THE FORMAL INTEGRALS OF HAMILTONIAN SYSTEMS

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1. We consider Hamiltonian Systems of the form

$$(1) \qquad \begin{cases} \dot{x} = H_y \\ \dot{y} = -H_x \end{cases}$$

where $H = H(x,y,t)$ has the period 2π in t and is a power series in x and y which contains no constant and linear terms. That means $x = y = 0$ is a solution which we call the reference solution. We also want to assume that this reference solution is stable according to a linear theory, i. e. according to the equations of variation. Under these circumstances it is no loss of generality to assume that H has the form

$$H = -\frac{\omega}{2}(x^2 + y^2) + \dots \quad .$$

It is well known that a system of this nature does not admit a non-trivial integral, that is a function $I(x,y,t)$ for which

$$(2) \qquad \frac{d}{dt} I = I_t + I_x H_y - I_y H_x = 0$$

holds, which has the period 2π in t and does not vanish identically. On the other hand it was proven in an earlier paper* that there

* J. Moser Stabilitätsverhalten kanonischer Differentialgleichungssysteme, Nachrichten der Akademie der Wissenschaften, Göttingen, 1955.

exists a formal power series expansion (which is divergent in general) for an integral which satisfies the equation (2) in the formal sense, that means the coefficients of the series in (2) are all zero. These formal expansions prove to be of importance in deciding the stability of the reference orbit. Even more they allow an asymptotic description of the motion near the reference orbit.

In the paper just mentioned only the existence of such expansions for I was proven. But this proof involves many transformations of coordinates and it would be desirable to construct these series directly. Such a construction has the following difficulty: If one tries to compute the coefficients of I from (2), one sees that they are not uniquely determined by this equation, the reason being that with I every function of I is an integral and hence satisfies (2). But if one chooses the undetermined coefficients, say of order n in I, in a definite way, one is not able to satisfy equation (2) for the terms of order greater than n and is led to a contradiction. In other words, one has to determine the coefficients of I in a definite way which guarantees that equation (2) can be satisfied for all terms. It is the aim of this note to give such a construction which is a purely formal problem. The method will be applicable just as well to construct one invariant for a Hamiltonian system of two degrees of freedom.

2. The condition for I to be an integral is given by equation (2). This equation does not determine I uniquely as we saw above. This caused the difficulty of constructing I. In order to choose the arbitrary coefficient in I in the right way - as to obtain compatibility of the terms of higher order - we will add a second equation for I which has to be fulfilled simultaneously with (2). It is this second equation that we want to reveal.

For this purpose we write the system in polar coordinates: Let

$$R = x^2 + y^2 \quad \text{and} \quad x = \sqrt{R}\cos\varphi, \quad y = \sqrt{R}\sin\varphi$$

and

$$-2H = K(R, \varphi, t) = \omega R + \dots \quad .$$

Then the system (1) transforms into

$$\dot{\varphi} = K_R$$
$$\dot{R} = -K_\varphi$$

and φ, R are canonical conjugate variables. The condition for $I(R, \varphi, t)$ to be an integral is

$$I_t + I_\varphi \dot{\varphi} + I_R \dot{R} = I_t + I_\varphi K_R - I_R K_\varphi = 0.$$

Introducing the notation

$$[I, K] = I_\varphi K_R - I_R K_\varphi$$

we have

$$I_t + [I, K] = 0.$$

The second equation which we will add artificially is

$$I(R,\varphi,t) + u_t + \omega u_\alpha = K(R,\varphi,t) - \omega R$$

where $u(R,\alpha,t)$ is a function which has to be determined additionally and where for φ we have to insert

$$\varphi = \alpha + u_R\,(R,\alpha,t).$$

The functions I and u have to be determined as to have the period 2π in t. The system we have to solve then reads as follows:

$$(3) \qquad \begin{cases} I_t + [I, K] = 0 \\ I(R,\alpha + u_R,t) + u_t + \omega u_\alpha = K(R,\alpha + u_R,t) - \omega R. \end{cases}$$

