AS-ITP-92-07

18

ASITP

INSTITUTE OF THEORETICAL PHYSICS **ACADEMIA SINICA**

AS–ITP–92–07

THE COVARIANT GENERATORS OF THE QUANTUM TRANSFORMATION GROUPS ON THE SPINOR SPACES

Xing–Chang SONG

P.O.Box 2735, Beijing 100080, The People's Republic of China

Telefax : (086)-1-2562587 **Telephone** : 2568348

Telex : 22040 **BAOAS CN** **Cable** : 6158

AS-ITP-92-07
January, 1992

The Covariant Generators of the Quantum Transformation Groups on the Spinor Spaces [1]

Xing-Chang SONG

CCAST(World Laboratory)
P.O.BOX 8730, Beijing 100080

Department of Physics[2]
Peking University, Beijing 100871

Institute of Theoretical Physics
Academia Sinica, Beijing 100080

Abstract

The covariant formalism of the generators for the quantum group $SL_q(2,C)$ on the quantum spinor space is constructed manifestly, very similar to the ones on the quantum group itself proposed by Woronowicz through his 4 D_+ calculus. Also constructed is the set of eight generators for the quantum Lorentz group on the bispinor space. In the limit $q \longrightarrow 1$, these generators reduce to those of the left and the right $SL(2,C)$ plus two corresponding Casimirs.

[1] Work supported in part by the National Natural Science Foundation of China and the Doctoral Programme Foundation Institution of High Education.

[2] Communication address.

I Introduction

The importance of the idea of the quantum groups [1] is now more and more extensively understood by most of physicists. This is due to its close connection to the Yang-Baxter equation [2] which plays a deep role in various physical problems. In our opinion, as for the classical group, for more direct physical application, the view considering the quantum group as the "quantum" symmetry of some basic physical objects, or "quantum" space, is more attractive.

The noncommutative differential calculus over the quantum group itself and the generators of the quantum group have been established by Woronowicz [1]. A general construction of quantum group as linear transformations upon the quantum plane has been suggested by Manin[3]. And the covariant differential calculus on the quantum plane was developed by Wess and Zumino [4] and generalized to the more general quantum spaces including the quantum orthogonal plane and symplectic plane[5], and more recently, to the quantum Minkowski space[6].

In this paper we would like to give the explicit covariant form of the generators of quantum groups $SL_q(2,C)$ and $SO_q(3,1)$ on these quantum spaces. The main tools are the consistent covariant differential calculus on these spaces[4,5] and the projection operator method developed in Ref[5-7]. We start with the linear representation of the $SL_q(2,c)$ on the spinor space in Section II. With the help of the differential calculus on the spinor space, we construct the differential realization of the covariant generators explicitly in Section III, just as the ones for the ordinary $SL(2,c)$ group. We have a set of four generators satisfying the relations similar to those given by Woronowicz [8] in considering the 4 D_+ calculus on quantum group $SL_q(2,C)$ itself and by Wu & Zhang recently [12] in developing RTF method [1] to discuss the differential calculus on quantum matrix groups. In $q \longrightarrow 1$ limit three of them reduce to the generators of classical $SL(2,C)$ and the fourth is connected with the Casimir operator(total angular momentum)[9].Then we turn to discuss the counterpart set of generators in conjugate spinor (dotted spinor) space in Section IV. Combining these two set we get the total eight generators of quantum Lorentz group $SO_q(3,1)$ in the bispinor space. The action of these generators on the spinors as well as the 4-vectors is presented in Section V.

II. Quantum Spinors and $\check{R}$ Matrix

We start from two-dimensional q-spinor $u^\alpha = (u^1, u^2) = (u, v)$ with its components obeying the q-deformed commutation relation

$$uv = q\, vu \ . \tag{2.1}$$

This relation is preserved under the transformation of the q-spinor (The summation over the repeated indeces is understood throughout this paper)

$$u^\alpha \longrightarrow u'^\alpha = M^\alpha{}_\beta u^\beta \ , \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.2}$$

if M is a $GL_q(2)$ matrix with its entries satisfying the definition relations

$$\begin{array}{lll} ab = q\, ba & ac = q\, ca & ad - da = (q - q^{-1})bc \\ bc = cb & bd = q\, db & cd = q\, dc \end{array} \tag{2.3}$$

and commuting with the components of the spinor, *i.e.*, $au = ua$ etc. The relations (2.1) and (2.3) can be put into the following form

$$u^\alpha u^\beta = q^{-1}\ \check{R}^{\alpha\beta}{}_{\gamma\delta} u^\gamma u^\delta \ , \tag{2.4a}$$

$$\check{R}^{\alpha\beta}{}_{\gamma\delta} M^\gamma{}_{\gamma'} M^\delta{}_{\delta'} = M^\alpha{}_{\alpha'} M^\beta{}_{\beta'} \check{R}^{\alpha'\beta'}{}_{\gamma'\delta'} \ , \quad (\check{R}_{12} M_1 M_2 = M_1 M_2 \check{R}_{12}) \tag{2.4b}$$

by introducing the numerical $\check{R}_{12}$ matrix associated with $GL_q(2)$

$$\check{R}(q) = \left(\check{R}^{\alpha\beta}{}_{\gamma\delta}\right) = \begin{pmatrix} q & & & \\ & q - q^{-1} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix} \tag{2.5}$$

which satisfies the Yang-Baxter equation (in the braid form)

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{2.6a}$$

and the reduction relation

$$(\check{R} - q)(\check{R} + q^{-1}) = 0 \ . \tag{2.6b}$$

The left-acting as well as the right-acting eigenvalue equations can be written as

$$\check{R}(q)^{\alpha\beta}{}_{\gamma\delta}\ t_m(q)^{\gamma\delta} = q\ t_m(q)^{\alpha\beta} \ , \quad \check{R}(q)^{\alpha\beta}{}_{\gamma\delta}\ s(q)^{\gamma\delta} = -q^{-1} s(q)^{\alpha\beta} \ , \tag{2.7a}$$

$$\bar{t}^m(q)_{\alpha\beta}\check{R}(q)^{\alpha\beta}{}_{\gamma\delta} = q\ \bar{t}^m(q)_{\gamma\delta} \ , \quad \bar{s}(q)_{\alpha\beta}\check{R}(q)^{\alpha\beta}{}_{\gamma\delta} = -q^{-1}\bar{s}(q)_{\gamma\delta} \tag{2.7b}$$

Now since the matrix $\check{R}$ is symmetric, the components of $\bar{t}^m(q)$ and $\bar{s}(q)$ may be taken the same as those of their left-acting counterparts $t_m(q)$ and $s(q)$, namely[3]

$$\begin{aligned} t_+(q)^{\alpha\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad t_3(q)^{\alpha\beta} = \begin{pmatrix} 0 & -q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix} [2]^{-1/2} \ , \\ t_-(q)^{\alpha\beta} &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad s(q)^{\alpha\beta} = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} [2]^{-1/2}, \end{aligned} \tag{2.8}$$

where the q-number is defined as $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. The q-analogue of the Levi-Civita symbols $\epsilon(q)_{\alpha\beta}$ and $\epsilon(q)^{\alpha\beta}$ are related to the singlet eigenvectors

$$\epsilon(q)_{\alpha\beta} = -[2]^{1/2}\bar{s}(q)_{\alpha\beta} \ , \quad \epsilon(q)^{\alpha\beta} = [2]^{1/2} s(q)^{\alpha\beta} \tag{2.9a}$$

and are normalized in the way such that

$$\epsilon(q)_{\alpha\beta}\epsilon(q)^{\beta\gamma} = \delta_\alpha{}^\gamma \ , \quad \epsilon(q)^{\alpha\beta}\epsilon(q)_{\beta\gamma} = \delta^\alpha{}_\gamma \ . \tag{2.9b}$$

