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MAR 2, 1992

QUANTUM GROUPS AND GEOMETRIC QUANTIZATION

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ASITP-92-08
CCAST-92-03
January 1992

QUANTUM GROUPS AND GEOMETRIC QUANTIZATION [1] [2]

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Abstract. The quantum groups are neither group nor necessarily to be physically quantum. Namely, they can be realized at both classical and quantum levels by means of symplectic geometry and geometric quantization method.

The quantum groups [1-5] are usually thought to appear in certain quantum physical systems and when the quantum systems revert to classical ones the quantum groups reduce to the corresponding Lie groups. We found [6-12], however, this is not the case in principle. Actually, the quantum groups are neither groups (this is well-known) nor necessarilly to be physically quantum. That is, they can be realized at both classical and quantum levels. In this talk I would briefly introduce this approach to the quantum groups by means of the symplectic geometry of the phase spaces of certain classical mechanical systems and the geometric quantization theory [13].

In order to realize the quantum group $SU_q(2)$ at classical level, We first present the classical q-deformed algebra of $SU(2)$, denoted $SU_{q,\hbar\to 0}(2)$, which is isomorphic to the so-called quantum algebra $SU_q(2)$, in both Poisson brackets and Lie

[1] Work supported in part by the National Natural Science Foundation of China.

[2] Invited talk to be given at The International Conference on Fluid Mechanics and Theoretical Physics In Honor of Professor Pei-Yuan CHOU's 90th Anniversary to be held in Beijing, June 1-3, 1992.

brackets. On the classical phace space of two linear harmonic oscillators C^2 and the line bundle over it, we construct a set of q-deformed observables which form the $SU_{q,\hbar\to 0}(2)$ algebra in the Poisson brackets and a set of corresponding operators which generate the $SU_{q,\hbar\to 0}(2)$ algebra in Lie brackets. Once such operators are given, the Hopf algebra structure can be easily set up and the entire structure of the quantum group $SU_q(2)$ follows.

In fact, these operators are nothing but the prequantization operators of the q-deformed observables. Since the prequantization line bundle over the phase space is still classical, so these operator together with unit provide a realization of the quantum group $SU_q(2)$ at classical level. Of course, there is a tiny but crucial difference between these operators and the usual prequantization operators [13]. Namely, there is no Planck constant $\hbar$ at this level in our approach. From physical point of view, the Planck constant $\hbar$ is of completely quantum characteristic so that it should appear only after suitable polarization is taken rather than at prequantization level. In other words, polarization turns the classical q-deformed algebra $SU_{q,\hbar\to 0}(2)$ to the quantum one of $SU_q(2)$, denoted $SU_{q,\hbar}(2)$, for which the Hopf algebra and the quantum group structure is well established.

Let us begin with a system of two harmonic oscillators with the Hamiltonian and the symplectic form on the phace space C^2

$$\begin{aligned} H &= \sum_{i=1}^{2} \bar{z}_i z_i, \\ \Omega &= -i \sum_{i=1}^{2} d\bar{z}_i \wedge dz_i, \end{aligned} \tag{1}$$

where $z_i = \frac{p_i + iq_i}{\sqrt{2}}$, $\bar{z}_i = \frac{p_i - iq_i}{\sqrt{2}}$.

A set of the classical observables $J_+ = z_1\bar{z}_2$, $J_- = z_2\bar{z}_1$ and $J_3 = \frac{1}{2}(z_1\bar{z}_1 - z_2\bar{z}_2)$

constitute the $SU(2)$ algebra in Poisson brackets:

$$[J_+, J_-]_{P.B.} = -i2J_3, \quad [J_3, J_\pm]_{P.B.} = -i(\pm J_\pm). \tag{2}$$

On the other hand, it is easy to see that a set of the q-deformed observables

$$J'_+ = z_1\bar{z}'_2, \quad J'_- = z_2\bar{z}'_1, \quad J'_3 = J_3, \tag{3}$$

generate the $SU_{q,\hbar\to 0}(2)$ algebra in Poisson brackets

$$[J'_+, J'_-]_{P.B.} = -i[2J'_3]_q, \quad [J'_3, J'_\pm]_{P.B.} = -i\left(\pm J'_\pm\right), \tag{4}$$

where

$$z'_i = \frac{1}{\sqrt{\gamma\sinh\gamma}}\frac{\sinh(\gamma z_i\bar{z}_i)}{z_i\bar{z}_i}z_i, \quad \bar{z}'_i = \frac{1}{\sqrt{\gamma\sinh\gamma}}\frac{\sinh(\gamma z_i\bar{z}_i)}{z_i\bar{z}_i}\bar{z}_i \tag{5}$$

and

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

On the $U(1)$ line bundle over the phase space C^2 with curvature Ω and connection (symplectic one-form)

$$\theta = -i(\bar{z}_1 dz_1 + \bar{z}_2 dz_2), \tag{6}$$

the prequantization operators of the observable f can be given by the map

$$f \longrightarrow \tilde{f} = -i(X_f - i\theta(X_f)) + f, \tag{7}$$

where X_f is the Hamiltonian vector field of f. For the reasons stated above, here we do not insert the Planck constant.

