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Higher Spin Symmetries and the Problem of Unification of all Interactions

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Higher Spin Symmetries and the Problem of Unification of all Interactions

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There are two problems of basic interest in modern theoretical physics. The first is to unify all the fundamental interactions including gravity at the quantum level. Nowadays it is a general point of view that the quantum gravity problem is in fact a problem of unification of all interactions.

The second is a problem of exact solvability in quantum field theory and quantum statistics. During the recent years there has been a considerable success in solving 2D conformal models, however the main problems in D>2 are still open. This is the exact theory of phase transitions in D=3 and a confinement problem in D=4.

The question arising here is: What new methods should be developed to approach these problems? The development of theoretical physics during the whole century shows that practically all new fundamental results were obtained on the basis of discovering some novel symmetries. So it may be said: symmetry is a guide for physicists.

Then, what symmetries may be expected to appear to help us to solve the above problems? At least one thing can be said for sure: they will be infinite-dimensional symmetries.

Two types (classes) of infinite dimensional (super)symmetries were proposed by our group at Lebedev Physical Institute:

(1) Higher Spin Anti-de Sitter (*Fradkin & Vasiliev (1986), (1987c), (1988); Vasiliev (1988), (1989a,b), (1991))* and Conformal (*Fradkin & Linetsky (1989a), (1990a); (1991a))* supersymmetries.

(2) Virasoro-like (super)symmetries in dimension D$\geq$2 or more generally a new class $AC(g)$ (and its central extension $\widetilde{AC}(g)$) of infinite-dimensional algebras, constructed by our method of infinite extension ("analytical continuation") of any semisimple finite Lie algebra (*Fradkin & Linetsky (1990b), (1991b,e,f))*.

From the mathematical point of view these infinite-dimensional algebras, containing the given semisimple finite-dimensional Lie algebras g as a maximal finite subalgebra, (for possibility of having a spontaneous broken phase with the lower-energy g-symmetry) are devided into two classes under the natural representation $[g,\cdot]$ of g. The first-class algebras (the higher spin or Kac-Moody algebras) are decomposed only into a direct sum of finite-dimensional irreducible g-modules. Second-class algebras (Virasoro like* or more generally $AC(g)$ and $\widetilde{AC}(g)$ algebras) involve also infinite-dimensional g-modules (irreducible or/and non-decomposable). We believe that these infinite-dim. algebras and their representation theory will play an important role in solving the problems of unification of all interactions and of exact solvability in QFT and statistics. In the present talk I shall try to give a brief account of our attempts to construct a local unified theory on the basis of the infinite-dimensional higher spin symmetries and the corresponding higher spin gauge theories proposed by us.

*The simplest example of second-class infinite-dimensional algebras is Virasoro-algebra ($\widetilde{AC}(sl_2)$).

1. The Problem of Unification of all Fundamental Interactions Including Gravity (Basic Results).

During the whole twentieth century the selfconsistent unification of all the fundamental interactions including gravity has been a central problem of theoretical physics. This problem came into existence right after Einstein formulated General Relativity as a theory of gravity. At the beginning the problem consisted in unification of gravity and electrodynamics in the framework of classical field theory (it is sufficient to mention the Weyl conformally-invariant approach and the Kaluza-Klein approach). At the stage it was realized that gauge invariance principles could give keys to the unification of interactions, and all the attempts to construct a Unified Theory actually had been a search of some extension of the general coordinate transformation groups to incorporate also the gauge group of vector fields in a natural way.

The progress in physics has led to the establishment of the following main criteria that a Unified Theory must satisfy (for more details see *Fradkin (1989)):*

(1) it should be selfconsistent at the quantum level;

(2) it should give an adequate description of low-energy physics.

