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SYMBOLIC DYNAMICS FOR THE TÉL MAP

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Abstract

In terms of contracting and expanding foliations the two-dimensional Tél map is decomposed into two coupled one-dimensional maps. Symbolic sequences are assigned to the two classes of foliations, and their ordering discussed. The pruning front is constructed. A necessary condition and a sufficient condition for admissible sequences are proposed.

PACS number: 05.45.+b

I. Introduction

It is believed that strange attractors of the Hénon-type are generic for low-dimensional dynamical systems. Owing to the nonlinearity of the Hénon map, the investigations are mainly restricted to numerical or perturbative methods [1]. In order to construct explicit results, piecewise linear maps are introduced. Among them is the Lozi map [2], which is expected to exhibit certain topological similarities to the Hénon map. The Lozi map may be viewed as a two-dimensional extension of the one-dimensional tent map. The symmetric tent map is equivalent to the sawtooth map. A two-dimensional extension of the sawtooth map has been introduced by Tél [3,4]. The map is defined by

$$x' = ax - \mathrm{sgn}(x) + by, \qquad y' = x, \tag{1}$$

where $0 < b < 1$ and $\mathrm{sgn}(x)$ denotes the sign of x. This map may be regarded as the simplest map of the Hénon type, since all the expanding or contracting foliations [5] of the phase space are of the same direction, while the two classes of foliations are still coupled together. In this paper we shall make a symbolic analysis of the Tél map.

II. Foliations and their ordering

For the Tél map the y-axis is the partition line for the phase plane, so the symbolic dynamics contains an L for $x < 0$, R for $x > 0$, and a C for $x = 0$. Equivalently, for the y-coordinate we may assign an L for $y < 0$, R for $y > 0$, and C for $y = 0$ [6]. For a given initial point (x_0, y_0) we encode its orbit according to the sign of x or y as

$$\cdots s_m \cdots s_{\bar{1}} s_{\bar{0}} \bullet s_0 s_1 s_2 \cdots s_n \cdots,$$

where s_n denotes the x-sign of its n-th image, s_m the y-sign of the m-th preimage, and the thick dot indicates the "present" position.

Assume that points sharing the same backward sequence $\cdots s_{n-2} s_{n-1} \bullet$ fall on the straight line $x - h_n y = \eta_n$. For a given point (x_n, y_n) on the line, denoting $e_n = \mathrm{sgn}(x_n)$,

from Eqs. (1) we have

$$x_n - h_n y_n = (a - h_n)x_{n-1} + by_{n-1} - \varepsilon_{n-1},$$

which may be written as

$$\eta_n = -bh_{n-1}^{-1}\eta_{n-1} - \varepsilon_{n-1}, \tag{2}$$

where $\varepsilon_n = \mathrm{sgn}(x_n)$, and

$$\eta_{n-1} = x_{n-1} - h_{n-1}y_{n-1}, \qquad h_n = a + b/h_{n-1}. \tag{3}$$

We may express h_n in the form of the continued fraction as

$$h_n = a + \frac{b}{a} \frac{b}{a} + \cdots \tag{4}$$

which is independent of the position, and converges to a value between a and $a + b/a$:

$$h_n = h \equiv (a + \sqrt{a^2 + 4b})/2. \tag{5}$$

Let us introduce the backward operator B and forward operator F defined by

$$Bu_n = u_{n-1}, \qquad Fu_n = u_{n+1}, \tag{6}$$

where $\{u_n\}$ is an arbitrary sequence. In terms of B a particular solution of Eq. (2) may be written as

$$\begin{aligned} \eta_n &= -(1 + bh^{-1}B)^{-1}\varepsilon_{n-1} = -\sum_{i=0}^{\infty}(-bh^{-1}B)^i\varepsilon_{n-1} \\ &= -\varepsilon_{n-1} + bh^{-1}\varepsilon_{n-2} - b^2h^{-2}\varepsilon_{n-3} + \cdots. \end{aligned} \tag{7}$$

Thus, we have determined the line and may assign the semi-infinite symbolic sequence to it. Such lines form one class of foliations of the phase plane [7].