For z_i and $\bar{z}_i$, we have

$$X_{\bar{z}_i} = i\frac{\partial}{\partial z_i}, \quad X_{z_i} = -i\frac{\partial}{\partial \bar{z}_i}, \quad i = 1, 2. \tag{8}$$

and their prequantization operators

$$\tilde{z}_i = -\frac{\partial}{\partial \bar{z}_i} + z_i, \quad \tilde{\bar{z}}_i = \frac{\partial}{\partial z_i}, \tag{9}$$

with the following commutation relations

$$[\tilde{z}_i, \tilde{\bar{z}}_j] = -\delta_{ij}. \tag{10}$$

To get the prequantization operators of $J'_\pm$ and J'_3, we need a suitable ordering of $\tilde{z}_i$ and $\tilde{\bar{z}}_i$. From formulae (3) and (4) we have

$$\begin{aligned} \tilde{J}_+{}' &= \frac{1}{\sqrt{\gamma\sinh\gamma}}\frac{\tilde{z}_1}{\tilde{z}_2}\sinh\left(\gamma\tilde{z}_2\tilde{\bar{z}}_2\right), \\ \tilde{J}_-{}' &= \frac{1}{\sqrt{\gamma\sinh\gamma}}\frac{\tilde{z}_2}{\tilde{z}_1}\sinh\left(\gamma\tilde{z}_1\tilde{\bar{z}}_1\right), \\ \tilde{J}'_3 &= \frac{1}{2}\left(\tilde{z}_1\tilde{\bar{z}}_1 - \tilde{z}_2\tilde{\bar{z}}_2\right). \end{aligned} \tag{11}$$

with commutation relations of $SU_{q,\hbar\to 0}(2)$ algebra in Lie brackets

$$[\tilde{J}_+{}', \tilde{J}_-{}'] = \frac{1}{\gamma}\sinh\left(2\gamma\tilde{J}'_3\right), \quad [\tilde{J}'_3, \tilde{J}'_\pm] = \pm\tilde{J}'_\pm. \tag{12}$$

With formulae (11) and (12) we can define the co-product Δ, co-unit ϵ and antipodal mapping S,

$$\begin{aligned} \Delta\left(\tilde{J}'_3\right) &= \tilde{J}'_3 \otimes 1 + 1 \otimes \tilde{J}'_3, \\ \Delta\left(\tilde{J}'_\pm\right) &= \tilde{J}'_\pm \otimes q^{\tilde{J}'_3} + q^{-\tilde{J}'_3} \otimes \tilde{J}'_3, \\ S\left(\tilde{J}'_3\right) &= -\tilde{J}'_3, \quad S\left(\tilde{J}_3\right) = -q^{\pm 1}\tilde{J}'_\pm, \\ \epsilon\left(\tilde{J}'_\pm\right) &= \epsilon\left(\tilde{J}'_3\right) = 0, \quad \epsilon(1) = 1. \end{aligned} \tag{13}$$

These three Hopf operations are algebra homomorphisms and anti-homomorphism, *i.e.*,

$$\begin{aligned} \Delta : A &\to A \otimes A, & \Delta(ab) &= \Delta(a)\Delta(b); \\ S : A &\to A, & S(ab) &= S(b)S(a); \\ \epsilon : A &\to C, & \epsilon(ab) &= \epsilon(a)\epsilon(b), \end{aligned} \tag{14}$$

where a, b are elements of algebra A (which is at present $SU_{q\hbar\to 0}(2)$), and C the field of complex numbers. The above three operations, supplemented by the identical mapping id and multiplication m, should be consistent, *i.e.*,

$$\begin{aligned}(\mathrm{id}\otimes\Delta)\Delta(a) &= (\Delta\otimes\mathrm{id})\Delta(a);\\ m(\mathrm{id}\otimes S)\Delta(a) &= m(S\otimes\mathrm{id})\Delta(a)=\epsilon(a)\cdot 1;\\ (\epsilon\otimes\mathrm{id})\Delta(a) &= (\mathrm{id}\otimes\epsilon)\Delta(a)=a,\end{aligned}\tag{15}$$

and compatible with the algebra relations.

