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DETERMINATION OF PARTITION LINES FROM DYNAMICAL FOLIATIONS FOR THE HÉNON MAP

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Abstract

The binary partition lines for the Hénon map are numerically constructed from tangencies between the contracting and expanding foliations. The ordering of foliations according to their symbolic sequences are examined.

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The Hénon map[1]

$$T(x,y) = (1 - ax^2 + by, x) \tag{1}$$

is one of the simplest non-trivial dissipative systems. It was proven that there exists a binary symbolic dynamics when the unstable orbits form a horseshoe and no chaotic attractor persists.[2] Grassberger and Kantz[3] found binary partitions for the map also when there is a strange attractor. They conjectured that a good partition should pass through all "primary" homoclinic tangencies between stable and unstable manifolds. Later, Gvitanović, Gunaratne and Procaccia,[4] using this partition, obtained "the pruning front" which separates the allowed and disallowed orbits. However, they were mainly interested in periodic orbits. Generally, symbolic sequences are closely related to dynamical foliations. In this paper we shall discuss the role played by the dynamical foliations, extend the concept of the homoclinic tangencies to the more general tangencies between two sets of foliations, and then describe the ordering of foliations according to symbolic sequences.

For the Hénon map, once the binary partition is determined, the phase plane is divided into the right(R) and left(L) halves by the partition line for x (denoted by $\bullet C$), and into the upper(R) and the lower(L) halves by the partition line for y (denoted by $C\bullet$). Regarding an initial point as the first point and applying the forward and backward maps, we have an orbit which may be coded with a doubly infinite symbolic sequence:

$$\cdots s_n \cdots s_2 s_1 \bullet s_1 s_2 \cdots s_n \cdots$$

where the thick dot indicates the present position, s_n is the code for the x-coordinate of the n-th point on the orbit along the forward direction, and s_n the code for the y-coordinate of the backward n-th point. A step of the forward map shifts the sequence to the left by one letter with respect to the dot, while a step of the backward map shifts the sequence to the right. We call the semi-sequence right of the dot the forward sequence, and the left one the backward sequence. We may introduce a metric representation of a symbolic sequence as follows.[4] A forward sequence $\bullet s_1 s_2 \cdots s_n \cdots$ is specified with a number defined by

$$\xi = \sum_{n=1}^{\infty} s_n 2^{-n}, \tag{2}$$

where ε_n is zero when the number of letters R in the string $s_1 s_2 \cdots s_n$ is even but otherwise is one. (When the letter C appears in a sequence, we may replace it with RRL^∞ or LRL^∞.) Similarly, we assign to a backward sequence $\cdots s_n \cdots s_2 s_1 \bullet$ the number η defined by

$$\eta = \sum_{n=1}^{\infty} \varepsilon_n 2^{-n}, \tag{3}$$

where ε_n is one when the number of L in the string $s_n \cdots s_2 s_1 \bullet$ is even but otherwise is zero. (A letter C may be replaced with $R^\infty LL$ or $R^\infty LR$.) For example,

$$\xi(\bullet RL^\infty) = \eta(R^\infty \bullet) = 1, \quad \xi(\bullet L^\infty) = \eta(R^\infty L \bullet) = 0,$$
$$\eta(R^\infty LR \bullet) = \eta(R^\infty LL \bullet) = \eta(C \bullet) = \tfrac{1}{2}.$$

The $\xi\eta$-plane may be called the symbolic plane since a point (ξ, η) in the plane corresponds to a symbolic sequence. Forward and backward sequences correspond to straight lines in the plane. These lines are closely related to the dynamical foliations of the phase plane.[5]