The anomaly cancellation conditions for all classical symmetries of the theory and the finiteness of the Unified Theory are understood as a selfconsistency. Simultaneously this criterium of anomaly cancellation imposes strong restrictions on the spectrum of elementary particles and the gauge group for vector fields. (*Fradkin E.S.* and *Tseytlin A.A. (1984), Green M.B.* and *Schwarz J.H. (1984)).*

An adequate description of low-energy physics must include a solution of the cosmological constant problem, an answer to the question why the world we live in has just four observable dimensions, and it must predict the observed spectrum of elementary particles and fundamental forces, including the Clashow-Salam-Weinberg model, quantum chromodynamics and gravity, in final spontaneously broken phase.

Conventionally the evolution of theoretical physics on the way towards a Unified The-

ory can be divided into two stages. The first one is a lower spin stage, including only lower spin ($s \leq 2$) fields. A number of important discoveries has been obtained in this stage (the discovery of global non-Abelian symmetries among elementary particles led to their successful classification, the discovery of non-Abelian Yang-Mills gauge symmetries allowed one to unify the vector interactions and led to the Unification of electromagnetic, weak and strong interactions in the framework of Grand Unification Models; the discovery of global supersymmetry allowed one to unify Bose and Fermi particles together in unified supermultiplets and led to the Supersymmetric Grand Unification Models. At last, by gauging the global supersymmetry, supergravity was discovered).

The golden age of supergravity theories at the end of the seventies was a culmination of the lower spin stage on the way towards a Unified Theory. There had been discovered three types of supersymmetries and, correspondingly, three types of supergravity theories: Poincaré, anti-de Sitter and conformal ones. They describe systems of interacting gauge fields with spins $s = 2$ (graviton), $s = 3/2$ (gravitino), $s = 1$ (Yang-Mills vector fields), as well as matter fields with $s = 0, 1/2$ in extended models. (More details see in the review articles *Nieuwenhuizen (1981); Fradkin & Tseytlin (1985a).)*

However it did not become the end of the way. Eventually it turned out none of the known supergravity theories completely met the strong requirements for a Unified Theory. They were neither finite (except for the $N = 4$ conformal supergravity) nor contained the observable spectrum of elementary particles because there appeared a restriction $N \leq 8$ on the degree of N-extended supersymmetry* (or $N \leq 4$ in conformally invariant case).

In this way, realizing that it does not seem possible to construct a selfconsistent quantum theory including gravity and observed spectrum of elementary particles without introducing an infinite number of higher spin particles which are required both to make the theory finite and to get more extended supersymmetries, one comes to the next stage on the way towards a Unified Theory, the higher spin stage. According to the

*$N > 8$ supersymmetries involve higher spins $S > 2$ in the supermultiplets.

classification of supersymmetries and supergravities, there may exist three kinds of theories involving infinite towers of higher spins along with the lower spin supergravity sector: Poincaré, anti-de Sitter and conformally-invariant theories.

The higher spin stage can be divided into two substages: a massive higher spin stage and massless gauge symmetry higher spin stage.

Superstring theory provides a first successful example which extends Einstein supergravity to the case of all massive higher spins. It describes gravity coupled with an infinite number of massive spin fields. In string theory there is a natural mass parameter inversely related with the square root of the string slope α'. In the zero-slope limit $\alpha' \to 0$ masses of all the higher spin excitations, as well as of the lower spin ones of higher levels, tend to infinity and only the massless (super)gravity and Yang-Mills sector remains as observed [in detail see: *Green, Schwarz* and *Witten*, String Theory (Cambridge (1987))].

However, as far as the higher energy domain is concerned, all the higher spin excitations of the string become equally essential and give their contributions to the interaction. This situation gives rise to several interesting higher energy phenomena. The interaction vertex in string theory contains increasingly higher derivatives (up to infinity). This in turn involves the negative power of the mass parameter to make the whole action dimensionless. As a result the interaction turns out to be non-analytical in the mass parameter* (which is proportional to $(\alpha')^{-1/2}$ in string theory) and this non-analyticity brings about the formal obstruction to the massless limit $\alpha' \to \infty$ on the flat background.