We want to show how this system can be solved by determining the coefficients step by step. For this purpose we expand I and u in a power series in $\sqrt{R}$, namely $\sum I_\kappa R^\kappa$, $\sum u_\kappa R^\kappa$.
Since I,u have the period 2π in t and φ resp. α, we can expand I_κ, u_κ in a Fourier series

$$I_\kappa = \sum_{\mu,\nu} I_{\kappa\mu\nu}\, e^{i(\mu\varphi - \nu t)}, \quad u_\kappa = \sum_{\mu,\nu} u_{\kappa\mu\nu}\, e^{i(\mu\alpha - \nu t)}$$

so that we write the general term in I,u, as

$$I_{\kappa\mu\nu}\, R^\kappa\, e^{i(\mu\varphi - \nu t)}, \quad u_{\kappa\mu\nu}\, R^\kappa\, e^{i(\mu\alpha - \nu t)}$$

where 2κ is a positive integer and μ, ν are integers.

How to determine the coefficients: We will assume that $\omega = \frac{p}{q}$ is a rational number. (In the case of irrational ω the procedure is even easier.) Let us assume we have determined $I_{\kappa\mu\nu}$, $u_{\kappa\mu\nu}$ for $\kappa < k$ already and try to determine them for $\kappa = k$. By expanding the functions in (3) we see that

$$\frac{\partial}{\partial t} I_k + \omega \frac{\partial}{\partial \varphi} I_k$$

$$I_k + \frac{\partial}{\partial t} u_k + \omega \frac{\partial}{\partial \alpha} u_k$$

are given functions which can be computed from the determined coefficients. In order to solve for I_k and u_k expand these functions in Fourier series with respect to φ, t, resp. α, t. We find that

$$i(\nu - \omega\mu)\, I_{k\mu\nu}$$

$$I_{k\mu\nu} - i(\nu - \omega\mu)\, u_{k\mu\nu}$$

are numbers which can be computed by known coefficients. If now $\nu - \omega\mu \neq 0$, we easily obtain $I_{k\mu\nu}$ from the first equation and $u_{k\mu\nu}$ from the second. If, however, $\nu - \omega\mu = 0$, then $I_{k\mu\nu}$ is determined by the second equation, while $u_{k\mu\nu}$ remains arbitrary, and <u>the first equation will be satisfied automatically.</u> This underlined statement will be proven below. In other words, the coefficients $u_{\kappa\mu\nu}$ for which $\nu - \omega\mu = 0$ are arbitrary and after they have been chosen, I is determined uniquely.

In order to achieve that I is a real power series, we have to choose the arbitrary coefficients $u_{\kappa\mu\nu}$ with $\nu - \omega\mu = 0$, so that

$$\overline{u_{k\mu\nu}} = u_{\kappa,-\mu,-\nu} .$$

Then I, u will be real automatically.

3. Before proving that this method actually never leads to a contradiction, let us add a few remarks:

a. Actually this method will be more interesting when ω is only close to a rational number, say $\omega \sim \frac{p}{q}$. In that case one would replace the equations (3) by

$$(4) \qquad \begin{cases} I_t + [I, K] = 0 \\ I(R, \alpha + u_R, t) + u_t + \frac{p}{q} u_\alpha = K - \frac{p}{q} R \end{cases}$$

and determine the first coefficients of I, u from these equations. Expanding into a series we obtain for I

$$I = (\omega - \frac{p}{q}) R + \dots \quad .$$

The higher terms will be responsible for the shape of the level lines of I.

b. This method can be applied to systems of two degrees of freedom just as well. The equations (3) have to be replaced by

$$I_t + \sum_{\nu=1}^{2} I_{\varphi_\nu} K_{R_\nu} - I_{R_\nu} K_{\varphi_\nu} = 0, \quad \varphi_\nu = \alpha_\nu + u_{R_\nu}$$
$$I + u_t + \omega_1 u_{\alpha_1} + \omega_2 u_{\alpha_2} = K - \omega_1 R_1 - \omega_2 R_2.$$

Thus we obtain one formal integral.

c. For actual computations the coordinates R, φ are somewhat inconvenient, but it is no difficulty to rewrite (3) in the old coordinates x, y. For this purpose one introduces a function $v(x, \eta, t) = x\eta + \dots$ and fulfills instead of (3) the equations

$$I_t + I_x H_y - I_y H_x = 0 \ , \quad y = v_x$$

and

$$I(x, v_x, t) + v_t + \omega(v_\eta^2 + \eta^2) = K(x, v_x, t).$$

Expanding with respect to x, η one can determine the coefficients of I, v successively.