As is used in Ref[6], $t_m(q)$ and $s(q)$ are grouped together to form the matrix-valued four vectors

$$t_\mu(q) = (t_0(q),\ \ t_m(q)) \ , \quad t_0(q)^{\alpha\beta} = q\ s(q)^{\alpha\beta} \ ; \tag{2.10a}$$

$$\bar{t}^\mu(q) = (\bar{t}^0(q),\ \ \bar{t}^m(q)) \ , \quad \bar{t}^0(q)^{\alpha\beta} = q^{-1}\ \bar{s}(q)_{\alpha\beta} \ . \tag{2.10b}$$

It is easy to check that $t_\mu(q)$ and $\bar{t}^\mu(q)$ satisfy the following orthonomality condition

$$t_\mu(q)^{\alpha\beta}\ \bar{t}^\nu(q)_{\alpha\beta} = \delta_\mu{}^\nu \tag{2.11a}$$

and the completeness relation

$$t_\mu(q)^{\alpha\beta}\ \bar{t}^\mu(q)_{\gamma\delta} = \delta^\alpha{}_\gamma \delta^\beta{}_\delta \equiv E^{\alpha\beta}{}_{\gamma\delta} \ . \tag{2.11b}$$

The projection operators for the triplet and the singlet can be defined as

$$Q^{(2)}(q)^{\alpha\beta}{}_{\gamma\delta} = t_m(q)^{\alpha\beta}\bar{t}^m(q)_{\gamma\delta} \ , \quad Q^{(1)}(q)^{\alpha\beta}{}_{\gamma\delta} = s(q)^{\alpha\beta}\bar{s}(q)_{\gamma\delta} \tag{2.12}$$

respectively, with the properties

$$Q^{(i)}Q^{(j)} = \delta^{ij}Q^{(j)} \ , \quad Q^{(1)} + Q^{(2)} = E \ , \tag{2.13}$$

[3]The convention we adopted here is different from that in Ref[6] by exchanging + and−.

$\check{R}$ matrix and other relevant matrices can be expressed as the linear combination of Q's,

$$\check{R}(q) = \lambda_2 Q^{(2)}(q) + \lambda_1 Q^{(1)}(q) = qQ^{(2)} - q^{-1}Q^{(1)} . \tag{2.14}$$

Coversely, the projectors can be re-expressed in terms of $\check{R}$:

$$Q^{(2)} = \frac{\check{R} - \lambda_1 E}{\lambda_2 - \lambda_1} , \quad Q^{(1)} = \frac{\check{R} - \lambda_2 E}{\lambda_1 - \lambda_2} . \tag{2.15}$$

From these relations and the Yang-Baxter relation (2.4b) and the Yang-Baxter equation (2.6a) we see immediately that

$$Q^{(i)}_{12} M_1 M_2 = M_1 M_2 Q^{(i)}_{12} , \tag{2.16a}$$

$$Q^{(i)}_{12} \check{R}_{23} \check{R}_{12} = \check{R}_{23} \check{R}_{12} Q^{(i)}_{23} , \quad \check{R}_{12} \check{R}_{23} Q^{(i)}_{12} = Q^{(i)}_{23} \check{R}_{12} \check{R}_{23} . \tag{2.16b}$$

The preservation of the q-commutation relation (2.1) comes from the fact that $\epsilon(q)_{\alpha\beta}$ and $\epsilon(q)^{\alpha\beta}$ are the eigenvectors of $M \otimes M$with $Det_q M$ being the associated eigenvalue:

$$M^\alpha{}_\gamma M^\beta{}_\delta \epsilon(q)^{\gamma\delta} = Det_q M \ \epsilon(q)^{\alpha\beta} , \quad \epsilon(q)_{\alpha\beta} M^\alpha{}_\gamma M^\beta{}_\delta = Det_q M \ \epsilon(q)_{\gamma\delta} \tag{2.17}$$

where $Det_q M = ad - qbc$ is the center of algebra generated by a, b, c and d. For M with $Det_q M = 1$ we say M is an $SL_q(2, C)$ matrix. In this case we see immediately from (2.17) that

$$M^\alpha{}_\gamma \epsilon(q)^{\gamma\delta} M^\beta{}_\delta \epsilon(q)_{\beta\alpha'} = \delta^\alpha{}_{\alpha'} \tag{2.18a}$$

$$\epsilon(q)_{\beta'\alpha} M^\alpha{}_\gamma \epsilon(q)^{\gamma\delta} M^\beta{}_\delta = \delta_{\beta'}{}^\beta \tag{2.18b}$$

This implies that

$$\epsilon(q)^{\gamma\delta} M^\beta{}_\delta \epsilon(q)_{\beta\alpha} = M^{-1\gamma}{}_\alpha \tag{2.19a}$$

$$\epsilon(q)_{\beta\alpha} M^\alpha{}_\gamma \epsilon(q)^{\gamma\delta} = (M^t)^{-1}{}_\beta{}^\delta , \tag{2.19b}$$

where M^t is the matrix transpose of M. Now since $\epsilon(q)_{\beta\alpha} (= -\epsilon(q^{-1})_{\alpha\beta} \neq -\epsilon(q)_{\alpha\beta})$ is not ordinary antisymmetric with respect to α and β, $(M^{-1})^t \neq (M^t)^{-1}$. This fact tells us that starting from the basic spinor u^α, we can build two different types of lower index spinors. The one is

$$u_\alpha \equiv u^\beta \epsilon(q)_{\beta\alpha} \longrightarrow u_\gamma M^{-1\gamma}{}_\alpha \tag{2.20a}$$

and the other

$$\tilde{u}_\alpha \equiv \epsilon(q)_{\alpha\beta} u^\beta \longrightarrow (M^t)^{-1}{}_\alpha{}^\gamma \tilde{u}_\gamma \tag{2.20b}$$

So two types of invariants can be formed

$$u_\alpha u^\alpha \longrightarrow u_\gamma M^{-1\gamma}{}_\alpha M^\alpha{}_\beta u^\beta = u_\beta u^\beta , \tag{2.21a}$$

$$u^\alpha \tilde{u}_\alpha \longrightarrow u^\beta M^t{}_\beta{}^\alpha (M^t)^{-1}{}_\alpha{}^\gamma \tilde{u}_\gamma = u^\beta \tilde{u}_\beta . \tag{2.21b}$$

In most of the cases we use lower index spinor u_α which transforms contravariantly to u^α as in (2.21a). The only exception is the derivative spinor $\partial_\alpha = \frac{\partial}{\partial u^\alpha}$, which transforms as in (2.21b) indeed:

$$\partial_\alpha \longrightarrow (M^t)^{-1}{}_\alpha{}^\gamma \partial_\gamma . \tag{2.22a}$$

Then

$$\delta_\alpha{}^\beta \equiv \partial_\alpha u^\beta \longrightarrow (M^t)^{-1}{}_\alpha{}^\gamma \partial_\gamma u^\delta M^t{}_\delta{}^\beta = (M^t)^{-1}{}_\alpha{}^\gamma \delta_\gamma{}^\delta M^t{}_\delta{}^\beta = \delta_\alpha{}^\beta . \tag{2.23}$$

Therefore the derivative spinor with upper index $\partial^\alpha \equiv \epsilon(q)^{\alpha\beta}\partial_\beta$ transforms just as the basic spinor u^α does:

$$\epsilon(q)^{\alpha\gamma}\partial_\gamma \equiv \partial^\alpha \longrightarrow M^\alpha{}_\beta \partial^\beta . \tag{2.22b}$$

The components of the derivative spinor obey a q-commutation relation similar to (2.1)

$$\partial_2 \partial_1 = q \ \partial_1 \partial_2 \tag{2.24a}$$

which can be re-written as

$$\partial_\beta \partial_\alpha = q^{-1} \ \check{R}(q)_{\alpha\beta}{}^{\gamma\delta} \partial_\delta \partial_\gamma . \tag{2.24b}$$