A striking example of the non-analyticity on the mass parameter $m \approx (\alpha')^{-1/2}$ of the interaction in string theory is the tree effectiv action for the electromagnetic field in string theory. As a consequence of the contribution of the interacting infinite tower of higher spin excitations in the effective action of strings (*Fradkin & Tseytlin (1985c-e)*), the effective Lagrangian of the electromagnetic field has a Born-Infeld form (see:

*However, it is natural to wait that a nontrivial phase transition with a drastical change in the structure of the flat vacuum in string theory makes possible the continuous massless limit for the higher spins.

Fradkin & Tseytlin (1985f))

$$L_{eff} = \frac{1}{2\pi\alpha'} \sqrt{det(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})} \,. \tag{1.1}$$

An other important and closely related consequence of the infinite interacting tower of spins is the upper bound* on the stress tensor $F_{\mu\nu}^{max}$ in (1.1). Moreover, when the electric field E tends to its critical value $E^{max} = (2\pi\alpha')^{-1}$ the mass of the higher spins tends to zero. This illustrates the possibility of a phase transition from the massively higher spin stage to the "non-broken" phase – the massless (gauge) higher spin stage. (Moreover it is natural to suppose that the masses of the higher spins in the massively higher spin stage are resulted by spontaneous breakdown of the higher spin gauge symmetry.)

The massless higher spin gauge stage of Unification of all interactions was initiated and developed in our group at Lebedev Physical Institute. Two types of such gauge theories describing infinite towers of massless higher spin gauge fields coupled to supergravity: Anti-de Sitter Higher Spin and Conformal Higher Sping Gauge Theories have been obtained.

Anti-de Sitter higher spin gauge theories in four space-time dimensions have been developed by *Fradkin* and *Vasiliev (1986), (1987a-c), (1988)* and *Vasiliev (1988), (1989a,b), (1990), (1991).* They are based on a new infinite-dimensional higher spin global supersymmetry generalizing the ordinary anti-de Sitter supersymmetry to all higher spins which was discovered by *Fradkin* and *Vasiliev (1986), (1987c)* and *Vasiliev (1988).* From the mathematical point of view this is an infinite-dimensional Lie superalgebra containing $osp(N \mid 4)$ as its maximal finite subalgebra. Its representation theory was studied by *Vasiliev (1987)* and *Konstein* and *Vasiliev (1989), (1990).* These global higher spin symmetries may be gauged similarly to the AdS supergravity. The corresponding gauge theory was constructed by *Fradkin* and *Vasiliev (1987a,b)* in the cubic approximation in the framework of Lagrangian field theory, and then extended to

*The critical bound of the physical quantities (electromagnetic stress tensor, gravity curvature, temperature, etc.), and the essential non-linearity of the tree effective actions for the fundamental fields is typical for a finite quantum theory and this makes finite also the classical limit of such theories.

all orders in the interaction by *Vasiliev (1990), (1991)* in the framework of the invariant equations of motion. Two most interesting qualitative physical properties of the theory are the presence of higher derivatives in the interaction vertices, as in string theory, and a peculiar non-analyticity of the interaction in the anti-de Sitter cosmological constant Λ. This non-analyticity does not allow one to pass to the naive flat limit $\Lambda \to 0$. The situation here is quite similar to one in the closed string theory where one is not allowed to pass to the massless limit $\alpha' \to \infty$ ($T \to 0$) due to the non-analyticity of string vertices in the string tension.

It should be mentioned also that our results disprove the common belief that consistent gauge-invariant interaction among massless higher spins and gravity does not exist (the so-called "no-go" theorem*). A key point here is just the above mentioned non-analyticity in the cosmological constant and the presence of higher derivatives interaction vertices in our higher spin gauge theories.