4. Finally we want to show that the equations (3) really can be solved step by step and do not lead to a contradiction. For this purpose we define a canonical transformation of R, φ into P, α by the equations

$$P = R + u_\alpha$$
$$\varphi = \alpha + u_R$$

where $R\alpha + u(R, \alpha, t)$ forms the generating function. By this transformation the system with the Hamiltonian $K(R, \varphi, t)$ will be transformed into a system with the Hamiltonian $F(P, \alpha, t)$ where

$$F = K - u_t.$$

We first rewrite the equations (3). Introducing the function $I(R, \varphi, t) = F - \omega P$ we have because of $P = R + u_\alpha$

$$\begin{aligned} I + u_t + \omega u_\alpha &= F - \omega P + u_t + \omega u_\alpha \\ &= F - \omega R + u_t \\ &= K - \omega R \end{aligned}$$

which is exactly the second equation of (3).

In order to rewrite the first equation, we observe that the Poisson bracket

$$[G, H] = G_\varphi H_R - G_R H_\varphi = G_\alpha H_P - G_P H_\alpha$$

has a meaning independent of the canonical variables used. Therefore we have - observing that F is a Hamiltonian -

$$F_t = \frac{d}{dt} F - [F, F] = \frac{d}{dt} F = \frac{d}{dt} (I + \omega P) = \frac{d}{dt} I + \omega [P, F]$$

and since

$$F_\alpha = [F, P] = -[P, F]$$

we have

$$F_t + \omega F_\alpha = \frac{d}{dt} I = I_t + [I, K] .$$

Therefore the equations (3) have the meaning that the transformation determined by the generating function $R\alpha + u$ transforms H into the Hamiltonian $F(P, \alpha, t)$ which satisfies

$$F_t + \omega F_\alpha = 0.$$

Now it is sufficient to show that the two equations

$$(5) \qquad \begin{cases} F(R + u_\alpha, \alpha, t) + u_t = K(R, u_R, t) \\ F_t + \omega F_\alpha = 0 \end{cases}$$

can be solved step by step. We expand

$$F = \sum F_\kappa P^\kappa , \quad u = \sum u_\kappa R^\kappa$$

and assume that F_κ, u_κ for $\kappa < k$ are determined already. Then we form the terms of order k in (5) and find from the first equation that

$$F_k(\alpha, t) + \left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \alpha}\right) u_k$$

is a given function, and from the second that

$$\frac{\partial}{\partial t} F_k + \omega \frac{\partial}{\partial \alpha} F_k = 0$$

By expanding F_k, u_k into a Fourier series:

$$F_k = \sum_{\mu, \nu} F_{k\mu\nu} e^{i(\mu\alpha - \nu t)} , \quad u_k = \sum_{\mu, \nu} u_{k\mu\nu} e^{i(\mu\alpha - \nu t)}$$

we see that

$$(\nu - \omega\mu)\, F_{k\mu\nu} = 0$$

and that

$$F_{k\mu\nu} - i(\nu - \omega\mu)\, u_{k\mu\nu}$$

is a given number. If $\nu - \omega\mu \neq 0$, we have $F_{k\mu\nu} = 0$ and $u_{k\mu\nu}$ is uniquely determined. If, however, $\nu - \omega\mu = 0$ then $F_{k\mu\nu}$ is determined from the second equation and the first one is satisfied automatically.

Since we saw that the equations (5) are equivalent to (3), we thus proved the compatibility of the equation (3).