Similarly, a forward symbolic sequence $\bullet s_n s_{n+1}\cdots$ corresponds a line $\xi_n = x - k_n y$. One can verify that

$$k_n = k \equiv \frac{b}{-a} \frac{b}{-a} + \cdots = -bh^{-1} = (a - \sqrt{a^2 + 4b})/2, \tag{8}$$

$$\begin{aligned} \xi_n &= -h^{-1}(\xi_{n+1} + \varepsilon_n) = h^{-1}(1 - h^{-1}F)^{-1}\varepsilon_n \\ &= h^{-1}\sum_{i=0}^{\infty}(h^{-1}F)^i\varepsilon_n = h^{-1}\varepsilon_n + h^{-2}\varepsilon_{n+1} + h^{-3}\varepsilon_{n+2} + \cdots. \end{aligned} \tag{9}$$

Such lines form the other class of foliations for the phase plane.

Here the two directions given by $(h, 1)$ and $(k, 1)$ are the eigen directions of the constant Jacobian matrix of the map. When $a + b > 1$, we have $|k| < 1 < h$. Lines of forward sequences are oriented along the contracting direction $(k, 1)$ of the map, while those of backward sequences along the expanding direction.

So far we have not mentioned sequences containing the letter C. First, the x-axis is $C\bullet$, and the y-axis is $\bullet C$. They have $h_n^{-1} = 0$ and $k_n = 0$, respectively. For a backward sequence ending with a C, the continued fraction representation (4) of h_n becomes finite, and $h_n^{-1} \in [0, (a + b/a)^{-1}]$ is always positive. Similarly, for a forward sequence ending with a C, its $k_n \in [-b/a, 0]$ is always negative.

We now discuss the ordering of the contracting foliations according to their symbolic sequences or words. Consider lines $\bullet RP$, $\bullet LP$ and their image $\bullet P$ together with the pair of lines adjacent to each of them, as shown in Fig. 1. Here string P consists of R and L only. In the figure, points 1 and 2 on a vertical line map to points $1'$ and $2'$ on a horizontal line. Since for a fixed x, from Eq. (1) a large y corresponds to a large x', the lines left and right of $\bullet RP$ respectively map to the same side of the line $\bullet P$. Similarly, from $\bullet LP$ to $\bullet P$, the ordering of lines is also preserved. Suppose that there are two lines $\bullet PR\cdots$ and $\bullet PL\cdots$, where P is a finite string of R and L, and both of them intersect with some line $Q\bullet$, where Q may contain C also. Thus, the ordering of these two lines is the same as that of the lines $\bullet R\cdots$ and $\bullet L\cdots$ intersecting with the line $QP\bullet$. The line $\bullet R\cdots$ is on the right of the line $\bullet L\cdots$. We may define the right of two lines of forward sequences to be greater, and state the rule for ordering as follows: the ordering of two contracting lines transversal with some expanding line is

$$\bullet PR\cdots > \bullet PL\cdots \tag{10}$$

where the common leading string P consists of R and L, or is blank. According to this rule the greatest word is $\bullet R^\infty$, and the smallest is $\bullet L^\infty$.

A similar argument may be applied to the ordering of expanding lines. From Fig. 2 we see that a leading letter, either L or R, always reverses the ordering. We may define the upper of any two expanding lines to be greater, and the rule for ordering is

$$\cdots RE\bullet > \cdots LE\bullet \qquad \cdots RO\bullet < \cdots LO\bullet \tag{11}$$

where the total numbers of letters in the common string E and O are even and odd, respectively. Thus, the greatest word is $(LR)^\infty\bullet$, while the smallest is $(RL)^\infty\bullet$.

We may also include words containing a C for ordering. A word containing a C is between the two words which are generated from the word by replacing the C with an R and an L.

It is often convenient to introduce a metric representation of words [6]. We assign to the word $\bullet s_1 s_2 \cdots s_n \cdots$ the number α

$$\alpha = \sum_{i=1}^{\infty} \mu_i 2^{-i}, \tag{12}$$

where

$$\mu_i = \begin{cases} 0 \\ 1 \end{cases} \quad \text{if} \quad s_i = \begin{cases} L, \\ R. \end{cases} \tag{13}$$

The number β assigned to the word $\cdots s_n \cdots s_2 s_1 \bullet$ is defined by

$$\beta = \sum_{i=1}^{\infty} \nu_i 2^{-i}, \tag{14}$$

where

$$\nu_i = \begin{cases} 0 \\ 1 \end{cases} \quad \text{for } i \text{ even and } s_i = \begin{cases} R, \\ L, \end{cases} \quad \text{or for } i \text{ odd and } s_i = \begin{cases} L, \\ R. \end{cases} \tag{15}$$