We have seen that Eq.(13) defines a neither commutative nor co-commutative Hopf algebra. As is well known in the quantum group theory, once such a Hopf algebra structure is given, the entire quantum group structure follows [1-5]. Thus the classical q-deformed algebra $SU_{q,\hbar\to 0}(2)$ possesses the nontrivial Hopf algebra structure as well as the entire structure of the quantum groups, but it has nothing to do with physical quantization at all.

In geometric quantization approach, to quantize a system on the symplectic space (C^2,Ω) is to introduce a suitable polarization. Let us take the Kähler polarization with basis $X=\left\{\frac{\partial}{\partial\bar z_1},\frac{\partial}{\partial\bar z_2}\right\}$. Then from the quantum map, we get the quantum operator of a given classical observable f (which is supposed to preserve the polarization):

$$f\longrightarrow\hat f=-i\hbar(X_f-i\theta(X_f))+f-\frac{i}{2}\hbar a,\tag{16}$$

where a is determined by the formula

$$[X_f,X]=aX.\tag{17}$$

The quantum operators of z_i and $\bar z_i$ then take the form

$$\hat z_i=z_i,\quad \hat{\bar z}_i=\hbar\frac{\partial}{\partial z_i},\tag{18}$$

with commutation relations

$$[\hat z_i,\hat{\bar z}_j]=-\hbar\delta_{ij}.\tag{19}$$

Similarly, the quantum versions of $J'_\pm$ and J'_3 are

$$\begin{aligned}\hat J'_+ &= \frac{1}{\sqrt{\gamma\sinh\gamma}}\frac{z_1}{z_2}\sinh\left(\gamma\hbar z_2\frac{\partial}{\partial z_2}\right)\\ \hat J'_- &= \frac{1}{\sqrt{\gamma\sinh\gamma}}\frac{z_2}{z_1}\sinh\left(\gamma\hbar z_1\frac{\partial}{\partial z_1}\right)\\ \hat J'_3 &= -\frac{\hbar}{2}\left(z_2\frac{\partial}{\partial z_2}-z_1\frac{\partial}{\partial z_1}\right)\end{aligned}\tag{20}$$

In terms of (19), $\hat J'_\pm$ and $\hat J'_3$ give rise to the quantum q-deformed algebra $SU_{q,\hbar}(2)$,

$$[\hat J'_+,\hat J'_-]=\frac{\sinh\gamma\hbar}{\gamma}\frac{\sinh\left(2\gamma\hat J'_3\right)}{\sinh\gamma},\quad [\hat J'_3,\hat J'_\pm]=\hbar\hat J'_\pm,\tag{21}$$

which is isomorphic to the usual quantum algebra $SU_q(2)$, but with an additional parameter $\hbar$ characterizing the physical quantization of the system.

Obviously, the Hopf algebraic structure of $SU_{q,\hbar}(2)$ is identical to that of $SU_{q,\hbar\to 0}(2)$ in form and the entire structure of the quantum groups follows as well.

So far we have constructed the Hopf algebraic structures for both $SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$ by the symplectic geometry and geometric quantization method. Although we have dealt with the particular quantum algebra $SU_q(2)$, it is easy to convince that the principle of the method can be applied to generic cases more or less straightforwardly.

It is worthwhile to point out that the deformations $z_i\to z'_i$ are quasiconformal deformations while q is real. And $(z',\bar z')$ describe a q-deformed oscillator which is a nonlinear object but can be exactly solved at both classical and quantum levels [6-9].

In [10-12], both $SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$ are realized on a line bundle over a q-deformed sphere, which is closely linked to the q-deformed rotator and the q-deformed spherical top. Equivalently, they can also be realized by a set of q-deformed functions, which satisfy the equation for the q-deformed sphere, on the sphere and by corresponding operators on the line bundle over the sphere. The Lie group theory shows that the sphere is known as the coadjoint orbit of the Lie group $SU(2)$ and for any given compact semisimple Lie group there always exists a natural symplectic structure on its coadjoint orbit. Therefore, we can find certain q-deformed functions defined on the coadjoint orbit such that they form a q-deformed algebra in Poisson brackets given by the symplectic structure. Correspondingly, we can also construct a set of q-deformed operators on the line bundle over the coadjoint orbit to get the q-deformed algebra in Lie brackts physically at both classical and quantum level.

Our approach also indicates that the quantum groups may be the "symmetries" of single deformed dynamical systems, such as the q-deformed oscillator, the q-deformed rotator as well as the q-deformed spherical top and so on, in addition to that of the solvable models in quantum field theory and statistical mechanics.

Finally, It should be mentioned that the Poisson bracket realization of the "quantum algebra" $sl_q(2)$ has also heen found by Flato and Lu [14].

I would like to thank Z. Chang, W. Chen, S.M. Fei and H. Yan for creative collaboration on this topic during last two years.

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