Consider a cluster of points on a circle surrounding the initial point of an orbit segment. In general, after a number of iterations of the forward map, the circle is stretched into an ellipse. If we fix this final point and increase the length of the orbit, the major axis of such ellipses appoach a limit direction, which may be called the forward direction of that given point. For most cases of interest there exists a field of such directions.[6] The integral curves of this field is the forward foliations, a simple example of which is the invariant unstable manifold of a unstable fixed point. Similarly, for the backward map there is another field of directions, and its integral curves are the backward foliations, an example of which is the invariant stable manifold of a unstable fixed point. Generally, a segment of a forward foliation is mapped to a contracted segment of another forward foliation under the backward map while a backward foliation is contracted under the forward map. Thus, the forward foliation and the backward foliation, may be called, respectively, the backward contracting manifold (BCM) and the forward contracting manifold (FCM).[7] (In Ref. 6 the BCM's were studied, while in Ref. 7 the FCM's were studied where the term "most stable manifolds" were used instead of FCM.)

An image of $C\bullet$ is a line on which every point shares the same backward sequence of the type $CQ\bullet$ where Q is a string consisting of R and L. Similarly, a preimage of $\bullet C$

is a line which may be marked with $\bullet PC$ where P stands for some string of R and L. Generally, around a point on a given foliation (FCM or BCM) there exists a segment on which every point shares the same forward or backward sequence as the given point. In the remainder of this paragraph we shall consider only the BCM. Similar discussion may be made for the FCM with a slight modification. For the BCM, such a segment is cut by two lines $CQ_1\bullet$ and $CQ_2\bullet$ at its two ends, and no other line of the type $CQ\bullet$ can intersect the segment. Furthermore, there are certain areas with the same sequences. The simplest example is shown in Fig. 1 where the shaded area belongs to the sequence $\bullet L^\infty$. Any preimage of the region is again an area which belons to a certain forward sequence ending with L^∞. This many-to-one correspondence is by no means a drawback of the symbolic dynamics. Points in such areas have similar behavior, so the coarse-grained representation better reflects the essential features of the dynamics. We may then adopt the convention that such an area is represented by its borders.

It has been used as the guiding principle for a good partition that to distinquish any two homoclinic intersections they must be separated by a partition line, or at least one of its images or preimages.[3] Thus, the division lines for partition should pass through points of homoclinic tangencies. For the symbolic analysis of the whole phase plane and not just of the strange attractor, it is natural to require that the division lines for partition must pass through all the tangencies between the representative FCM and BCM, where the word "representative" means that for an area sharing a single sequence we consider only its borders. Inside any area of a single sequence a division line may go rather freely. For $b \to 0$ the y-axis is a partition line. By increasing b gradually, we can trace the "primary" tangencies which are generate from the y-axis. We can then obtain the corresponding partition line for a finite b. We require that this line contains a minimum set of tangencies, i.e. all the other tangencies are images or preimages of the set, and the set contains no images or preimages of itself.

Using this generalized guiding principle we have determined the partition for several cases of different parametes:

i) $a = 0.3$, $b = 0.3$ (see Fig. 2)

There are two fixed points $H_\pm$ of T, of which the one (H_-) is a saddle point, and the

other (H_+) a sink. No tangencies of representative lines appear. The only requirement for the determination of the partition line $\bullet C$ is that it must separate the two fixed points H_+ and H_- on its two sides. Thus, the line $\bullet C$ can go freely between H_+ and H_-.

ii) $a = 0.88$, $b = 0,3$ (see Fig. 3)

After period doubling bifurcations the period-2 orbit (F_1, F_2) now appears to be stable. Both the fixed points become saddles. Still no tangencies appear. The partition line $\bullet C$ must separate H_+ and F_1 on the one side, and H_- and F_2 on the other side, but no points on a specific route of $\bullet C$ are uniquely fixed.

iii) $a = 1.4$, $b = 0.3$ (see Fig. 4)

This is the most often studied case. Here some tangencies not on the attractor have been included in the partition line $\bullet C$, e.g. the tangency on the line $L^\infty\bullet$. Notice that lines $\bullet RRL^\infty$ and $\bullet LRL^\infty$ have the same $\xi = \frac{1}{2}$, so the segment of $\bullet C$ between them may be drawn rather freely.

