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TRANSVERSE SPACE CHARGE EFFECTS

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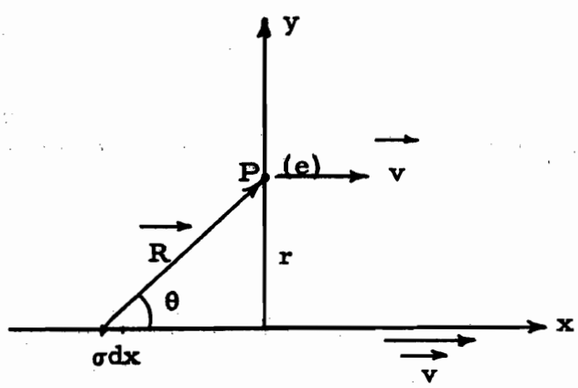
ABSTRACT

The conventional calculation of transverse space charge defocusing effect in circular magnetic accelerators for circular beam cross-section is improved and extended to take into account of the realistic elliptical shape of the beam cross-section. This results in an entirely different trajectory of migration of the operating point (ν_x, ν_y) due to space charge effect and, hence, a different set of limiting resonance lines. The result indicates that the space charge limit of the beam intensity in the ZGS is approximately between 4×10^{13} and 7×10^{13} protons/pulse depending on which resonance line is fatal and the distribution of protons along the equilibrium orbit at injection. Several effects tending to reduce the space charge limit are discussed. Close examination shows also that compensation of the space charge defocusing effect by varying the focusing action of the guide magnetic field is impractical. On the other hand it is likely that by artificially injecting free electrons into the vacuum chamber total neutralization of the transverse space charge defocusing effect can be achieved. This is studied.

TRANSVERSE SPACE CHARGE EFFECTS

The transverse defocusing effect due to space charge is circular magnetic accelerators has been investigated by many people, and the space charge limit on the beam intensity for the ZGS has been quoted to be 5×10^{13} protons/pulse. However the details of the investigation such as the approximations and assumptions made in carrying out the calculation have not been clearly discussed. This report is a detailed discussion of the various steps in the calculation of the space charge effect starting from first principles.

A. Force on a moving charge due to an infinite line of moving charges.



Let the infinite line of charges be on the x-axis moving with velocity $\vec{v}$ in the +X direction and having a linear charge density σ. Let there be a point charge e situated at P(o, r) moving also with velocity $\vec{v}$. Then the electric and the magnetic fields at P due to the charge element σdx may be obtained by a simple Lorentz transformation from the Coulomb field for a stationary charge element and are given by

$$\begin{cases} \vec{E} = \dfrac{(\sigma dx)\,\hat{R}}{R^2} \dfrac{1-\beta^2}{(1-\beta^2 \sin^2\theta)^{3/2}} \\[2ex] \vec{B} = \beta \hat{v} \times \vec{E} = \dfrac{\sigma dx}{R^2} \dfrac{\beta(1-\beta^2)}{(1-\beta^2 \sin^2\theta)^{3/2}} \quad (\hat{v} \times \hat{R}) \end{cases} \tag{1}$$

where $\vec{R}$ is the vecton from σdx to P, θ is the angle between $\vec{R}$ and $\vec{v}$ and $\beta \equiv \frac{v}{c}$ (c = speed of light) (For derivation see e.g. "The Classical Theory of Fields" Landau and Lifshitz, p. 99). The force on the charge e at P is, therefore,

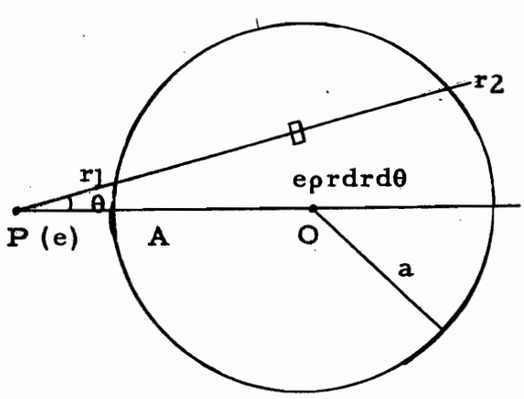
$$
\begin{aligned}
d\vec{f} &= e\,(\vec{E} + \beta\,\hat{v} \times \vec{B}) \\
&= \frac{e(\sigma dx)}{R^2}\;\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\;\left[(1-\beta^2)\,\hat{R} + \beta^2\,(\hat{v}\bullet\hat{R})\,\hat{v}\right] \\
&= \frac{e(\sigma dx)}{R^2}\;\frac{(1-\beta^2)^2\,\sin\theta}{(1-\beta^2\sin^2\theta)^{3/2}}\;\hat{y} + \frac{e(\sigma dx)}{R^2}\;\frac{(1-\beta^2)\cos\theta}{(1-\beta^2\sin^2\theta)^{3/2}}\;\hat{x}
\end{aligned}
\tag{2}
$$

where $\hat{x}$ and $\hat{y}$ are unit vectors along x and y directions respectively. To get the force due to the entire line--charge on the x-axis we integrate (2) to get $\vec{f} \equiv \int d\vec{f}$. It is clear from symmetry that $f_x = o$, and for f_y we have

$$
\left\{
\begin{aligned}
f_y &= e\sigma\,(1-\beta^2)^2 \int_{-\infty}^{\infty} \frac{\sin\theta}{R^2\,(1-\beta^2\sin^2\theta)^{3/2}}\,dx \\
&= \frac{e\sigma}{r}\,(1-\beta^2)^2 \int_{o}^{\pi} \frac{\sin\theta\,d\theta}{(1-\beta^2\sin^2\theta)^{3/2}} \\
&= \frac{2e\sigma}{r}\,(1-\beta^2) \equiv \frac{2e\sigma}{\gamma^2}\,\frac{1}{r} \\
f_x &= o
\end{aligned}
\right.
\tag{3}
$$