The conformal higher spin gauge theory has been developed by *Fradkin* and *Linetsky (1989a), (1990a), (1991a)*. It generalizes the conformal supergravity (supersymmetric extension of the C^2 conformal gavity discovered by Hermann Weyl) and describes a conformally invariant interaction among higher spin fields in four space-time dimensions. This gauge theory is based on the global infinite-dimensional higher-spin conformal supersymmetry discovered by *Fradkin* and *Linetsky (1990a)*. From the mathematical point of view it is an infinite-dimensional Lie superalgebra containing $SU(2,2 \mid N)$ as its miaximal finite subalgebra. The invariant Lagrangian generalizing the Weyl Lagrangian $C^2_{\mu\nu\rho\sigma}$ (C-Weyl tensor) to the interaction of all higher spins was constructed by *Fradkin*

*It was demonstrated that the minimal gravitational interaction introduced by means of covariantizing the derivatives violated the higher spin gauge symmetry under which the free higher spin massless action was invariant. Precisely, in the higher spin gauge variation of the higher spin action there appear some terms including gravitational Weyl tensor (already in the cubic order). Such terms cannot be compensated not to violate the gauge symmetry of the higher spin action via any modifications of the transformational law of the gravitational field (metric) because the variation of the Einstein action is proportional only to the Ricei tensor, no longer the whole Riemann one. (More details see: *C. Aragone* and *S. Deser*, Phys. Lett. B86 (1979) 161; *S. Christensen* and *M. Dutt*, Nucl. Phys. B154 (1979) 301; *F.A. Berends et al.*, J. Phys. A13 (1980) 1643; *B. de Wit* and *D.Z. Freedman*, Phys. Rev. D21 (1980).) This no-go theorem had served for a long time as an obstruction to all the attempts to build up a massless higher spin gauge theory coupled with gravity.

and *Linetsky (1989a), (1991a)* in the cubic order approximation. Most important qualitative feature of this theory is the absence of any scale parameters. The only arbitrary parameter in the theory is a dimensionless Weyl gravity coupling constant. Hence such a theory might be regarded as an asymptotic theory of gravity and higher spins in the ultra-high energy domain where any mass parameters become insufficient.

In particular, all the three theories, string theory, anti-de Sitter higher spin gauge theory and conformal higher spin theory, though they are essentially different, might turn out eventually to be actually three different phases of one and the same Unified field theory (see *Fradkin* and *Linetsky (1991g))* with new forces conditioned by higher spin gauge fields.

2. Anti-de Sitter Higher Spin Gauge Theory

Below I will draw more details upon the ideas, methods and the structure of the Higher Spin Gauge Theories constructed by us. The cornerstone of our approach is a new class of symmetries, so-called higher spin symmetries and their localization.

The mathematical basis to describe AdS higher spin symmetries, generalizing the usual finite space-time (super)symmetries is an infinite-dimensional Lie superalgebra which contains $osp(N \mid 4)$ as its maximal finite-dimensional subalgebra.

2.1. The superalgebras for higher spins in AdS_4.

There is more than one way to construct the infinite-dimensional global superalgebras. One way is to postulate the superalgebra spectrum with respect to a finite-dimensional subalgebra g and, by imposing some additional restrictions, solve the Jacobi identities for the structure constants of the infinite-dimensional superalgebra. In this way we (*Fradkin & Vasiliev (1986), (1987c))* first obtained the simplest superalgebra of higher spins in AdS_4, denoted by $shs(1 \mid 4)$ (*shs* means super higher spin). The anti-de Sitter superalgebra $osp(1 \mid 4)$ is its maximum finite-dimensional subalgebra. The gauge fields

of this superalgebra have the labelling $\omega_{\mu,\alpha(n),\dot\beta(m)}$ with arbitrary $n, m \geq 0$. The fields ω with an even (odd) number of spinor indices are commuting (anticommuting). They also satisfy the hermiticity relation*

$$\omega^{+}_{\mu,\alpha(n),\dot\beta(m)} = \omega_{\mu,\beta(m),\dot\alpha(n)} \tag{2.1}$$