For consistent differential calculus, we also need the relations between coordinates and derivatives. The result is first given by Wess and Zumino[4]:

$$\partial_\alpha u^\beta = \delta_\alpha{}^\beta + C^{\beta\alpha'}{}_{\alpha\beta'} \ u^{\beta'} \partial_{\alpha'} \tag{2.25}$$

Of the two possible choises in the following discussion we choose $C = q^{-1}\check{R}^{-1}$. It can be easily checked that the above relation is covariant with respect to the transformation in (2.2) and (2.22)

The conjugate spinor $\bar{u}_{\dot{\alpha}} \equiv (u^\alpha)^*$transforms[6] according to the hermitian conjugate of the quantum matrix M:

$$\bar{u}_{\dot{\alpha}} \longrightarrow \bar{u}_{\dot{\beta}} M^{+\dot{\beta}}{}_{\dot{\alpha}} . \tag{2.26a}$$

Then

$$\bar{u}^{\dot{\alpha}} \equiv -\bar{u}_{\dot{\beta}}\epsilon(q)^{\dot{\beta}\dot{\alpha}} \longrightarrow \bar{M}^{\dot{\alpha}}{}_{\dot{\gamma}}\, \bar{u}^{\dot{\gamma}} \tag{2.26b}$$

from a similar relation for $\bar{M}$ as in (2.18). Quantum matrix $\bar{M} = (M^+)^{-1}$ satisfies the Yang-Baxter relation similar to (2.4b),

$$\check{R}^{\dot{\alpha}\dot{\beta}}{}_{\dot{\gamma}\dot{\delta}}\bar{M}^{\dot{\gamma}}{}_{\dot{\gamma}'}\bar{M}^{\dot{\delta}}{}_{\dot{\delta}'} = \bar{M}^{\dot{\alpha}}{}_{\dot{\alpha}'}\bar{M}^{\dot{\beta}}{}_{\dot{\beta}'}\check{R}^{\dot{\alpha}'\dot{\beta}'}{}_{\dot{\gamma}'\dot{\delta}'} \tag{2.27}$$

and an additional relation[9] with M:

$$\check{R}^{\dot{\alpha}\dot{\beta}}{}_{\alpha\dot{\beta}}M^{\alpha}{}_{\gamma}\bar{M}^{\dot{\beta}}{}_{\dot{\delta}} = \bar{M}^{\dot{\alpha}}{}_{\dot{\gamma}}M^{\beta}{}_{\delta}\check{R}^{\dot{\gamma}\delta}{}_{\gamma\dot{\delta}} \ . \tag{2.28}$$

Similar to (2.2b), for $\bar{\partial}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{u}_{\dot{\alpha}}}$, we see $\bar{\partial}_{\dot{\alpha}} \equiv -\epsilon(q)_{\dot{\alpha}\dot{\beta}}\bar{\partial}^{\dot{\beta}}$ transforms just as the dotted spinor $\bar{u}_{\dot{\alpha}}$ does. Now considering q real and taking the complex conjugate of Eq(2.1) and (2.24) we obtain that the components of the dotted spinor $\bar{u}_{\dot{\alpha}}$ and components of the dotted derivative spinor $\bar{\partial}^{\dot{\alpha}}$ obey the relations

$$\bar{u}_{\dot{2}}\bar{u}_{\dot{1}} = q\bar{u}_{\dot{1}}\bar{u}_{\dot{2}} \ , \quad \bar{\partial}^{\dot{1}}\bar{\partial}^{\dot{2}} = q\bar{\partial}^{\dot{2}}\bar{\partial}^{\dot{1}} \tag{2.29a}$$

which can be recast to the $\check{R}$ commutation form

$$\bar{u}_{\dot{\beta}}\bar{u}_{\dot{\alpha}} = q^{-1}\check{R}^{\dot{\gamma}\dot{\delta}}{}_{\dot{\alpha}\dot{\beta}}\bar{u}_{\dot{\delta}}\bar{u}_{\dot{\gamma}} \ , \quad \bar{\partial}^{\dot{\beta}}\bar{\partial}^{\dot{\alpha}} = q^{-1}\check{R}^{\dot{\beta}\dot{\alpha}}{}_{\dot{\delta}\dot{\gamma}}\bar{\partial}^{\dot{\delta}}\bar{\partial}^{\dot{\gamma}} \tag{2.29b}$$

Also we have relations among $\bar{u}_{\dot{\beta}}$ and $\bar{\partial}^{\dot{\alpha}}$

$$\bar{u}_{\dot{\beta}}\bar{\partial}^{\dot{\alpha}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}} + q^{-1}\check{R}^{-1}(q)^{\dot{\alpha}\dot{\beta}'}{}_{\dot{\beta}\dot{\alpha}'}\bar{u}^{\dot{\alpha}'}\bar{\partial}_{\dot{\beta}'} \ . \tag{2.30}$$

In discussing the bispinor comprising both u^{α} and $\bar{u}_{\dot{\alpha}}$, we also need the cross commutation relations between u^{α} and $\bar{u}_{\dot{\alpha}}$, between ∂_{α} and $\bar{u}_{\dot{\alpha}}$ ($\bar{\partial}^{\dot{\alpha}}$ and u^{α}) and between ∂_{α} and $\bar{\partial}^{\dot{\alpha}}$. The results are found to be

$$u^{\alpha}\bar{u}_{\dot{\alpha}} = \bar{u}_{\dot{\beta}}u^{\beta}q\check{R}^{-1}(q)^{\dot{\beta}\dot{\alpha}}{}_{\dot{\beta}\dot{\alpha}} \ , \tag{2.31a}$$

$$\partial^{\alpha}\bar{u}_{\dot{\alpha}} = \bar{u}_{\dot{\beta}}\partial^{\beta}\check{R}^{-1}(q)^{\dot{\beta}\dot{\alpha}}{}_{\dot{\beta}\dot{\alpha}} \ , \quad u^{\alpha}\bar{\partial}_{\dot{\alpha}} = \bar{\partial}_{\dot{\beta}}u^{\beta}\check{R}^{-1}(q)^{\dot{\beta}\dot{\alpha}}{}_{\dot{\beta}\dot{\alpha}} \ , \tag{2.31b}$$

$$\partial^{\alpha}\bar{\partial}_{\dot{\alpha}} = \bar{\partial}_{\dot{\beta}}\partial^{\beta}q\check{R}^{-1}(q)^{\dot{\beta}\dot{\alpha}}{}_{\dot{\beta}\dot{\alpha}} \ . \tag{2.31c}$$

The consistency of all these relations can be directly checked by considering the triplet product of operators chosen from (u^{α}, ∂^{β}, $\bar{u}_{\dot{\alpha}}$, $\bar{\partial}_{\dot{\beta}}$) and altering the order in two different ways.

Starting from the given 4×4 $\check{R}$ matrix (2.5) we can obtain different higher dimensional $\check{R}$ matrices by using different "fusion": for example

$$\check{R}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'} = \check{R}^{\alpha\gamma}{}_{\alpha'\gamma'}\check{R}^{\beta\delta}{}_{\beta'\delta'}(= P_{23}\check{R}_{12}\check{R}_{34}P_{23}) \tag{2.32a}$$

is the $\check{R}$ matrix corresponding to the quantum group $SO_q(4)$,

$$\tilde{\mathcal{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'} = q^{-1}\check{R}^{\beta\gamma}{}_{bc}\check{R}^{\alpha b}{}_{\alpha'b'}\check{R}^{c\delta}{}_{c'\delta'}\check{R}^{-1}(q)^{b'c'}{}_{\beta'\gamma'}(= q^{-1}\check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}_{23}^{-1}) \tag{2.32b}$$

is the $\check{R}$ matrix for the quantum Lorentz group $SO_q(3,1)$ [6,10], and

$$\check{\mathbf{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'} = q^{-2}\check{R}^{\beta\gamma}{}_{bc}\check{R}^{\alpha b}{}_{\alpha'b'}\check{R}^{c\delta}{}_{c'\delta'}\check{R}^{b'c'}{}_{\beta'\gamma'}(= q^{-2}\check{R}_{23}\check{R}_{12}\check{R}_{34}\check{R}_{23}) \tag{2.32c}$$