B. Force on a moving charge due to an infinite cylinder of moving charges with uniform density

I. Circular cross-section



Let the charge e be the pole of a set of polar coordinates r and θ, and the center o of the cross-sectional circle of radius a be situated on the polar axis at distance $\overline{PO} \equiv A$ from P. From section A. above we see that the force on P is in the plane of the polar coordinates, and from symmetry it is clear that the force

is along the polar axis PO. The magnitude of the component of the force on P along PO due to the charge element $e\rho r dr d\theta$ where ρ is the volumetric particle density in the beam and e is the charge on each particle, is according to (3)

$$df_{PO} = \frac{2e\,(e\,\rho\, r dr d\theta)}{r\gamma^2} \cos\theta \tag{4}$$

The magnitude of the total force on P is, then

$$f = \int df_{PO} = \frac{2e^2\rho}{\gamma^2} \int_{circle} \cos\theta\, dr d\theta = \frac{2e^2\rho}{\gamma^2} \int_{circle} (r_2 - r_1) \cos\theta d\theta \tag{5}$$

The equation of the circle is

$$(r\cos\theta - A)^2 + r^2 \sin^2\theta = a^2$$

or

$$r^2 - 2(A\cos\theta)\, r + (A^2 - a^2) = o \tag{6}$$

solving for r we have

$$r = A\cos\theta \pm \sqrt{a^2 - A^2 \sin^2\theta} \tag{7}$$

<u>Case 1</u> . $A < a$ viz the point P is inside the beam.
Only the positive sign in (7) is meaningful and we have

$$r_2 = A\cos\theta + \sqrt{a^2 - A^2 \sin^2\theta} \tag{8}$$

and that r_1 is obtained by replacing θ by $\pi + \theta$ or

$$r_1 = -A\cos\theta + \sqrt{a^2 - A^2 \sin^2\theta} \tag{9}$$

substituting (8) and (9) in (5) we get

$$f_{A<a} = \frac{4e^2\rho}{\gamma^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A\cos^2\theta\, d\theta = \frac{2e^2(\pi a^2\rho)}{a^2\gamma^2}\cdot A \qquad (10)$$

Case 2 $A > a$ viz point P is outside the beam. Now we have

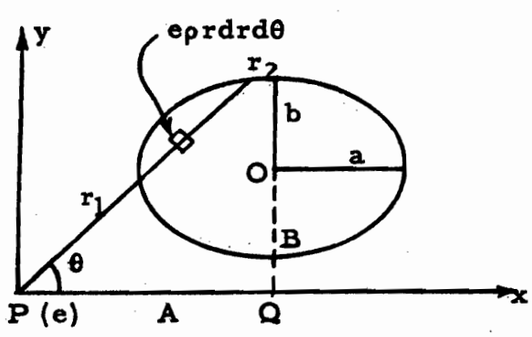
$$\begin{cases} r_2 = A\cos\theta + \sqrt{a^2 - A^2\sin^2\theta} \\ r_1 = A\cos\theta - \sqrt{a^2 - A^2\sin^2\theta} \end{cases} \qquad (11)$$

substituting (11) in (5) we get

$$f_{A>a} = \frac{4e^2\rho}{\gamma^2}\int_{-\sin^{-1}\frac{a}{A}}^{\sin^{-1}\frac{a}{A}} \sqrt{a^2 - A^2\sin^2\theta}\,\cos\theta\, d\theta \qquad (12)$$

$$= \frac{2e^2(\pi a^2\rho)}{\gamma^2}\,\frac{1}{A}$$

II. Elliptical cross-section



Here again we put the charge e at the pole and draw the polar axis PQ // to the major axis of the cross-sectional ellipse. We shall assume that the center of the ellipse is at $x = \overline{PQ} \equiv A$, $y = \overline{QO} \equiv B$ from P and that it's semi-major and -minor axes are a and b respectively. Again the force on P is in the coordinate plane. However, now, the component of force ⊥ to PQ (F_y) will, in general, not vanish. The magnitudes of the two components of the force are now

$$\begin{cases} f_x = \int df_x = \dfrac{2e^2\rho}{\gamma^2} \displaystyle\int_{\text{ellipse}} \cos\theta\, dr\, d\theta = \dfrac{2e^2\rho}{\gamma^2} \displaystyle\int_{\text{ellipse}} (r_2 - r_1)\cos\theta \\ f_y = \int df_y = \dfrac{2e^2\rho}{\gamma^2} \displaystyle\int_{\text{ellipse}} \sin\theta\, dr\, d\theta = \dfrac{2e^2\rho}{\gamma^2} \displaystyle\int_{\text{ellipse}} (r_2 - r_1)\sin\theta \end{cases} \tag{13}$$