According to the definition, we have

$$\begin{aligned} \alpha(\bullet R^\infty) &= \beta((LR)^\infty\bullet) = 1, \\ \alpha(\bullet L^\infty) &= \beta((RL)^\infty\bullet) = 0, \\ \alpha(\bullet RL^\infty) &= \alpha(\bullet LR^\infty) = 1/2, \\ \beta((LR)^\infty R\bullet) &= \beta((RL)^\infty L\bullet) = 1/2. \end{aligned} \tag{16}$$

To find the α-value of a forward word containing a C we may replace the letter C with LR^∞ or RL^∞. The letter C in a backward word may be replaced with $(RL)^\infty L$ or $(LR)^\infty R$. In this representation any word corresponds to a number between zero and one, and a doubly infinite symbolic sequence corresponds to a point in the unit square of the (α, β)-plane, the so-called symbolic plane. In the plane, contracting and expanding lines become vertical and horizontal lines, respectively.

III. The pruning front and admissibility conditions

An example of expanding and contracting lines is shown in Fig. 3. In the phase plane the segment of the unstable manifold passing through the period-2 point in the second quadrant of the phase plane has the word $(LR)^\infty\bullet$. Above the line there is a region of $(LR)^\infty\bullet$ since no word can be greater than $(LR)^\infty\bullet$. From the anti-symmetry of the map in the lower right corner of the phase plane there is a region of $(RL)^\infty\bullet$, the border of which passes through the other period-2 point in the fourth quadrant. These two border lines map to the lines $\beta = 1$ and $\beta = 0$ in the symbolic plane. Each is the preimage of the other. In the figure an image $(LR)^\infty R\bullet$ of the one border line $(LR)^\infty\bullet$ and an image $(RL)^\infty L\bullet$ of the other are shown. The whole area enclosed by the lines $(RL)^\infty L\bullet$, $(LR)^\infty R\bullet$, and $C\bullet$ (the x-axis) maps to a single line $\beta = 1/2$ in the symbolic plane. There are many regions belonging to a single sequence of the type $(LR)^\infty Q\bullet$. We shall focus only on the border lines of such regions. In the figure the expanding and contracting lines passing through the fixed points are also shown. There are regions of $\bullet R^\infty$ and $\bullet L^\infty$.

The ordering rules of foliations put restrictions on allowed symbolic sequences. Let us look at the case shown in Fig. 4a, where the expanding line $QL\bullet$ is tangential with the contracting line $\bullet RP$ at the x-axis. None of the expanding lines inside the angle spanned by $QL\bullet$ and the x-axis can intersect with any contracting lines right of $\bullet RP$, i.e. no crossing between lines $(QL)_+\bullet$ and $\bullet(RP)_+$ happens, where $(QL)_+\bullet$ denotes a word greater than $QL\bullet$ and leading with an L, and the meaning of $\bullet(RP)_+$ is analogous. In the symbolic plane the tangency then rules out a forbidden rectangle, the area bounded

by $\alpha = 1$, $\alpha = \alpha(\bullet RP)$, $\beta = \frac{1}{2}$, and $\beta = \beta(QL\bullet)$, as shown in Fig. 4b.

The condition for the tangency of $(RL)^\infty\bullet$ and $\bullet R^\infty$ can be derived from Eqs. (7) and (9) as

$$\eta((RL)^\infty\bullet) = \frac{1}{1-bh^{-1}} = \frac{h^{-1}}{1-h^{-1}} = \xi(\bullet R^\infty), \tag{17}$$

which leads to [3]

$$a = a_c \equiv 2\sqrt{1-b}. \tag{18}$$

When $a < 2\sqrt{1-b}$, the two lines are separated. Some other line, say $\bullet RK$, smaller than $\bullet R^\infty$ is tangential with the line $(RL)^\infty\bullet$. The word $\bullet RK$ determines the first forbidden rectangle in the symbolic plane. In any allowed symbolic sequences no string greater than $\bullet RK$ may appear immediately after a letter L (except for $(RL)^\infty\bullet$). For a string W we denote by $\bar{W}$ its mirror image which is generated from W by switching $R \leftrightarrow L$. From the anti-symmetry of the map no string smaller than $\bullet L\bar{K}$ appears immediately after a letter R. The parallelogram formed by $\bullet RK$, $\bullet L\bar{K}$, $(LR)^\infty\bullet$ and $(RL)^\infty\bullet$ may be chosen as a fundamental trapping region of the map. When moving from the tangent point of $\bullet RK$ and $(RL)^\infty\bullet$ to the right, we meet tangent points on the x-axis one by one, which rule out a series of forbidden rectangles, shorter and shorter in the vertical direction. in the symbolic plane. The last tangent point is reached at $(RL)^\infty L\bullet$. Without loss of generality we may consider only the case of $bh^{-1} < \frac{1}{2}$, or $2a > 4b - 1$, when the line $(RL)^\infty L\bullet$ has a positive x-intercept. In this case the line $(RL)^\infty L\bullet$ is tangential to some line denoted by $\bullet RJ$. The whole forbidden area caused by the tangencies located between $\bullet RK$ and $\bullet RJ$ is shaped as a staircase as shown in Fig. 5. The ragged side of the staircase may be called the pruning front [6]. Every flat segment of the pruning front corresponds to a region sharing a single word of the type of $(RL)^\infty Q\bullet$ in the phase space. From the anti-symmetry of the map there is another forbidden staircase symmetric to the above with respect to the center $(\frac{1}{2}, \frac{1}{2})$ of the unit square.