represents a reducible $\check{R}$ matrix[7,11]. By multiplying $\bar{t}^{\mu}{}_{\alpha\beta}$, $\bar{t}^{\nu}_{\gamma\delta}$, $t^{\alpha'\beta'}_{\kappa}$ and $t^{\gamma'\delta'}_{\lambda}$, Eq(2.32c) leads to

$$\begin{aligned}\check{\mathbf{R}}^{\mu\nu}{}_{\kappa\lambda} &\equiv \bar{t}^{\mu}(q)_{\alpha\beta}\bar{t}^{\nu}_{\gamma\delta}\check{\mathbf{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'}t^{\alpha'\beta'}_{\kappa}t^{\gamma'\delta'}_{\lambda} \\ &= \check{\mathbf{R}}^{00}{}_{00}\oplus\check{\mathbf{R}}^{m0}{}_{0l}\oplus\check{\mathbf{R}}^{0n}{}_{k0}\oplus\check{\mathbf{R}}^{mn}{}_{kl} \ .\end{aligned} \tag{2.33}$$

Here the reduction of $\check{\mathbf{R}}^{\mu\nu}{}_{\kappa\lambda}$ comes from the repeated use of (2.16b), and $\check{\mathbf{R}}^{mn}{}_{kl}$ is a $\check{R}$ matrix associated with $SO_{q^2}(3)$, corresponding quantum matrix being[11]

$$D^{m}{}_{k} = \bar{t}^{m}(q)_{\alpha\gamma}M^{\alpha}{}_{\beta}M^{\gamma}{}_{\delta}t_{k}(q)^{\beta\delta} \ . \tag{2.34}$$

Similarly, from Eq(2.32b) we obtain

$$\tilde{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} \equiv \bar{t}^{\mu}_{\alpha\beta}\bar{t}^{\nu}_{\gamma\delta}\tilde{\mathcal{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'}t^{\alpha'\beta'}_{\kappa}t^{\gamma'\delta'}_{\lambda} \tag{2.35}$$

which is indeed the $\check{R}$ matrix for quantum Lorentz group with its singlet eigenvector

$$g_{\mu\nu} = (g_{+-}, g_{33}, g_{00}, g_{-+}) = (-q^{-1}, -1, 1, -q) \tag{2.36}$$

being identified as the q-deformed Lorentz metric[6].

III. $SL_q(2,C)$ Generators on Spinor Space

Now we are ready to construct the set of generators on the spinor space. As for ordinary angular momentum operators in the classical spinor space[9], we consider the combination operators

$$L^{\alpha\beta} = u^{\alpha}\partial^{\beta} = u^{\alpha}\epsilon(q)^{\beta\gamma}\partial_{\gamma} \ . \tag{3.1}$$

From (2.24) and (2.25)

$$\partial_\beta\partial_\alpha = q^{-1}\check{R}^{\gamma\delta}{}_{\alpha\beta}\partial_\delta\partial_\gamma\ , \quad \partial_\alpha u^\beta = \delta_\alpha{}^\beta + q^{-1}\check{R}^{-1}(q)^{\beta\alpha'}{}_{\alpha\beta'}u^{\beta'}\partial_{\alpha'}$$

and the important relations

$$\epsilon(q)^{\beta\delta}\check{R}^{\alpha'\gamma}{}_{\delta\beta'}\epsilon(q)_{\gamma\alpha} = q\check{R}^{-1}(q)^{\beta\alpha'}{}_{\beta'\alpha} = \epsilon(q)_{\beta'\delta'}\check{R}^{\delta'\beta}{}_{\alpha\gamma'}\epsilon(q)^{\gamma'\alpha'} \tag{3.2}$$

we obtain

$$\partial^\alpha u^\beta = \epsilon(q)^{\alpha\beta} + q^{-2}\check{R}^{\alpha\beta}{}_{\beta'\alpha'}u^{\beta'}\partial^{\alpha'} \tag{3.3a}$$

$$\partial^\alpha\partial^\beta = q^{-1}\check{R}^{\alpha\beta}{}_{\beta'\alpha'}\partial^{\beta'}\partial^{\alpha'} \tag{3.3b}$$

together with the relation (2.4a)

$$u^\alpha u^\beta = q^{-1}\check{R}^{\alpha\beta}{}_{\beta'\alpha'}u^{\beta'}u^{\alpha'} \tag{3.4}$$

Then by a straightforward but tedious derivation we obtain

$$\begin{aligned} &L^{\alpha\beta}L^{\gamma\delta} - q^{-1}\check{\mathcal{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'}L^{\alpha'\beta'}L^{\gamma'\delta'} \\ &= q\ \check{R}^{-1}(q)^{\alpha\beta}{}_{\alpha'\beta'}\epsilon(q)^{\beta'\gamma'}\check{R}^{-1}(q)^{\gamma\delta}{}_{\gamma'\delta'}\check{R}^{\alpha'\delta'}{}_{\xi\eta}L^{\xi\eta} + \epsilon(q)^{\beta\gamma}L^{\alpha\delta} \end{aligned} \tag{3.5}$$

where $\check{\mathcal{R}}^{\alpha\beta,\gamma\delta}{}_{\alpha'\beta',\gamma'\delta'}$ is the $\check{R}$ matrix defined in Eq(2.32b) which can be transferred to $\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}$ as in Eq(2.35). As is discussed in Ref[6], the 16×16 $\check{R}$ matrix $\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}$ has three distinctive eigenvalues: the single one $\lambda_0(q) = q^{-3}$, the sixfold one $\lambda_1(q) = -q^{-1}$ and ninefold one $\lambda_2(q) = q$.

$$\begin{aligned} &\bar{v}(q)_{\mu\nu}\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} = \lambda_0(q)\bar{v}(q)_{\kappa\lambda}\ , \\ &\bar{u}^{ms}(q)_{\mu\nu}\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} = \lambda_1(q)\bar{u}^{ms}(q)_{\kappa\lambda}\ , \\ &\bar{w}^{mn}(q)_{\mu\nu}\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda} = \lambda_2(q)\bar{w}^{mn}(q)_{\kappa\lambda} \end{aligned} \tag{3.6}$$

where for $\bar{u}(q)$, $m = +, 3, -$ and $s = \pm$, for $\bar{w}(q)$ (m,n)= $(2,0)$, $(1,\pm 1)$, $(0,\pm 2)$, $(0,0)$,$(\bar{1},\pm 1)$ and $(\bar{2},0)$. To write down the explicit form of $\check{\mathcal{R}}$, we order the Lorentz index $\mu = (+,3,0,-)$ and define a "charge" for each index: $c(+) = +1$, $c(-) = -1$, $c(3)$=$c(0)$=0. Then $\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}$ is "charge" conservative $c(\mu)+c(\nu) = c(\kappa)+c(\lambda) = m$, and breaks into the block diagonal form according to the total "charge" m.

$$\check{\mathcal{R}}(q) = S_1^{(2)} \oplus S_4^{(1)} \oplus S_6^{(0)} \oplus S_4^{(\bar{1})} \oplus S_1^{(\bar{2})} \tag{3.7a}$$

where $S_d^{(m)}$ is a $d\times d$ matrix with total "charge" m. The following standard order for the indeces pair (μ,ν) or (κ,λ) is adopted throughout this paper:

$$\begin{aligned} &(\mu,\nu) = (++)\quad for\quad m = 2, \\ &(\mu,\nu) = (+3,+0,3+,0+)\quad for\quad m = 1, \\ &(\mu,\nu) = (+-,33,30,03,00,-+)\quad for\quad m = 0, \\ &(\mu,\nu) = (3-,0-,-3,-0)\quad for\quad m = \bar{1}, \\ &(\mu,\nu) = (--)\quad for\quad m = \bar{2}. \end{aligned} \tag{3.7b}$$