The equation of the ellipse is

$$\frac{(r\cos\theta - A)^2}{a^2} + \frac{(r\sin\theta - B)^2}{b^2} = 1$$

or

$$r^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) - 2\left(\frac{A}{a^2}\cos\theta + \frac{B}{b^2}\sin\theta\right) r + \left(\frac{A^2}{a^2} + \frac{B^2}{b^2} - 1\right) = \tag{14}$$

solving for r we have

$$r = \frac{\dfrac{A}{a^2}\cos\theta + \dfrac{B}{b^2}\sin\theta \pm \sqrt{\left(\dfrac{A}{a^2}\cos\theta + \dfrac{B}{b^2}\sin\theta\right)^2 - \left(\dfrac{A^2}{a^2} + \dfrac{B^2}{b^2} - 1\right)\left(\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}\right)}}{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}$$

$$= \frac{a^2 B\sin\theta + b^2 A\cos\theta \pm ab\sqrt{a^2\sin^2\theta + b^2\cos^2\theta - (A\sin\theta - B\cos\theta)^2}}{a^2\sin^2\theta + b^2\cos^2\theta} \tag{15}$$

<u>Case 1</u> $\dfrac{A^2}{a^2} + \dfrac{B^2}{b^2} < 1$ viz point P is inside the beam. Only the positive sign in (15) is meaningful and we have

$$r_2 = \frac{a^2 B \sin\theta + b^2 A \cos\theta + ab\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta - (A \sin\theta - B \cos\theta)^2}}{a^2 \sin^2\theta + b^2 \cos^2\theta} \tag{16}$$

and that r_1 is obtained by replacing θ by $\theta+\pi$ or

$$r_1 = \frac{-(a^2 B \sin\theta + b^2 A \cos\theta) + ab\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta - (A \sin\theta - B \cos\theta)^2}}{a^2 \sin^2\theta + b^2 \cos^2\theta} \tag{17}$$

substituting (16) and (17) in (13) we have, denoting $a^2 - b^2$ by c^2

$$f_x = \frac{4 e^2 \rho}{\gamma^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a^2 B \sin\theta \cos\theta + b^2 A \cos^2\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta} \, d\theta$$

$$= \frac{4 e^2 \rho}{\gamma^2} \left[o + b^2 A \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{(a^2 + b^2) - c^2 \cos 2\theta} \, d\theta \right]$$

$$= \frac{4 e^2 \rho}{\gamma^2} \frac{b^2 A}{c^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{2 a^2}{(a^2 + b^2) - c^2 \cos 2\theta} - 1 \right] d\theta$$

$$= \frac{4 e^2 (\pi a b \rho)}{a(a+b) \gamma^2} A \tag{18}$$

$$f_y = \frac{4 e^2 \rho}{\gamma^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a^2 B \sin^2 \theta + b^2 A \sin \theta \cos \theta}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta$$

$$= \frac{4 e^2 \rho}{\gamma^2} \left[a^2 B \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - \cos 2\theta}{(a^2 + b^2) - c^2 \cos 2\theta} \, d\theta + o \right]$$

$$= \frac{4 e^2 \rho}{\gamma^2} \frac{a^2 B}{c^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[1 - \frac{2 b^2}{(a^2 + b^2) - c^2 \cos 2\theta} \right] d\theta$$

$$= \frac{4 e^2 (\pi a b \rho)}{b (a + b) \gamma^2} B \tag{19}$$

<u>Case 2</u> $\frac{A^2}{a^2} + \frac{B^2}{b^2} > 1$ viz. point P is outside the beam. In this case we have

$$\begin{cases} r_2 = \dfrac{a^2 B \sin \theta + b^2 A \cos \theta + a b \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta - (A \sin \theta - B \cos \theta)^2}}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \\[2ex] r_1 = \dfrac{a^2 B \sin \theta + b^2 A \cos \theta - a b \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta - (A \sin \theta - B \cos \theta)^2}}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \end{cases} \tag{20}$$

and

$$\left\{\begin{aligned} f_x &= \frac{4e^2\rho}{\gamma^2}\, a\, b \int \frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta - (A\sin\theta - B\cos\theta)^2}}{a^2\sin^2\theta + b^2\cos^2\theta}\, \cos\theta\, d\theta \\ f_y &= \frac{4e^2\rho}{\gamma^2}\, a\, b \int \frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta - (A\sin\theta - B\cos\theta)^2}}{a^2\sin^2\theta + b^2\cos^2\theta}\, \sin\theta\, d\theta \end{aligned}\right. \tag{21}$$

where the integrations are over all values of θ for which the integrands are real. These integrals are not easily evaluated in general. However, in cases where either A or B vanishes the integrals are readily evaluated to give:

$\underline{B = o}$

$$\left\{\begin{aligned} f_x &= \frac{4e^2\rho}{\gamma^2}\, a\, b \int \frac{\sqrt{(a^2 - A^2)\sin^2\theta + b^2\cos^2\theta}}{a^2\sin^2\theta + b^2\cos^2\theta}\, d(\sin\theta) \\ &= \frac{4e^2\rho}{\gamma^2}\, a\, b \int_{\sin\theta = -\frac{b}{\sqrt{A^2 - c^2}}}^{\frac{b}{\sqrt{A^2 - c^2}}} \frac{\sqrt{b^2 - (A^2 - c^2)\sin^2\theta}}{b^2 + c^2\sin^2\theta}\, d(\sin\theta) \\ &= \frac{4e^2(\pi a b \rho)}{c^2\gamma^2}\left(A - \sqrt{A^2 - c^2}\right) \\ f_y &= 0 \end{aligned}\right. \tag{22}$$

