end{aligned} \tag{63}$$

It should be emphasized that if we identify $|0\rangle_q$ with $|0\rangle$ then

$$\frac{\left(a_q^\dagger\right)^n}{\sqrt{[n]_q!}}|0\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0\rangle \tag{64}$$

i.e., $|n\rangle_q = |n\rangle$ with $n = 1, 2, \cdots$. Henceforth we will not distinguish $|n\rangle_q$ and $|n\rangle$. This is obvious if we consider the realization of $h_q(4)$ by $h(4)$ [12][13][14]

$$a_q = \sqrt{\frac{[\hat{N}+1]_q}{\hat{N}+1}}\, a , \quad a_q^\dagger = a^\dagger\sqrt{\frac{[\hat{N}+1]_q}{\hat{N}+1}} . \tag{65}$$

The excitations of the systems are not bosons, but their self-interacting composites. The q-bosons are not described by linear operators, but nonlinear ones. And from the above equations, the nonlinear operators are certain combinations of the linear operators.

The Hamiltonian for the spin chain in the constant magnetic field reads

$$H'_{XXZ} = H_{XXZ} + B\mu_B \sum_{i=1}^{N+1} J_{q\,i}^3 , \tag{66}$$

but $\sum_{i=1}^{N+1} J^3_{q\,i}$ is nothing but the operator of the quantum bi-algebra $\Delta^{(N)}(J^3_q)$. The nonzero quantity B induces the breaking of the $su_q(2)$-symmetry, i.e., the degeneracies of the states in solution spaces are lifted.

The action of the Hamiltonian on each state in space ${}_qV_0$ yields a different energy eigenvalue,

$$\begin{aligned} H'_{XXZ}|j=\tfrac{N+1}{2}, j^3=m, E=0\rangle &= H'_{XXZ}\left[\Delta^{(N)}(J^-)\right]^m|0\rangle \\ &= mB\mu_B|j=\tfrac{N+1}{2}, j^3=m, E=0\rangle \end{aligned} \tag{67}$$

where $0 \le m \le N+1$. In space ${}_qV_1^k$, the action of the Hamiltonian gives

$$\begin{aligned} H'_{XXZ}|j=\tfrac{N-1}{2}, j^3=m, E_k\rangle &= H'_{XXZ}\left[\Delta^{(N)}(J_q^-)\right]^m|1\rangle_q \\ &= (E_k+mB\mu_B)\,|j=\tfrac{N-1}{2}, j^3=m, E_k\rangle \end{aligned} \tag{68}$$

where $0 \le m \le N$.

At the thermodynamic limit, the Hamiltonian for the chain of XXZ type is

$$H'_{XXZ} = H_{XXZ} + \mu_B B\hat{N}\ . \tag{69}$$

§9. The q-Coherent States and Phonon Squeezing

To see clearly the strange but physical object: q-boson, it is worthwhile to see its minimal uncertainty property, and to do so we have to introduce the concept of q-analogous coherent state.

The q-coherent states $|z\rangle_q$ with $z \in \mathbf{C}$ are defined as follows

$$|z\rangle_q = \mathcal{N}^{-1}\exp\left(za^\dagger\right)|0\rangle = \mathcal{N}^{-1}\sum_{\ell=0}^{\infty}\frac{z^\ell}{\sqrt{[\ell]_q!}}|\ell\rangle\ , \tag{70}$$

where $\mathcal{N}^2 = \exp_q(|z|^2)$, $\exp_q$ is the Beidenharn exponential function [12] and it can be checked directly that

$$a_q|z\rangle_q = z|z\rangle_q\ . \tag{71}$$

The adjoint states are

$${}_q\langle z| = (|z\rangle_q)^\dagger\ , \tag{72}$$

and

$${}_q\langle z|a_q^\dagger = z^*\,{}_q\langle z|\ . \tag{73}$$

The resolution of unity is supplied in [15]. As the q-bosons are nonlinear composition of bosons, we should observe the behaviour of the uncertainty of the bosons on these q-coherent states.

The coordinate $Q = \dfrac{a^\dagger + a}{\sqrt{2}}$ and momentum $P = i\dfrac{a^\dagger - a}{\sqrt{2}}$ of the boson excitation has the following square root differences on the q-coherent states,

$$\begin{aligned} \Delta P &= \sqrt{\tfrac{1}{2} + \langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle + \mathrm{Re}\left[\langle a^{\dagger 2}\rangle - \langle a^\dagger\rangle^2\right]} \\ \Delta Q &= \sqrt{\tfrac{1}{2} + \langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle - \mathrm{Re}\left[\langle a^{\dagger 2}\rangle - \langle a^\dagger\rangle^2\right]} \end{aligned} \tag{74}$$

When $q \to 1$, these two values approach $\frac{1}{2}$, and the limit of $q \to 0$ or equivalently $q \to \infty$ gives

$$\Delta Q = \sqrt{\frac{1}{2} + z^2\frac{z^2-1}{(z^2+1)^2}}\ , \tag{75}$$

and for a finite value of q there should be phonon squeezing.

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