Then the singlet eigenvector(with total "charge" zero)

$$\bar{v}(q)_{\mu\nu} = (-q^{-1},-1,0,0,1,-q)[2]^{-1/2}\ , \tag{3.8}$$

is proportional to the Lorentz metric $g(q)_{\mu\nu}$. And the sextet eigenvectors are chosen as

$$\begin{aligned} &\bar{u}^{1+}(q)_{\mu\nu} = (-q^{-1},q^{-1},q,-q)[2]^{-1}, \\ &\bar{u}^{1-}(q)_{\mu\nu} = (-q^{-1},-q,q,q^{-1})[2]^{-1}\ ; \\ &\bar{u}^{3+}(q)_{\mu\nu} = (1,q-q^{-1},q^{-1},-q,0,-1))[2]^{-1}, \\ &\bar{u}^{3-}(q)_{\mu\nu} = (1,q-q^{-1},-q,q^{-1},0,-1)[2]^{-1}\ ; \\ &\bar{u}^{\bar{1}+}(q)_{\mu\nu} = (-q^{-1},-q,q,q^{-1})[2]^{-1}, \\ &\bar{u}^{\bar{1}-}(q)_{\mu\nu} = (-q^{-1},q^{-1},q,-q)[2]^{-1}\ . \end{aligned} \tag{3.9}$$

Then by changing the bispinor index (α,β) into 4-vector index μ and defining

$$L^\mu = \bar{t}^\mu(q)_{\alpha\beta}L^{\alpha\beta} \tag{3.10}$$

we obtain

$$\begin{aligned} &L^\mu L^\nu - q^{-1}\check{\mathcal{R}}^{\mu\nu}{}_{\kappa\lambda}L^\kappa L^\lambda = f^{\mu\nu}{}_\rho L^\rho \\ &= [2]^{1/2}[u_{l+}(q)^{\mu\nu} + u_{l-}(q)^{\mu\nu}]L^l + [2]^{1/2}(q-q^{-1})v(q)^{\mu\nu}L^0 \end{aligned} \tag{3.11}$$

where $u_{ls}(q)^{\mu\nu}$ and $v(q)^{\mu\nu}$ are the left acting eigenvectors of $\check{\mathcal{R}}^{\mu\nu}{}_{\lambda\kappa}$ [6] satisfying

$$\begin{aligned} &\bar{u}^{mr}(q)_{\mu\nu}u_{ls}(q)^{\mu\nu} = \delta^m{}_l\delta^r{}_s\ , \\ &\bar{v}(q)_{\mu\nu}v(q)^{\mu\nu} = 1\ , \\ &\bar{u}^{mr}(q)_{\mu\nu}v(q)^{\mu\nu} = \bar{v}(q)_{\mu\nu}u_{ls}(q)^{\mu\nu} = 0 \end{aligned} \tag{3.12}$$

and so on. Then Eq(3.11) can be rewritten in the form

$$\begin{aligned}
\bar{u}^{l+}(q)_{\mu\nu}L^\mu L^\nu(1+q^{-2}) &= [2]^{1/2}L^l\ ,\\
\bar{u}^{l-}(q)_{\mu\nu}L^\mu L^\nu(1+q^{-2}) &= [2]^{1/2}L^l\ ,\\
\bar{v}(q)_{\mu\nu}L^\mu L^\nu(1-q^{-4}) &= [2]^{1/2}(q-q^{-1})L^0\ .
\end{aligned} \tag{3.13}$$

Or more explicitly

$$\begin{aligned}
&[L^0, L^m] = 0\ ,\\
&q(L^3-L^0)L^+ - q^{-1}L^+(L^3-L^0) = q[2]^{1/2}L^+\ ,\\
&qL^-(L^3-L^0) - q^{-1}(L^3-L^0)L^- = q[2]^{1/2}L^-\ ,\\
&L^+L^- - L^-L^+ + (q-q^{-1})L^3(L^3-L^0) = q[2]^{1/2}L^3\ ,\\
&-q^{-1}L^+L^- - L^3L^3 + L^0L^0 - qL^-L^+ = q^2[2]^{1/2}L^0\ .
\end{aligned} \tag{3.14}$$

This set of relations are obviously equivalent to those given in [8] from the 4 D_+ differential calculus on quantum group $SU_q(2)$ itself [8]. So the operators L^μ defined in (3.1) and (3.10) are indeed the derivative realization of the $SL_q(2,C)$ generators on the spinor space.

In the limit $q \longrightarrow 1$, it is easily seen that

$$\begin{aligned}
&L^+ \sim u^1\partial_2\ ,\quad L^- \sim u^2\partial_1\ ,\\
&L^3 \sim (u^1\partial_1 - q^2u^2\partial_2)/\sqrt{2}\ ,\quad L^0 \sim (u^1\partial_1 + u^2\partial_2)/\sqrt{2}\ .
\end{aligned} \tag{3.15}$$

This means that L^m are the generators of $SL(2,C)$ and L^0 an operator related to the Casimir operator L^2.

IV. Generators on Conjugate Space

On the conjugate spinor space, corresponding right-acting generators can be defined in a similar way. Consider

$$\bar{L}_{\dot\beta\dot\alpha} = \bar{\partial}_{\dot\beta}\bar{u}_{\dot\alpha} = (L^{\alpha\beta})^*\ , \tag{4.1}$$

then we see

$$\begin{aligned}
&\bar{L}_{\dot\delta\dot\gamma}\bar{L}_{\dot\beta\dot\alpha} - q^{-1}\bar{L}_{\dot\delta'\dot\gamma'}\bar{L}_{\dot\beta'\dot\alpha'}\check{\mathcal{R}}^t(q)^{\dot\alpha'\dot\beta',\dot\gamma'\dot\delta'}{}_{\dot\alpha\dot\beta,\dot\gamma\dot\delta}\\
&= \bar{L}_{\dot\eta\dot\xi}\check{R}(q)^{\dot\xi\dot\eta}{}_{\dot\alpha'\dot\gamma'}\check{R}^{-1}(q)^{\dot\gamma'\dot\delta'}{}_{\dot\gamma\dot\delta}\ q\ \epsilon(q)_{\dot\beta'\dot\gamma'}\check{R}^{-1}(q)^{\dot\alpha'\dot\beta'}{}_{\dot\alpha\dot\beta} - \epsilon(q)_{\dot\beta\dot\gamma}\bar{L}_{\dot\delta\dot\alpha}
\end{aligned} \tag{4.2}$$

where $\check{\mathcal{R}}^t(q)^{\dot\alpha'\dot\beta',\dot\gamma'\dot\delta'}{}_{\dot\alpha\dot\beta,\dot\gamma\dot\delta}$ is the matrix transpose of $\check{\mathcal{R}}(q)^{\dot\alpha\dot\beta,\dot\gamma\dot\delta}{}_{\dot\alpha'\dot\beta',\dot\gamma'\dot\delta'}$:

$$\check{\mathcal{R}}^t(q) = q^{-1}\check{R}_{23}^{-1}\check{R}_{12}\check{R}_{34}\check{R}_{23}$$

. In deriving (4.2) use has been made of the relations

$$\bar{u}_{\dot\beta}\bar{u}_{\dot\alpha} = q^{-1}\check{R}^{\dot\gamma\dot\delta}{}_{\dot\alpha\dot\beta}\bar{u}_{\dot\delta}\bar{u}_{\dot\gamma}\ ,\quad \bar{\partial}_{\dot\beta}\bar{\partial}_{\dot\alpha} = q^{-1}\check{R}^{\dot\gamma\dot\delta}{}_{\dot\alpha\dot\beta}\bar{\partial}_{\dot\delta}\bar{\partial}_{\dot\gamma}\ , \tag{4.3a}$$

$$\bar{u}_{\dot\beta}\bar{\partial}_{\dot\alpha} = -\epsilon(q)_{\dot\beta\dot\alpha} + q^2\check{R}^{-1}(q)^{\dot\beta'\dot\alpha'}{}_{\dot\alpha\dot\beta}\bar{\partial}_{\dot\alpha'}\bar{u}_{\dot\beta'}\ . \tag{4.3b}$$