$\underline{A = 0}$

$$
\begin{cases}
f_x = 0 \\
f_y = -\dfrac{4e^2\rho}{\gamma^2}\, ab \displaystyle\int \frac{\sqrt{a^2\sin^2\theta + (b^2 - B^2)\cos^2\theta}}{a^2\sin^2\theta + b^2\cos^2\theta}\, d(\cos\theta) \\
\quad = -\dfrac{4e^2\rho}{\gamma^2}\, ab \displaystyle\int_{\cos\theta = \frac{a}{\sqrt{B^2+c^2}}}^{\frac{-a}{\sqrt{B^2+c^2}}} \frac{\sqrt{a^2 - (B^2+c^2)\cos^2\theta}}{a^2 - c^2\cos^2\theta}\, d(\cos\theta) \\
\quad = \dfrac{4e^2(\pi ab\rho)}{c^2\gamma^2}\left(\sqrt{B^2+c^2} - B\right)
\end{cases}
\tag{23}
$$

It is interesting to check that (22) and (23) agree with (18) and (19) on the surface of the beam, i.e. A = a in (22) or B = b in (23), and that (17) (18) (22) and (23) agree with the case of circular cross-section (10) and (12) when a = b or c = o.

C. Space charge limit of beam intensity in a circular magnetic accelerator

We shall make the following assumptions:

1. The magnetic focusing force is smoothed out to distribute uniformly around the equilibrium orbit. The frequencies of the betatron oscillations in the absence of the space charge effect are ν_{xo} and ν_{yo}. (For ZGS $\nu_{xo} = 0.750$, $\nu_{yo} = 0.875$). The equations for the betatron oscillations can be written as

$$
\frac{d^2x}{d\theta^2} = -\nu_{xo}^2\, x, \qquad \frac{d^2y}{d\theta^2} = -\nu_{yo}^2\, y \tag{24}
$$

where

$$
d\theta \equiv \frac{d\,(\text{distance along equilibrium orbit})}{\text{equivalent radius of equil. orbit}} = \frac{ds}{R}
$$

2. The circulating beam has a constant velocity and a particle volumetric density ρ uniform throughout a cross-section of fixed shape (circular or elliptical). The radius of curvature of the equilibrium orbit is everywhere much larger than the dimensions of the cross-section of the beam so that the calculations made above for the space charge forces in infinite straight beams can be used as good approximations.

For Proton Beam with Circular Cross-section

The equations for the lateral motions of a proton in the beam due to space charge force alone are, from (10)

$$\left\{\begin{aligned} m\gamma \frac{d^2x}{dt^2} &= \frac{2e^2(\pi a^2 \rho)}{a^2\gamma^2}\, x \\ m\gamma \frac{d^2y}{dt^2} &= \frac{2e^2(\pi a^2 \rho)}{a^2\gamma^2}\, y \end{aligned}\right. \tag{25}$$

or

$$\left\{\begin{aligned} \frac{d^2x}{d\theta^2} &= \frac{2e^2}{mc^2}\left(\frac{R}{a}\right)^2 \frac{\pi a^2 \rho}{\eta^2\gamma}\, x \\ \frac{d^2y}{d\theta^2} &= \frac{2e^2}{mc^2}\left(\frac{R}{a}\right)^2 \frac{\pi a^2 \rho}{\eta^2\gamma}\, y \end{aligned}\right. \tag{26}$$

where $\eta \equiv \beta\gamma$ and m = rest-mass of proton. When both the magnetic focusing action and the space-charge force are present equations (24) and (26) give

$$\left\{\begin{aligned} \frac{d^2x}{d\theta^2} &= -\left[\nu_{xo}^2 - \frac{2e^2}{mc^2}\left(\frac{R}{a}\right)^2 \frac{\pi a^2 \rho}{\eta^2\gamma}\right] x \equiv -\nu_x^2\, x \\ \frac{d^2y}{d\theta^2} &= -\left[\nu_{yo}^2 - \frac{2e^2}{mc^2}\left(\frac{R}{a}\right)^2 \frac{\pi a^2 \rho}{\eta^2\gamma}\right] y \equiv -\nu_y^2\, y \end{aligned}\right. \tag{27}$$

Thus, we see that for a beam of circular cross-section as the particle density ρ varies the operating point (ν_x, ν_y) moves on the hyperbola

$$\nu_x^2 - \nu_{xo}^2 = \nu_y^2 - \nu_{yo}^2$$

or

$$\nu_y^2 - \nu_x^2 = \nu_{yo}^2 - \nu_{xo}^2 = (0.875)^2 - (0.750)^2 = .203125 \tag{28}$$

This curve is shown in Figure 1 together with all the important resonance lines in the neighborhood. The important resonance lines are those given by

$$p\,\nu_x + q\,\nu_y = n$$

with positive integral p, q, and n where $p + q \leqslant 3$. Starting from the point $\nu_x = \nu_{xo} = 0.750$, $\nu_y = \nu_{yo} = 0.875$ we see that as ρ increases the operating point encounters in succession the following resonance lines

$$3\,\nu_x = 2, \quad 2\,\nu_x + \nu_y = 2, \quad \nu_x + 2\,\nu_y = 2, \quad 2\,\nu_x = 1$$

$$3\,\nu_y = 2, \quad \nu_x + \nu_y = 1, \quad 3\,\nu_x = 1, \quad 2\,\nu_x + \nu_y = 1$$