iv) $a = 1.0$, $b = 0.54$ (see Fig. 5)

In this case the partition line $\bullet C$ is quite different from the y-axis.

v) $a = 0.95$, $b = 0.54$ (see Fig. 6)

There exists a four-piece attractor. Tangencies not on the attractor also play a role in the construction of the partition line.

vi) $a = 2.85$, $b = 0.3$ (see Fig. 7)

The border lines of $\bullet RRL^\infty$ and $\bullet LRL^\infty$ are tangential with the line $R^\infty L\bullet$. There is no attractor. The partition line $\bullet C$ goes through the tangent point and between $\bullet RRL^\infty$ and $\bullet LRL^\infty$. The symbolic dynamics becomes complete.

We now discuss the ordering of symbolic sequences. Ordering is essentially a concept in one dimension. In terms of the one-dimensional foliations we may talk about ordering. We have checked ordering of FCM on a segment of the BCM of a certain backward sequence. Assume that the segment is cut by $CQ_1\bullet$ and $CQ_2\bullet$, and at the two ends the forward sequences of the FCM are $\bullet P_1$ and $\bullet P_2$, respectively. It is found that on that segment the forward sequences of the FCM are ordered in the same way as the order of $\bullet P_1$ and $\bullet P_2$ according to their η values. An example is shown in Fig. 8. Similarly, on a segment of the FCM of a certain forward sequence the lines of BCM are ordered according to their ξ

values. This is true for all the cases shown in Figs. 2-7.

Finally, we point out that as far as the ordering of symbolic sequences is concerned the determination of the two partition lines is not unique. For example, instead of lines $\bullet C$ and $C\bullet$, we may take line $C\bullet$ and the joint line of $CR\bullet$ and $CL\bullet$ as two new partition lines $\bullet C'$ and $C'\bullet$. The tranformation between the two sets of symbolic sequences is rather simple:

$$\cdots s'_n \cdots s'_2 s'_1 \bullet s'_1 s'_2 \cdots s'_n \cdots = \cdots s_n \cdots s_2 \bullet s_1 s_2 \cdots s_n \cdots.$$

In the above we have discussed the role of the two sets of dynamical foliations, the FCM and the BCM, in the determination of the partition and in the ordering of symbolic sequences. One can see that the determination of the partition from the foliations is more general and efficient than the usual method by invariant manifolds.

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Figure Captions

Fig.1 The shaded region left of the line $\bullet L^\infty$ passing through the left fixed point H_- belongs to the single sequence $\bullet L^\infty$. Its preimages are also regions having single sequences. Three such regions marked with $\bullet RL^\infty$, $\bullet RRL^\infty$ and $\bullet LRL^\infty$ are shown.

Fig.2 The case of $a = 0.3$ and $b = 0.3$. The partition line may be drawn anywhere between the two fixed points H_+ (a sink) and H_- (a saddle).

Fig.3 The case of $a = 0.88$ and $b = 0.3$. There is a stable period-2 orbit (F_1 and F_2). The partition line separates the two fixed points and the two period-2 points as well. No points on the partition line are uniquely determined.

Fig.4 The most often studied case of $a = 1.4$ and $b = 0.3$. The partition line goes through tangent points of the two kinds of foliation lines. A tangent point on the line $L^\infty\bullet$, which is not on the attractor, is marked with an open circle.

Fig.5 The case of $a = 1.0$ and $b = 0.54$. The partition line is quite different from the y-axis.

Fig.6 The case of $a = 0.95$ and $b = 0.54$. The attractor becomes a four-piece one. Many tangent points determining the partition line are not on the attractor.

Fig.7 The case of $a = 2.85$ and $b = 0.3$. There is essentially only one tangency $R^\infty L\bullet RRL^\infty$ left for determining the partition line.

Fig.8 An example of the ordering of some FCM's on the line $R^\infty\bullet$.