Therefore by introducing

$$\bar{L}_\mu = (L^\mu)^* = \bar{\partial}_{\dot\beta}\bar{u}_{\dot\alpha}\bar{t}^\mu(q)^*_{\alpha\beta} \tag{4.4}$$

we have relations similar to those in (3.14)

$$\begin{aligned}
&[\bar{L}_0, \bar{L}_m] = 0\ ,\\
&q(\bar{L}_3-\bar{L}_0)\bar{L}_+ - q^{-1}\bar{L}_+(\bar{L}_3-\bar{L}_0) = q\ [2]^{1/2}\bar{L}_+\ ,\\
&q\ \bar{L}_-(\bar{L}_3-\bar{L}_0) - q^{-1}(\bar{L}_3-\bar{L}_0)\bar{L}_- = q\ [2]^{1/2}\bar{L}_-\ ,\\
&\bar{L}_-\bar{L}_+ - \bar{L}_+\bar{L}_- + (q-q^{-1})\bar{L}_3(\bar{L}_3-\bar{L}_0) = q\ [2]^{1/2}\bar{L}_3\ ,\\
&-q\bar{L}_+\bar{L}_- - \bar{L}_3\bar{L}_3 + \bar{L}_0\bar{L}_0 - q^{-1}\bar{L}_-\bar{L}_+ = q^2\ [2]^{1/2}\bar{L}_0\ .
\end{aligned} \tag{4.5}$$

Now by making use of the relations between one undotted object with dotted one

$$\begin{aligned}
&u^\alpha\bar{u}_{\dot\alpha} = \bar{u}_{\dot\beta}u^\beta\ q\ \check{R}^{-1}(q)^{\dot\beta\alpha}{}_{\beta\dot\alpha}\ ,\\
&\partial^\alpha\bar{u}_{\dot\alpha} = \bar{u}_{\dot\beta}\partial^\beta\ \check{R}^{-1}(q)^{\dot\beta\alpha}{}_{\beta\dot\alpha}\ ,\quad u^\alpha\bar{\partial}_{\dot\alpha} = \bar{\partial}_{\dot\beta}u^\beta\ \check{R}^{-1}(q)^{\dot\beta\alpha}{}_{\beta\dot\alpha}\ ,\\
&\partial^\alpha\bar{\partial}_{\dot\alpha} = \bar{\partial}_{\dot\beta}\partial^\beta\ q\ \check{R}^{-1}(q)^{\dot\beta\alpha}{}_{\beta\dot\alpha}\ ,
\end{aligned} \tag{4.6}$$

we see immediately that

$$\begin{aligned}
&u^\alpha\partial^\beta\bar{\partial}_{\dot\beta}\bar{u}_{\dot\alpha} = u^\alpha\bar{\partial}_{\dot\beta'}\partial^{\beta'}\ q\ \check{R}^{-1}(q)^{\dot\beta'\beta}{}_{\beta'\dot\beta}\bar{u}_{\dot\alpha}\\
&= \bar{\partial}_{\dot\delta}u^{\alpha'}\check{R}^{-1}(q)^{\dot\delta\alpha}{}_{\alpha'\dot\beta'}\ q\ \check{R}^{-1}(q)^{\dot\beta'\beta}{}_{\beta'\dot\beta}\bar{u}_{\dot\alpha'}\partial^\delta\check{R}^{-1}(q)^{\dot\alpha'\beta'}{}_{\delta\dot\alpha}\\
&= \bar{\partial}_{\dot\delta}\check{R}^{-1}(q)^{\dot\delta\alpha}{}_{\alpha'\dot\beta'}\bar{u}_{\dot\gamma}u^\gamma q\check{R}^{-1}(q)^{\dot\gamma\alpha'}{}_{\gamma\dot\alpha'}\ q\ \check{R}^{-1}(q)^{\dot\beta'\beta}{}_{\beta'\dot\beta}\partial^\delta\check{R}^{-1}(q)^{\dot\alpha'\beta'}{}_{\delta\dot\alpha}\ .
\end{aligned} \tag{4.7}$$

This gives

$$L^{\alpha\beta}\bar{L}_{\dot\beta\dot\alpha} - \bar{L}_{\dot\delta\dot\gamma}L^{\gamma\delta}\check{\mathbf{R}}^{-1}(q)^{\dot\gamma\dot\delta,\alpha\beta}{}_{\gamma\delta,\dot\alpha\dot\beta} = 0 \tag{4.8}$$

which means that $L^{\alpha\beta}$ is $\check{R}$ commuting with $\bar{L}_{\dot\beta\dot\alpha}$. Here $\check{\mathbf{R}}^{\dot\gamma\dot\delta,\alpha\beta}{}_{\gamma\delta,\dot\alpha\dot\beta}$ is the $\check{R}$ matrix given in Eq(2.32c), which can be transferred to the vector index form as in (2.33)

$$\check{\mathbf{R}}^{\mu\nu}{}_{\kappa\lambda} = \check{\mathbf{R}}^{00}{}_{00} \oplus \check{\mathbf{R}}^{m0}{}_{0l} \oplus \check{\mathbf{R}}^{0n}{}_{k0} \oplus \check{\mathbf{R}}^{mn}{}_{kl} \tag{4.9}$$

with $\check{\mathbf{R}}^{00}{}_{00} = 1$, $\check{\mathbf{R}}^{m0}{}_{0l} = \delta^m{}_l$, $\check{\mathbf{R}}^{0n}{}_{k0} = \delta^n{}_k$, and $\check{\mathbf{R}}^{mn}{}_{kl}$ being an $SO_{q^2}(3)$ $\check{R}$ matrix.

$$(\check{\mathbf{R}}^{mn}{}_{kl}) = q^2 \oplus \begin{pmatrix} \theta & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} (1-q^{-2})\theta & -q^{-1}\theta & q^{-2} \\ -q^{-1}\theta & 1 & 0 \\ q^{-2} & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} \theta & 1 \\ 1 & 0 \end{pmatrix} \oplus q^2 \tag{4.10}$$

where θ stands for $q^2 - q^{-2}$. This leads to

$$[L^0, \bar{L}_0] = [L^n, \bar{L}_0] = [L^0, \bar{L}_m] = 0 \tag{4.11a}$$

and

$$L^n \bar{L}_l = \bar{L}_m L^k \check{\mathbf{R}}^{mn}{}_{kl} \ . \tag{4.11b}$$

The latter relation can be written out more explicitly as follows

$$\begin{aligned}
L^+\bar{L}_+ &= q^2 \bar{L}_+ L^+ \ , \\
L^+\bar{L}_3 &= \bar{L}_3 L^+ \ , \\
L^+\bar{L}_- &= q^{-2} \bar{L}_- L^+ \ , \\
L^3\bar{L}_+ &= \bar{L}_+ L^3 \ , \\
L^3\bar{L}_3 &= \theta \bar{L}_+ L^+ + \bar{L}_3 L^3 \ , \\
L^3\bar{L}_- &= -q^{-1}\theta \bar{L}_3 L^+ + \bar{L}_- L^3 \ , \\
L^-\bar{L}_+ &= q^{-2} \bar{L}_+ L^- \ , \\
L^-\bar{L}_3 &= -q^{-1}\theta \bar{L}_+ L^3 + \bar{L}_3 L^- \ , \\
L^-\bar{L}_- &= (1-q^{-2})\theta \bar{L}_+ L^+ + \theta \bar{L}_3 L^3 + q\ \bar{L}_- L^- \ .
\end{aligned} \tag{4.12}$$

The relations (3.6), (4.2) and (4.11), or equivalently (3.14), (4.5), (4.11a) and (4.12) complete our cross commutation relations for quantum Lorentz algebra. Among total eight generators (L^μ, $\bar{L}_\mu$), two of them, L^0 and $\bar{L}_0$, are centers which must be added to complete the algebra.