$$2\,\nu_y = 1, \quad \nu_x + 2\,\nu_y = 1, \quad \nu_x = o$$

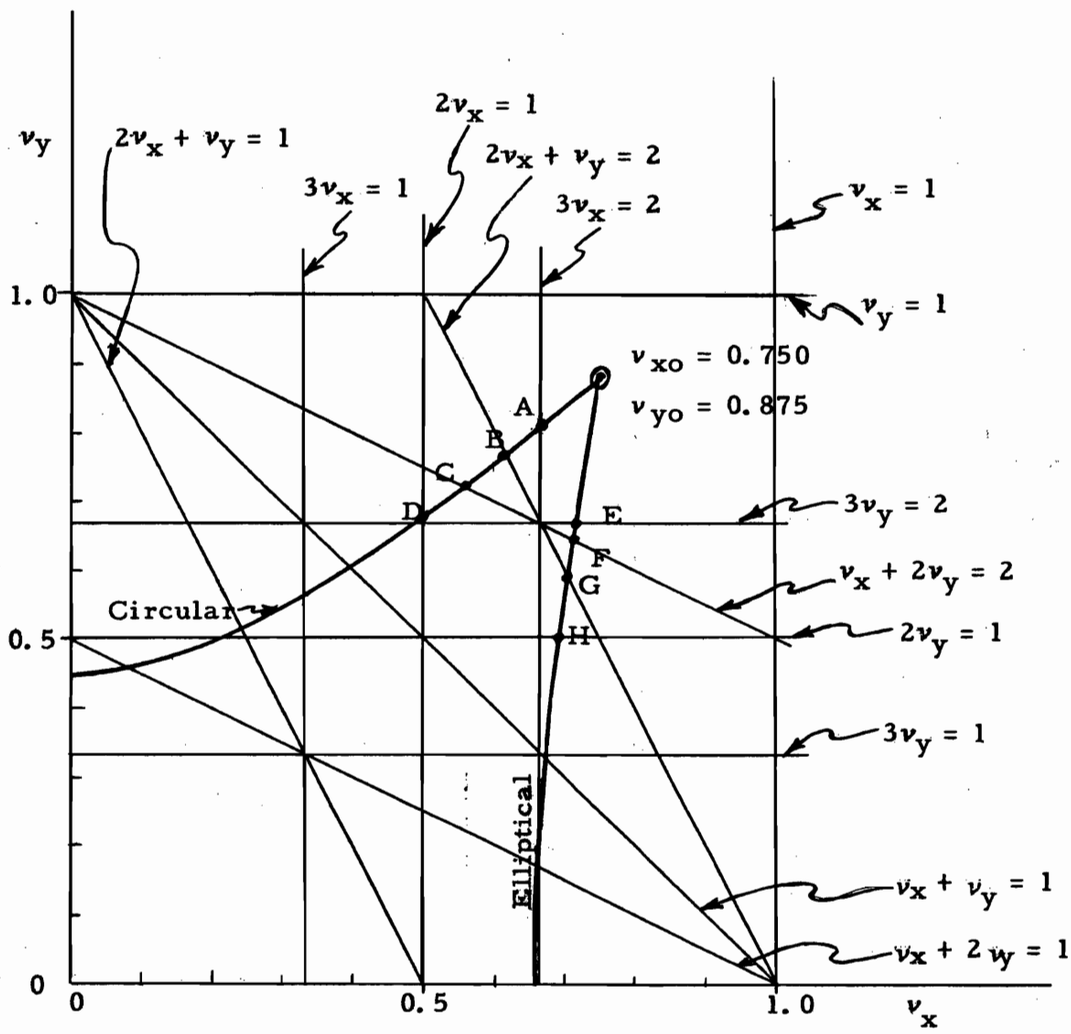
where the first 4 points of intersection are

$$\text{A} \begin{cases} \nu_x = .6667 \\ \nu_y = .8047 \end{cases} \qquad \text{B} \begin{cases} \nu_x = .6177 \\ \nu_y = .7646 \end{cases}$$

$$\text{C} \begin{cases} \nu_x = .5609 \\ \nu_y = .7195 \end{cases} \qquad \text{D} \begin{cases} \nu_x = .5000 \\ \nu_y = .6731 \end{cases} \tag{29}$$

Assuming that the protons are distributed uniformly all around the equilibrium orbit we can, then, write for the total number N of protons, from (27), as

FIGURE 1


ν_y
$2\nu_x + \nu_y = 1$
$2\nu_x = 1$
$2\nu_x + \nu_y = 2$
$3\nu_x = 1$
$3\nu_x = 2$
$\nu_x = 1$
$\nu_y = 1$
1.0
$\nu_{xo} = 0.750$
$\nu_{yo} = 0.875$
A
B
C
D
E
F
G
H
$3\nu_y = 2$
$\nu_x + 2\nu_y = 2$
Circular
0.5
$2\nu_y = 1$
$3\nu_y = 1$
Elliptical
$\nu_x + \nu_y = 1$
$\nu_x + 2\nu_y = 1$
0
0
0.5
1.0
ν_x

As ρ varies the operating point now moves on the hyperbola

$$a\,(\nu_x^2 - \nu_{xo}^2) = b\,(\nu_y^2 - \nu_{yo}^2)$$

or (33)

$$a\,\nu_x^2 - b\,\nu_y^2 = a\,\nu_{xo}^2 - b\,\nu_{yo}^2$$

For ZGS, $\nu_{xo} = 0.750$, $\nu_{yo} = 0.875$ and a and b are respectively the half radial-width and the half height of the vacuum chamber viz a = 40.64 cm., b = 6.667 cm. Equation (33) now becomes

$$40.64\,\nu_x^2 - 6.667\,\nu_y^2 = 17.755 \tag{34}$$

and is plotted also in Fig. 1. As ρ increases, the operating point encounters in succession the following resonances

$$3\,\nu_y = 2,\quad \nu_x + 2\,\nu_y = 2,\quad 2\,\nu_x + \nu_y = 2,\quad 2\,\nu_y = 1$$

$$3\,\nu_y = 1,\quad \nu_x + \nu_y = 1,\quad 3\,\nu_x = 2,\quad \nu_x + 2\,\nu_y = 1$$

$$\nu_y = o$$

with the coordinates of the first 4 points of intersection given by

$$E \begin{cases} \nu_x = .7140 \\ \nu_y = .6667 \end{cases} \qquad F \begin{cases} \nu_x = .7107 \\ \nu_y = .6447 \end{cases}$$

$$G \begin{cases} \nu_x = .7033 \\ \nu_y = .5934 \end{cases} \qquad H \begin{cases} \nu_x = .6913 \\ \nu_y = .5000 \end{cases} \tag{35}$$