Suppose that there is a tangency between $QVL\bullet$ and $\bullet RP$ on the x-axis, and the tangent point is $QVL\bullet RP$. Since the tangent point falls on the x-axis, we may mark it

as $C\bullet RP$ as well. This tangency generates an infinite number of tangencies in the phase space. For example, if we iterate the map backwards for a number of times which is equal to the length of the string VL, the original tangent point $C\bullet P$ is mapped to $Q\bullet VLRP$. It is clear that a segment of $C\bullet$ including the original tangent point is mapped to $\bullet VC$, so the generated tangent point may equally be marked by $Q\bullet VC$. Since the line $\bullet VLRP$ cannot cross $\bullet VC$, point $Q\bullet VLRP$ must be a tangent point too. We may also iterate forwards to generate new tangent points. Since any foliation line must end at a line $\bullet PC$ or $CQ\bullet$ where P and Q are some finite strings of R and L, and a tangency must happen at an end of one of the two lines involved, any tangency must have an image or preimage on the x-axis, or itself is already on the axis. Consequently, all the images and preimages of the above two fudamental forbidden staircases including themselves form the complete forbidden zones. We may ignore those derived forbidden zones, and focus mainly on the two fundamental zones. We shall explain briefly that points outside the forbidden zones certainly correspond to allowed sequences. Consider a point A outside of the forbidden zones of the symbolic plane. From Eqs. (7) and (9) we can calculate its ξ and η. From the definition of ξ and η we have

$$x = (h\xi - k\eta)/(h-k), \tag{19a}$$

$$y = (\xi - \eta)/(h-k). \tag{19b}$$

The problem is then to check the consistency of the signs of x and y with their codes. Assume that $y < 0$. From Eq. (19b) we must have $\xi < \eta$. This is guaranteed since the pruning front corresponds to $\xi = \eta$ and A is outside of the left fundamental forbidden zone. The proof of the consistence of y for the case of $y > 0$ is similar. To verify the consistency for x we may shift the present dot of A to the right by one letter with respect to the symbolic sequence. For the new point A' which is the image of A, the x-coordinate of A becomes the y-coordinate of A', and we can carry on the same argument to verify the consistency for this new y, and hence for the x-coordinate. In this way we can verify the

consistence for the whole sequence.

For a given doubly infinite symbolic sequence we may put the "present" dot somewhere to divide it into a pair of semi-infinite sequences, which correspond to a point in the symbolic plane. By shifting the dividing dot leftwards or rightwards we obtain an infinite number of points, which may be called shifts of the given sequence. We state the admissibility condition as follows: A symbolic sequence is admissible if and only if its shifts never fall inside the two forbidden staircases. We have mentioned a necessary condition for an admissible sequence determined by the first tangency at $\bullet RK$. We state a sufficient condition given by the last tangency at $\bullet RJ$: a symbolic sequence is admissible if its shifts are always between $\bullet RJ$ and $\bullet LJ$. Another sufficient condition is: For a sequence $\cdots s_{n-2}s_{n-1}LRs_{n+1}s_{n+2}\cdots$, if $\cdots s_{n-2}s_{n-1}L\bullet$ and $\cdots \bar{s}_{n-2}\bar{s}_{n-1}R\bullet$ are respectively the minimal and maximal of all the shifted backward semi-sequences, and at the same time $Rs_{n+1}s_{n+2}\cdots$ and $L\bar{s}_{n+1}\bar{s}_{n+2}\cdots$ are respectively the maximal and minimal of all the shited forward sequences, then the given sequence is admissible as long as $\eta(\cdots s_{n-2}s_{n-1}L\bullet) > \xi(\bullet Rs_{n+1}s_{n+2}\cdots)$.