In the limit $q \longrightarrow 1$, two of these relations bocome the definiton of L^0 and $\bar{L}_0$ which are decoupling from the other six generators and these six generators fall into two commuting sets of angular momentum operators. This is just the case for classical Lorentz algebra.

V. The Action of Generators

Now we are in the position to give the explicit results when the generators are acting upon the spinors and 4-vectors. The cross commutation relations given in Section II are enough to give all the following results.

The action of generators on the basic spinor u^γ gives

$$(u^\alpha \partial^\beta) u^\gamma = u^\alpha \epsilon(q)^{\beta\gamma} + q^{-3} \check{R}^{\beta\gamma}{}_{\gamma'\beta'} \check{R}^{\alpha\gamma'}{}_{\delta\alpha'} u^\delta (u^{\alpha'} u^{\beta'}) \ . \tag{5.1}$$

This yields

$$L^0 u^\gamma = -q^{-1}[2]^{-1/2} u^\gamma + q^{-2} u^\gamma L^0 \ , \tag{5.2a}$$

$$L^m u^\gamma = \bar{t}^m(q)_{\alpha\beta} u^\alpha \epsilon(q)^{\beta\gamma} + q^{-2} \check{R}^{\gamma m}{}_{l\delta} u^\delta L^l \tag{5.2b}$$

where $\check{R}(q)^{\gamma m}{}_{l\delta}$ is the $\check{R}$ matrix between spin 1 and spin 1/2 . If we set

$$R(q)^{m\gamma}{}_{l\delta} \equiv \check{R}^{\gamma m}{}_{l\delta} \ ,$$

then

$$R(q)^{m\gamma}{}_{l\delta} = \begin{pmatrix} q & & & & & \\ & q^{-1} & \omega & & & \\ & 0 & 1 & & & \\ & & & 1 & -\omega & \\ & & & 0 & q^{-1} & \\ & & & & & q \end{pmatrix} \tag{5.3}$$

where $\omega = (q^{-1} - q) q^{-1/2} [2]^{1/2}$, and the indeces pair (m, γ) or (l, δ) is ordered $(+,1)$, $(+,2)$, $(3,1)$, $(3,2)$, $(-,1)$ and $(-,2)$. Then (5.2b) can be written out more explicitly

$$\begin{aligned}
L^+ u^1 &= q^{-1} u^1 L^+ , \\
L^+ u^2 &= q^{-1/2} u^1 - (q - q^{-1}) q^{-5/2} [2]^{-1/2} u^1 L^3 + q^{-3} u^2 L^+ , \\
L^3 u^1 &= q[2]^{-1/2} u^1 + q^{-2} u^1 L^3 , \\
L^3 u^2 &= -q^{-1}[2]^{-1/2} u^2 + (q - q^{-1}) q^{-5/2} [2]^{-1/2} u^1 L^- + q^{-2} u^2 L^3 , \\
L^- u^1 &= q^{1/2} u^2 + q^{-3} u^1 L^- , \\
L^- u^2 &= q^{-1} u^2 L^- \ .
\end{aligned} \tag{5.2c}$$

The similar results can be obtained when the generators are acting upon the conjugate spinor $\bar{u}_{\dot{\gamma}}$. Indeed we have

$$(u^\alpha \partial^\beta)\bar{u}_{\dot{\gamma}} = q\ \check{R}^{-1}(q)^{\dot{\delta}\alpha}{}_{\alpha'\dot{\gamma}'}\check{R}^{\dot{\gamma}'\beta}{}_{\beta'\dot{\gamma}}\bar{u}_{\dot{\delta}}(u^{\alpha'}\partial^{\beta'})\ . \tag{5.4}$$

This gives

$$L^0\bar{u}_{\dot{\gamma}} = \bar{u}_{\dot{\gamma}}L^0\ , \tag{5.5a}$$

$$L^m\bar{u}_{\dot{\gamma}} = \check{R}^{-1}(q)^{m\dot{\delta}}{}_{\dot{\gamma}l}\bar{u}_{\dot{\delta}}L^l\ . \tag{5.5b}$$

Here $\check{R}^{-1}(q)^{m\dot{\delta}}{}_{\dot{\gamma}l}$ is the inverse of the $\check{R}$ matrix appearing in (5.2b). And (5.5b) becomes

$$\begin{aligned}
L^+\bar{u}_{\dot{1}} &= q^{-1}\bar{u}_{\dot{1}}L^+ + (q-q^{-1})q^{1/2}[2]^{1/2}\bar{u}_{\dot{2}}L^3,\\
L^+\bar{u}_{\dot{2}} &= q\ \bar{u}_{\dot{2}}L^+,\\
L^3\bar{u}_{\dot{1}} &= q^{-1}\bar{u}_{\dot{1}}L^3 - (q-q^{-1})q^{1/2}[2]^{1/2}\bar{u}_{\dot{2}}L^-,\\
L^3\bar{u}_{\dot{2}} &= \bar{u}_{\dot{2}}L^3,\\
L^-\bar{u}_{\dot{1}} &= q\ \bar{u}_{\dot{1}}L^-,\\
L^-\bar{u}_{\dot{2}} &= q^{-1}\bar{u}_{\dot{2}}L^-\ .
\end{aligned} \tag{5.5c}$$

Then when we consider the coordinates 4-vector as product of a basic spinor u^γ with a conjugate spinor $\bar{w}_{\dot{\gamma}}$ transformed in the same way as $\bar{u}_{\dot{\gamma}}$:

$$X^\gamma{}_{\dot{\gamma}} = \sigma_\mu{}^\gamma{}_{\dot{\gamma}}x^\mu \sim u^\gamma\bar{w}_{\dot{\gamma}}\ . \tag{5.6}$$

We can easily obtain that

$$\begin{aligned}
&(u^\alpha\partial^\beta)(u^\gamma\bar{w}_{\dot{\gamma}})\\
&= u^\alpha\bar{w}_{\dot{\gamma}}\epsilon(q)^{\beta\gamma} + q^{-2}\check{R}^{\beta\gamma}{}_{\gamma'\beta'}\check{R}^{\alpha\gamma'}{}_{\delta\alpha'}u^\delta\bar{w}_{\dot{\delta}}u^{\alpha''}\partial^{\beta''}\check{R}^{-1}(q)^{\dot{\delta}\alpha'}{}_{\alpha''\dot{\gamma}'}\check{R}^{-1}(q)^{\dot{\gamma}'\dot{\beta}}{}_{\beta''\dot{\gamma}}\ .
\end{aligned} \tag{5.7}$$

It follows that

$$\begin{aligned}
L^\mu x^\nu &= C^{\mu\nu}{}_\sigma x^\sigma + q^{-2}\check{\mathbf{R}}^{\mu\nu}{}_{\kappa\lambda}x^\kappa L^\lambda\ ,\\
C^{\mu\nu}{}_\sigma &= \bar{t}^\mu(q)_{\alpha\beta}\epsilon(q)^{\beta\gamma}\bar{t}^\nu(q)_{\gamma\dot{\gamma}}t(q)_\sigma^{\alpha\dot{\gamma}}\ .
\end{aligned} \tag{5.8}$$