Again assuming that the protons are distributed uniformly all along the equilibrium orbit we have, from (32)

$$N = 2\pi R(\pi a b \rho) = \frac{\pi m c^2}{e^2} R \frac{b(a+b)}{2R^2} \eta^2 \gamma (\nu_{yo}^2 - \nu_y^2)$$
$$= 2.047 \times 10^{16} R \frac{b(a+b)}{2R^2} \eta^2 \gamma (\nu_{yo}^2 - \nu_y^2) \tag{36}$$

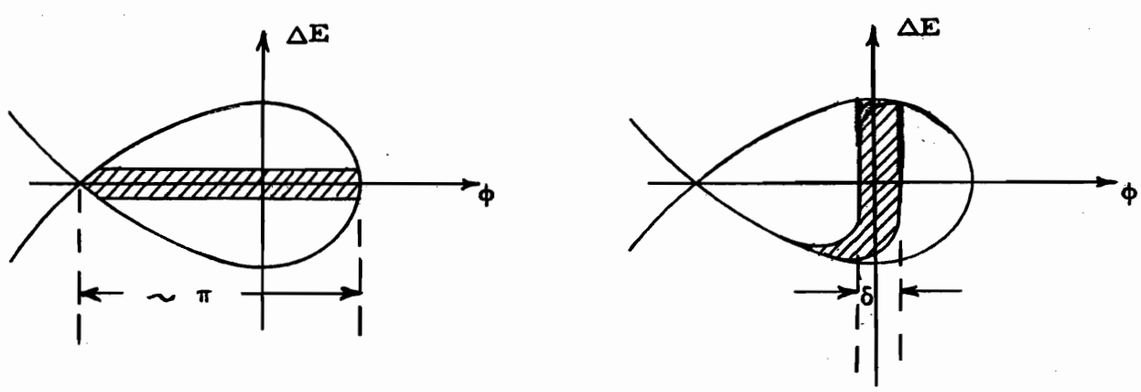
For ZGS at injection for the 4 values of ν_y given in (35) we have in the same order

$$N_E = 4.371 \times 10^{13} \qquad N_F = 4.763 \times 10^{13}$$
$$N_G = 5.627 \times 10^{13} \qquad N_H = 7.017 \times 10^{13} \tag{37}$$

We see, therefore, that the space charge limit of the beam intensity in the ZGS is approximately between 4.4×10^{13} and 7.0×10^{13} depending on whether the 3rd integral resonances are fatal. Again if we take the stable RF phase into consideration we should reduce the numbers given in (37) by a factor of approximately 2.

D. Further discussions

I. The above calculation assumes that the protons are uniformly distributed all along the equilibrium orbit. The consideration of the stable RF phase range already cuts the space charge limits given by (37) by a factor of 2. This is, again, assuming that the entire stable phase region in the RF phase plane (energy spread ΔE vs. RF phase angle ϕ) is filled. However, in reality, at injection if the energy spread of the protons from the injector Linac is very small only the shaded strip of the stable phase region (Fig. 2A) will be filled. A quarter of a synchrotron oscillation



(A) (B)

FIGURE 2

later the shaded region is deformed to that shown in Fig 2B when the protons are essentially concentrated over a $\frac{\delta}{2\pi}$ fraction of the entire length of the equilibrium orbit. In this case the space charge limits given in (37) should be reduced by a factor $\frac{\delta}{2\pi}$ where δ depends on the energy spread of the injected beam and could be very small for very small energy spread. This phenomenon can be remedied by artificially increasing the energy spread of the injected beam. The optimum condition is clearly that where the energy spread of the injected beam is about equal to the ΔE width of the stable phase region.

II. Equation (36) may be written as

$$N \backsim \frac{b(a+b)}{R} \eta^2 \gamma \Delta (\nu_y^2)$$

For the change in ν_y^2 namely $\Delta (\nu_y^2)$ we can write approximately $\Delta (\nu_y^2) = 2 \nu_y \Delta \nu_y$. Since the allowable change in ν_y before encountering a fatal resonance line, namely $\Delta \nu_y$ is approximately 1/4 whatever the value of ν_y, we can write for the ratio between the space charge limits of 2 accelerators (subscripts 1 and 2) as

$$\frac{N_1}{N_2} \cong \frac{b_1 (a_1 + b_1)}{b_2 (a_2 + b_2)} \frac{R_2}{R_1} \frac{\nu_{y_1}}{\nu_{y_2}} \frac{\eta_1^2 \gamma_1}{\eta_2^2 \gamma_2} \tag{38}$$

If we, now, compare ZGS with Brookhaven AGS ($R_2 = 12.846 \times 10^3$ cm., $a_2 = 7.620$ cm., $b_2 = 3.335$ cm. $\nu_{y_2} = 8.75$ and same injection energy) we get $\frac{N_{ZGS}}{N_{AGS}} = 4.056$, namely that the space charge limit for ZGS is only about 4 times better than that of AGS. The reason that in spite of the much smaller vacuum chamber aperture the space charge limit of AGS is rather good is twofold.