We now examine the special case of $a = a_c = 2\sqrt{1-b}$ when the one fixed point is heteroclinic to the period-2. When b approach zero, we have $a_c \longrightarrow 2$, $h = \sqrt{1-b}+1 \longrightarrow 2$, and $k = 1-\sqrt{1-b} \longrightarrow 0$, so from Eqs. (7) and (9) $\xi(\bullet R^\infty) \to 1$, and $\eta(WL\bullet) \to 1$ for any string W. The right fundamental forbidden staircase shrinks to the line $\xi = 1$. Any sequence not ending with RL^∞ or LR^∞ is then allowed. When b increase from zero, $\eta((RL)^\infty L\bullet)$ reduces. From Eq. (7) we have

$$\eta((RL)^\infty L\bullet) = 1 - \frac{bh^{-1}}{1-bh^{-1}} = 2-(1-b)^{-1/2}, \tag{20a}$$

$$\xi(\bullet R^m C) = \frac{h^{-1}-h^{-m-1}}{1-h^{-1}} = (1-h^{-m})(1-b)^{-1/2}. \tag{20b}$$

Suppose that $1-2^{-l-1} < \sqrt{1-b} < 1-2^{-l}$, where l is some integer. It can be verified that when $m \le l$

$$\sqrt{1-b}[\eta((RL)^\infty L\bullet) - \xi(\bullet R^m C)] = 2\sqrt{1-b} + h^{-m} - 2 > 2^{-m} - 2^{-l} > 0, \tag{21}$$

which guarantees the intersection between $(RL)^\infty L\bullet$ and $\bullet R^m C$, hence $\bullet RJ > \bullet R^l C$ and $\bullet LJ < \bullet L^l C$. Thus, symbolic sequences consisting of only segments R^iL^j with $i,j < l$ are always admissible.

From the admissibility conditions we may discuss the existence of a periodic orbit of the type $(R^mLWLL)^\infty$, where W is some string of R and L. The existence of the orbit requires the lines $\bullet(R^mLWLL)^\infty$ and $(R^mLWLL)^\infty\bullet$ to intersect. For $1-2^{-l-1} < \sqrt{1-b} < 1-2^{-l}$, we have that

$$\xi(\bullet(R^mLWLL)^\infty) > \xi(\bullet R^{m-1}C) > 1-2^{-m+1}, \tag{22a}$$

$$\begin{aligned}\eta((R^mLWLL)^\infty\bullet) < \eta((RL)^\infty LL\bullet) &= 1 - bh^{-1} + \frac{b^2h^{-2}}{1-bh^{-1}} \\ &< 1-2^{-l-1}+2^{-2l+1}.\end{aligned} \tag{22b}$$

When $m > l+2$ the value of ξ is greater than η, the condition for intersecting is broken, so the period cannot exist. However, the period $(R^mLWRL)^\infty$ may still exist because now the value of ξ is almost unchanged, while η may increase by an amount equal to about bh^{-1}.

When $1-b < a < 2\sqrt{1-b}$, any periodic orbits become unstable, so the existence of the trapping region implies the existence of a strange attractor. For $h > 1 > b$ we have that

$$\xi(\bullet(RL)^\infty) = \frac{h^{-1}}{1+h^{-1}} < 1 < \frac{1}{1-bh^{-1}} = \eta((RL)^\infty\bullet), \tag{23}$$

which implies the existence of the period-2 orbit $(RL)^\infty$. However, when b and h are very close to one, we have that

$$\xi(\bullet(RRLL)^\infty) = \frac{h^{-1}(1+h^{-1})}{1+h^{-2}} > \frac{1-bh^{-1}}{1+b^2h^{-2}} = \eta((RRLL)^\infty\bullet), \tag{24}$$

so the period-doubled orbit $(RLLR)^\infty$ does not exist.

IV. Concluding remarks

In the above we have discussed the expanding and contracting foliations of the phase space for the Tél map. They are ordered according to the backward and forward sequences

assigned to them. In terms of these two kinds of foliations the map is decomposed into two coupled one-dimensional maps. Tangencies between two different kinds of foliations rule out forbidden regions in the symbolic plane, among which two are fundamental and can generate all the others. The fundamental forbidden zones determine the admissibility condition for allowed symbolic sequences. At first sight, it seems that the Tél map differs qualitatively from the Lozi map . In fact, all the above arguments can be staightforwardly extended and applied to the Lozi map. Of course, for the Lozi map the slopes h_n and k_n of foliations are not constant any more, which makes the anlysis somewhat more complicated. A discussion of the Lozi map will be published elsewhere.