Both $C^{\mu\nu}{}_\rho$ and $\check{\mathbf{R}}^{\mu\nu}{}_{\kappa\lambda}$ are reducible in the way such that

$$\begin{aligned}
\check{\mathbf{R}}^{\mu\nu}_{\kappa\lambda} &= \check{\mathbf{R}}^{00}_{00} \oplus \check{\mathbf{R}}^{m0}_{0l} \oplus \check{\mathbf{R}}^{0n}_{k0} \oplus \check{\mathbf{R}}^{mn}_{kl}\ ,\\
C^{\mu\nu}{}_\sigma &= C^{00}{}_0 \oplus C^{m0}{}_s \oplus C^{0n}{}_s \oplus C^{mn}{}_0 \oplus C^{mn}{}_s
\end{aligned} \tag{5.9}$$

with

$$\begin{aligned}
C^{0\nu}{}_\sigma &= C^{\nu 0}{}_\sigma = -q^{-1}[2]^{-1/2}\delta^\nu{}_\sigma\ ,\\
C^{mn}{}_0 &= -q[2]^{-1/2}\mathbf{g}(q^2)^{mn}\ ,\\
C^{mn}{}_s &= \tfrac{[4]^{1/2}}{[2]}\mathbf{u}_s(q^2)^{mn}\ ,
\end{aligned} \tag{5.10}$$

where $\mathbf{g}(q^2)^{mn}$ and $\mathbf{u}_s(q^2)^{mn}$ are respectively the singlet and triplet eigenvectors [7] of the $\check{\mathbf{R}}^{mn}{}_{kl}$. Then Eq(5.8) is equivalent to

$$\begin{aligned}
L^0x^0 &= -q^{-1}[2]^{-1/2}x^0 + q^{-2}x^0L^0\ ,\\
L^mx^0 &= -q^{-1}[2]^{-1/2}x^m + q^{-2}x^0L^m\ ,\\
L^0x^n &= -q^{-1}[2]^{-1/2}x^n + q^{-2}x^nL^0\ ,\\
L^mx^n &= -q\ [2]^{-1/2}\mathbf{g}(q^2)^{mn}x^0 + \tfrac{[4]^{1/2}}{[2]}\mathbf{u}_s(q^2)^{mn}x^s + q^{-2}\check{\mathbf{R}}^{mn}{}_{kl}x^kL^l\ .
\end{aligned} \tag{5.11}$$

The reality of the 4-vector x^μ can be expressed as

$$(x^0)^* = x^0\ ,\quad (x^m)^* = x^n\mathbf{g}_{nm}(q^2)\ . \tag{5.12}$$

And similarly we define

$$\bar{L}^0 = \bar{L}_0 = (L^0)^*\ ,\quad \bar{L}_m = \bar{L}^n\mathbf{g}_{nm}(q^2) = (L^m)^*\ . \tag{5.13}$$

Then the action of the generators $\bar{L}^\mu$ on x^ν now becomes

$$\begin{aligned}
x^0\bar{L}^0 &= -q^{-1}[2]^{-1/2}x^0 + q^{-2}\bar{L}^0x^0\ ,\\
x^n\bar{L}^0 &= -q^{-1}[2]^{-1/2}x^n + q^{-2}\bar{L}^0x^n\ ,\\
x^0\bar{L}^m &= -q^{-1}[2]^{-1/2}x^m + q^{-2}\bar{L}^mx^0\ ,\\
x^n\bar{L}^m &= -q^{-1}[2]^{-1/2}\mathbf{g}(q^2)^{nm} + \tfrac{[4]^{1/2}}{[2]}\mathbf{u}_s(q^2)^{nm}x^s + q^{-2}\check{\mathbf{R}}^{nm}{}_{lk}\bar{L}^lx^k\ .
\end{aligned} \tag{5.14}$$

Relations (5.11) and (5.14) complete the action of whole set of generators on 4-vectors.

In the limit $q \longrightarrow 1$, (5.11) and (5.13) reduce to

$$
\begin{aligned}
&[L^+, x^+] = 0, && [\bar{L}^+, x^+] = 0,\\
&[L^+, x^3] = -\tfrac{1}{\sqrt{2}}x^+, && [\bar{L}^+, x^3] = -\tfrac{1}{\sqrt{2}}x^+,\\
&[L^+, x^-] = \tfrac{1}{\sqrt{2}}(x^3 - x^0), && [\bar{L}^+, x^-] = \tfrac{1}{\sqrt{2}}(x^3 + x^0),\\
&[L^3, x^+] = \tfrac{1}{\sqrt{2}}x^+, && [\bar{L}^3, x^+] = \tfrac{1}{\sqrt{2}}x^+,\\
&[L^3, x^3] = -\tfrac{1}{\sqrt{2}}x^0, && [\bar{L}^3, x^3] = \tfrac{1}{\sqrt{2}}x^0,\\
&[L^3, x^-] = -\tfrac{1}{\sqrt{2}}x^-, && [\bar{L}^3, x^-] = -\tfrac{1}{\sqrt{2}}x^-,\\
&[L^-, x^+] = -\tfrac{1}{\sqrt{2}}(x^3 + x^0), && [\bar{L}^-, x^+] = -\tfrac{1}{\sqrt{2}}(x^3 - x^0),\\
&[L^-, x^3] = \tfrac{1}{\sqrt{2}}x^-, && [\bar{L}^-, x^3] = \tfrac{1}{\sqrt{2}}x^-,\\
&[L^-, x^-] = 0, && [\bar{L}^-, x^-] = 0,\\
&[L^m, x^0] = -x^m/\sqrt{2}, && [\bar{L}^m, x^0] = x^m/\sqrt{2}.
\end{aligned}
\tag{5.15}
$$

It implies that $J^m = \frac{1}{\sqrt{2}}(L^m + \bar{L}^m)$ are the rotation generators while $K^m = \frac{1}{\sqrt{2}}(\bar{L}^m - L^m)$ the boost generators.

Note Added

After completing this manuscript we saw a paper by W.B. Schmidke et al (Z.Phys. C 52(1991)471) in which the ansatz-consistency method is used to give the generators of the quantum Lorentz group acting upon spinors and 4-vectors similar to those in Section V. We believe that their results will be equivalent to ours if they used an ansatz corresponding to the 4 D_+ differential calculus rather than the 3 D calculus they adopted. This is also the reason why their results were less compact and less explicitly covariant.

References

[1] V.G. Drinfeld, Quantum groups, Proceedings of ICM, 1986, Vol.1,798;

M. Jimbo, Lett. Math. Phys. 10(1986)63, 11(1986)247, Commun. Math. Phys. 102(1986)537;

N.Yu. Reshetikhin, LOMI preprint E-4-87, E-17-87 (1987);

N.Yu. Reshetikhin, L.A. Takhtajan and L.D. Faddeev, Leningrad Math. J. 1(1990)193;

S.L. Woronowicz, Publ. RIMS, Kyoto Univ., Vol.23, (1987)117, Commun. Math. Phys. 111(1987)613.

[2] C.N. Yang, Phys. Rev. Lett. 19(1967)1312; R.J. Baxter, Exactly Solved Models, Academic Press, London, 1982.

[3] Yu.I. Manin, Quantum groups and non-commutative geometry, Les publications du Cetre de Recherches Mathematiques. Universite de Montreal, 1988.

[4] J. Wess and B. Zumino, Covariant differential calculus on the quantum hyperplane, CERN-TH-5697-90.

[5] U. Carow-Watamura, M. Schlieker and S. Watamura, Z. Phys.C49(1991)439;

X.C. Song and L. Liao, preprint CCAST-91-04, 91-29, to appear in J. Phys. A.

[6] X.C. Song, Covariant Differential Calculus on Quantum Minkowski Space and The q-Analogue of Dirac Equation, Preprint CCAST-91-63, Beijing.

[7] X.C. Song, forthcoming paper.

[8] P. Podles, Lett. Math. Phys. 18(1989)107; S.L. Woronowicz, Commun. Math. Phys.122(1989)125.

[9] See for example, A.R. Edmonds, Angular Momentum in Quantum Mechanics, Sec. 2.6, Princeton University Press, 1957.

[10] P. Podles and S.L. Woronowicz, Commun. Math. Phys. 130(1990)381.

U. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura, Z. Phys.C48(1991)159.

[11] X.C. Song, Spinor Analysis for Quantum Group $SU_q(2)$, preprint CCAST-91-28, Beijing.

[12] K. Wu & R. Zhang, Differential Calculus on Quantum Matrix Lie Groups, ASITP-91-72, CCAST-91-69, Beijing.