In the first place, the betatron focusing in AGS is much stronger than that in ZGS, namely $\nu_{y_2} = 8.75$ as against $\nu_{y_1} = 0.875$. In the second place, it is clear from the above derivation that for a given vacuum chamber cross-sectional area the space charge limit is larger the nearer the cross-sectional area is to being circular. The major to minor axes ratio is $\frac{a_2}{b_2} = 2.285$ for AGS as against $\frac{a_1}{b_1} = 6.095$ for ZGS.

III. The space charge limit increases as the protons gain energy in the synchrotron. For ZGS assuming uniform distribution of protons along the equilibrium orbit we have

t (sec.)	γ	N_E	N_H
0	1.0533	4.371×10^{13}	7.017×10^{13}
.00913	1.1	8.760×10^{13}	1.406×10^{14}
.02382	1.2	2.002×10^{14}	3.214×10^{14}
.03582	1.3	3.401×10^{14}	5.461×10^{14}
.04650	1.4	5.096×10^{14}	8.182×10^{14}
.05640	1.5	7.110×10^{14}	1.141×10^{15}
.06692	1.6	9.465×10^{14}	1.519×10^{15}
.07919	1.75	1.369×10^{15}	2.197×10^{15}
.1004	2.0	2.275×10^{15}	3.653×10^{15}
.1789	3.0	9.101×10^{15}	1.461×10^{16}
.3273	5.0	4.550×10^{16}	7.305×10^{16}
.6892	10.0	3.754×10^{17}	6.027×10^{17}
1.0000	14.323	1.109×10^{18}	1.780×10^{18}

where t is the time after injection. From this table we see that even if we take the optimistic value N_H, the space charge limit will not be as large as 10^{14} until

$t = .00432$ sec when $\gamma = 1.0738$ and will not be as large as 5×10^{14} until $t = .03367$ when $\gamma = 1.2811$. In order to get beam intensities as high as 5×10^{14} protons/pulse or even 1×10^{14} protons/pulse, therefore, something has to be done to compensate for the space charge defocusing action during the time intervals specified on the preceding page..

IV. Remembering the lowest order smooth approximation formulas for the betatron focusing of a synchrotron with straight section length to magnet length ratio $\frac{S}{M}$ and effective field index n, namely

$$\begin{cases} \nu_x^2 \cong (1 - n)(1 + \frac{S}{M}) \\ \nu_y^2 \cong n(1 + \frac{S}{M}) \end{cases} \tag{39}$$

we see that by changing n we can only cause the operating point to move on a circle centered at $\nu_x = \nu_y = o$ with radius $\sqrt{1 + \frac{S}{M}}$. Referring to Fig. 1 again, we notice that changing n alone can not help to increase the space charge limit. What we would like to do is to increase ν_y^2 and at the same time leave ν_x^2 more or less unchanged. Putting $\nu_x^2 = \text{constant} = \nu_{xo}^2$ equations (39) give

$$\nu_y^2 = \nu_{xo}^2 \frac{n}{1 - n} = (1 + \frac{S}{M}) - \nu_{xo}^2 \tag{40}$$

showing that in order to change ν_y^2 both n and $\frac{S}{M}$ have to be changed, and that for a change of ν_y^2 from $\nu_{yo}^2 = .765625$ to $\nu_{yo}^2 + (.765625 - .250000) = 1.281250$ (to compensate for the space charge defocusing effect which shifts ν_y^2 from .765625 to .250000) n has to change from .5765 to .6949 which is relatively easy to achieve but also $\frac{S}{M}$ has to change from .3281 to .8438 which is quite impossible to do in practice. This shows conclusively that compensation of the space charge defocusing effect by varying the focusing action of the guide magnetic field is impractical. There remains the possibility of neutralizing the space

charge of the proton beam by free electrons. There is indication that partial space charge neutralization does occur in existing synchrotrons even without artificial injection of free electrons.

E. Neutralization of transverse space charge effect.

The same calculations as those in sections (A) and (B) and (C) above can be made to give the space charge force on one of the moving protons in a synchrotron beam of elliptical cross-section uniformly filled by moving protons of density ρ and by stationary electrons of density ρ_e. The result is

$$\begin{cases} f_x = \dfrac{4\,e^2\,(\pi\,a\,b)}{a\,(a+b)}\left(\dfrac{\rho}{\gamma^2} - \rho_e\right) x \\[2ex] f_y = \dfrac{4\,e^2\,(\pi\,a\,b)}{b\,(a+b)}\left(\dfrac{\rho}{\gamma^2} - \rho_e\right) y \end{cases} \tag{41}$$

The space charge force $\vec{F}$ on one of the stationary electrons in the same beam can be similarly calculated and is given by

$$\begin{cases} F_x = \dfrac{4\,e^2\,(\pi\,a\,b)}{a\,(a+b)}\,(\rho_e - \rho)\,x \\[2ex] F_y = \dfrac{4\,e^2\,(\pi\,a\,b)}{b\,(a+b)}\,(\rho_e - \rho)\,y \end{cases} \tag{42}$$