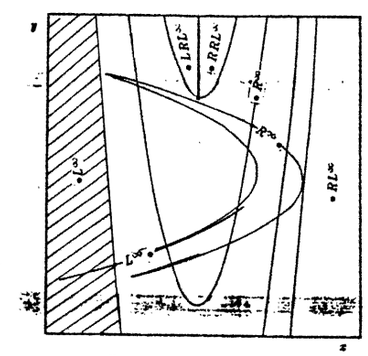


Fig. 1

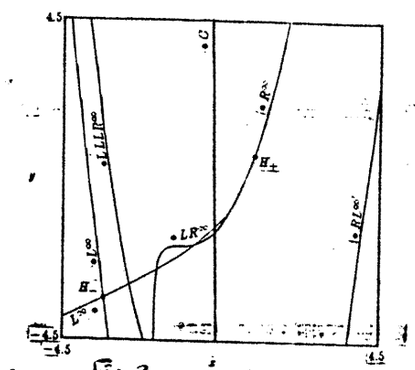


Fig. 2

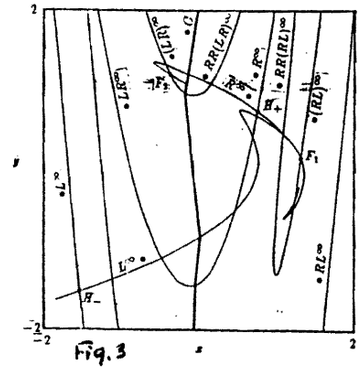
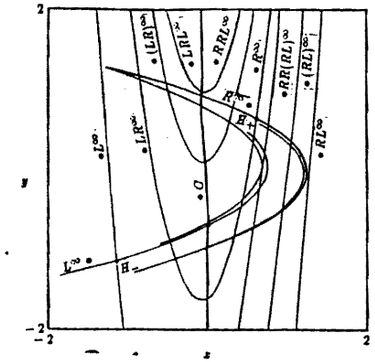
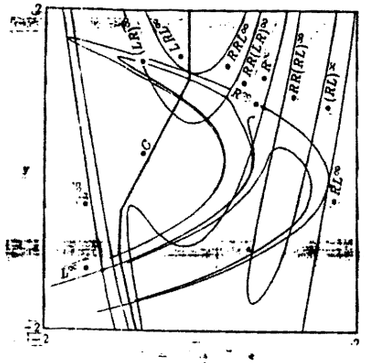
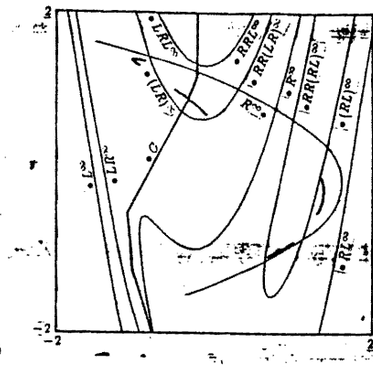


Fig. 3







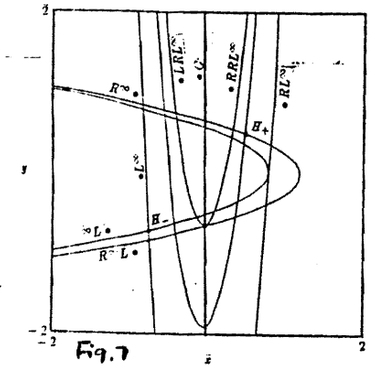


Fig. 7

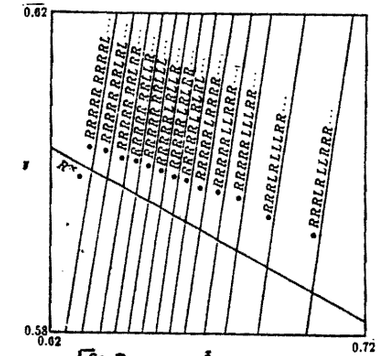


Fig. 8