This work was partially supported by the CNSF, China.

Figure Captions

Fig.1 Forward sequences are assigned to contracting foliations. From $\bullet LP$ or $\bullet RP$ to $\bullet P$ the ordering is preserved. Points marked with numbers 1, 2, 3 and 4 map to points $1'$, $2'$, $3'$ and $4'$, respectively.

Fig.2 Backward sequences are assigned to expanding foliations. From $QL\bullet$ or $QR\bullet$ to $Q\bullet$ the ordering is reversed. Points marked with numbers 1, 2, 3 and 4 respectively map to $1'$, $2'$, $3'$ and $4'$ by the inverse map.

Fig.3 Some expanding and contracting lines. Four lines $(RL)^{\infty}\bullet$, $\bullet RK$, $(RL)^{\infty}\bullet$ and $\bullet L\bar{K}$ form the boundary of the fundamental trapping region. Above the line $(LR)^{\infty}\bullet$ is the region with a common single sequence $(LR)^{\infty}\bullet$. The shaded region corresponds to the single line $\beta = \frac{1}{2}$ in the symbolic plane.

Fig.4 The tangency between lines $QL\bullet$ and $\bullet RP$.

a) In the phase space expanding lines inside the shaded region bounded by $C\bullet$ and $QL\bullet$ cannot intersect with the shaded region right of line $\bullet RP$.

b) The tangency rules out an area (shaded rectangle) in the symbolic plane. By the anti-symmetry of the map in the upper left of the symbolic plane there is another forbidden rectangle, which is also shaded.

Fig.5 A sketch of the two fundamental forbidden zones generated by the tangencies on the x-axis.

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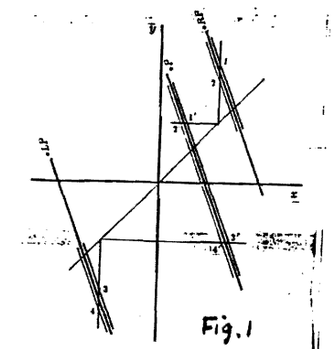

Fig. 1

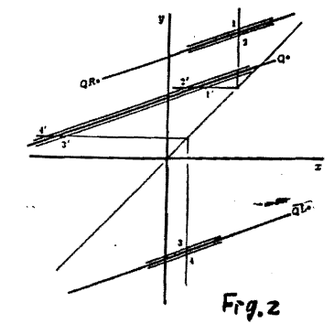

Fig. 2

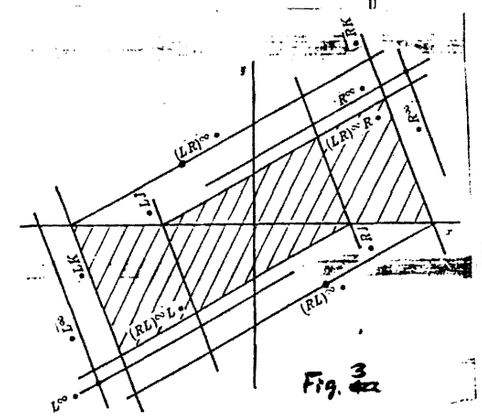

Fig. 3

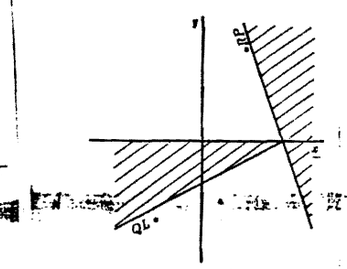

Fig. 4a

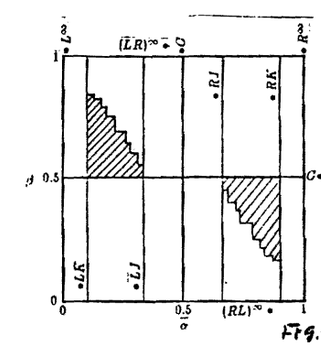

Fig. 4b

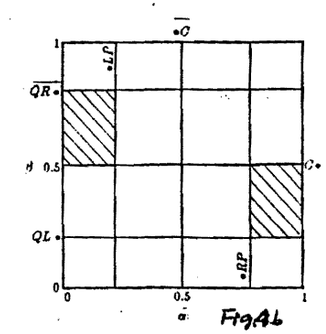

Fig. 4b