In the absence of other forces the protons and the electrons in the beam will be bound together if

$$\frac{1}{\gamma^2} < \frac{\rho_e}{\rho} < 1 \tag{43}$$

This, of course, is the same condition for the onset of the "pinch effect" when the focusing action of the guide magnetic field on the protons is taken into account, the equations corresponding to (32) for the transverse motion of the protons in the beam are

$$\begin{cases} \dfrac{d^2x}{d\theta^2} = -\left[\nu_{xo}^2 - \dfrac{4e^2}{mc^2} \dfrac{R^2(\pi ab)}{a(a+b)} \dfrac{\gamma}{\eta^2} \left(\dfrac{\rho}{\gamma^2} - \rho_e \right) \right] x \equiv -\nu_x^2 x \\ \dfrac{d^2y}{d\theta^2} = -\left[\nu_{yo}^2 - \dfrac{4e^2}{mc^2} \dfrac{R^2(\pi ab)}{b(a+b)} \dfrac{\gamma}{\eta^2} \left(\dfrac{\rho}{\gamma^2} - \rho_e \right) \right] y \equiv -\nu_y^2 y \end{cases} \tag{44}$$

The space charge defocusing effect is completely neutralized if

$$\rho_e = \frac{\rho}{\gamma^2} \tag{45}$$

The trouble with this procedure is that it is difficult to keep condition (45) satisfied for all values of γ throughout the acceleration cycle. If condition (45) is satisfied at injection ($\gamma = \gamma_o = 1.0533$ for ZGS) and if nothing is done to extract the free electrons from the beam, then, since condition (43) is satisfied at all times during the acceleration period, all the electrons will remain in the beam. Rewriting the second equation of (44) as

$$\begin{aligned} \nu_{y_o}^2 - \nu_y^2 &= \frac{2e^2}{\pi mc^2} \frac{R}{b(a+b)} \frac{\gamma}{\eta^2} \left(\frac{N}{\gamma^2} - N_e \right) \\ &= \frac{2e^2}{\pi mc^2} \frac{R}{b(a+b)} \frac{\gamma}{\eta^2} \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_o^2} \right) N \end{aligned} \tag{46}$$

where $N \equiv 2\pi R\,(\pi ab\rho)$ and $N_e \equiv 2\pi R\,(\pi ab\rho_e)$ are the total numbers of protons and electrons in the beam assuming again uniform distribution all along the equilibrium orbit we get for ZGS at $N = 10^{14}$ and $N = 5 \times 10^{14}$ (over)

t (sec.)	γ	ν_y ($N=10^{14}$)	ν_y ($N=5 \times 10^{14}$)
0	1.0533	.875	.875
.00913	1.1	.901	1.000
.02382	1.2	.929	1.118
.03582	1.3	.943	1.175
.04650	1.4	.951	1.208
.05640	1.5	.956	1.228
.06692	1.6	.960	1.244
.07919	1.75	.962	1.252
.1004	2.0	.964	1.258
.1789	3.0	.958	1.234
.3273	5.0	.941	1.168
.6892	10.0	.918	1.072
1.0000	14.323	.908	1.029

Here we have used the adiabatically damped beam cross-sectional dimensions a and b calculated in LCT-6. This table shows that if the trapped electrons are not extracted from the beam during the acceleration cycle ν_y (and also ν_x) will rise very quickly from ν_{y_o} to a maximum value at about $\gamma = 2.0$. For $N = 10^{14}$ this maximum value for ν_y is about .964 which, although being uncomfortably close to the $\nu_y = 1$ resonance, may still be tolerable and for $N = 5 \times 10^{14}$ the maximum value of ν_y is about 1.258 which means a crossing of the $\nu_y = 1$ resonance and is, therefore, not permissible. The values of ν_y calculated above are affected by the following considerations. In the first place, here again, we assumed uniform distribution of protons and electrons along the equilibrium orbit. The discussion in (D) (I) should be applied here. This means that the ν_y values given in the above table may be too low. In the second place, we have assumed that N remains constant throughout the acceleration. In actuality, of course, there is

a big reduction of the number of protons during the first few synchrotron oscillations. This indicates also that the ν_y values given above are too low. In the third place, with the ν_y so close to $\nu_y = 1$ resonance the beam cross-sectional dimensions a and b may well be larger than those computed under the assumption of pure adiabatic damping. This would make the ν_y values given above too high. However this can at most be only a small effect.
When all are taken into account we can now make a semi-quantitative conclusion. When space charge neutralization is employed to get synchrotron beam intensities above that given by the space charge limit free electrons should be injected (This can be accomplished by either a hot therm-ionic emitting wire or a photo-electric emitting surface running the whole length close to the top or the bottom of the vacuum chamber) into the vacuum chamber at the time of injection of the protons. Clearing electrodes should be provided across the vacuum chamber so that when the acceleration of the protons commences the trapped electrons in the beam can be gradually extracted so that the relation

$$\text{total number of electrons} = \frac{1}{\gamma^2}\,(\text{total number of protons})$$

is satisfied at all times.

We have, in the above, discussed only one of the many problems we have to face in order to get beam intensities higher than, say, 10^{14} protons/pulse. We shall now list some of the other problems:

1. The longitudinal space charge effect -- this amounts to a defocusing action on the phase oscillation.
2. The image effects -- the charge and current images in the vacuum chamber walls and the magnet poles of the very intense beam will produce non-negligible defocusing actions on the beam itself.
3. The radiation effects -- both the coherent and the incoherent radiations from the very intense beam may now be quite noticeable.

4. The beam-cavity interaction -- the heavy loading of the intense beam on the accelerating RF cavity may shift the synchronous phase (hence, the energy gain per turn) and reduce the stable phase area in the synchrotron oscillation phase plane.

All these problems should be investigated and solved before beams more intense than 10^{14} protons/pulse can be obtained.

LCT:bl

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