The curvatures of $shs(1 \mid 4)$ are of the form (*Fradkin & Vasiliev (1987c)*)

$$\begin{aligned} R_{\mu\nu,\alpha(n),\dot\beta(m)} &= \partial_{[\mu}\omega_{\nu],\alpha(n),\dot\beta(m)} + \\ &+ \sum_{p,q,s,k,l,t=0}^{\infty} i^{s+t-1} \frac{n!m!}{p!q!s!k!l!t!} \times \\ &\times \lambda^{1+\frac{1}{2}(|n-m|-|p+s-k-t|-|q+s-l-t|)} \times \\ &\times \delta\left(|\,(p+k)(q+l) + (p+k)(s+t) + (q+l)(s+t) + 1\,|_2\right) \times \\ &\times \delta\,(n-p-q)\,\delta(m-k-l)\,\omega_{\mu,\alpha(p)\gamma(s),\dot\beta(k)\dot\delta(t)} \times \\ &\times \omega_{\nu,\alpha(q)}{}^{\gamma(s)}{}_{,\dot\beta(l)}{}^{\dot\delta(t)}\,. \end{aligned} \tag{2.2}$$

Here $\delta(n)$ equals 1 or 0 for $n = 0$ and $n \neq 0$ respectively, and $\delta(|\,n\,|_2)$ equals 1 or 0 for n even and n odd respectively. The real parameter λ coincides with the inverse radius of the background anti-de Sitter space. The cosmological constant Λ is proportional to $-\lambda^2$.

The curvatures (2.2) contain both positive and negative powers of λ. This is the reason for the non-analyticity of the higher-spin interaction in the cosmological constant. Note that the linearized curvatures, unlike the full ones, are analytic in λ and admit the flat limit $\lambda \to 0$.

*Spins s of the field $\omega_{\nu,\alpha(n),\dot\beta(m)}$ are connected with n and m by the relation $n+m = 2(s-1)$. Each set of spinor indices $\alpha(n)$ and $\dot\beta(m)$ takes the values 1,2 and is symmetrized separately. The indices are raised and lowered by the respective antisymmetric symbols $\epsilon_{\alpha\beta}$; $\epsilon_{\dot\alpha,\dot\beta}$. The dynamical fields are components of $\omega_{\nu,\alpha(n),\dot\beta(m)}$ with $|\,n-m\,| \leq 1$. The components with $n-m = 2$ are auxiliary fields. The components with $|\,n-m\,| > 2$ are called extra fields. The extra and auxiliary fields can be expressed through the dynamical fields and are needed for the construction of the curvatures. For example, in the case of gravitational fields, $\omega_{\nu,\alpha\dot\beta}$ is the vierbein, and the auxiliary fields $\omega_{\nu\alpha(2)}$, $\omega_{\nu\dot\beta(2)}$ are components of the Lorentz connection. For more details of other notations see appendix.

This method, connected with the explicit solving of the Jacobi identities, is, however, very cumbersome and, therefore, practically inapplicable to more complicated cases such as the conformal theory. Another method, based on the operator realization of the algebra, is much simpler and more illuminating. This method for the construction of the infinite-dimensional superalgebra G containing a given finite-dimensional subalgebra g consists of the following (*Vasiliev (1988)*). First, one chooses an appropriate operator realization of the subalgebra g by quadratic and linear combinations (generally, with additional constraints) of the generators $\hat{Z}_A$ of the Heisenberg-Clifford superalgebra. (The supercommutator of $\hat{Z}_A$ is $[\hat{Z}_A, \hat{Z}_B] = \mathbb{C}_{AB}$ where $\mathbb{C}_{AB}$ is a constant orthosymplectic matrix.) Next, one considers an associative algebra of polinoms of all powers in the operators $\hat{Z}_A$. In this algebra one introduces the Grassman parity $\varepsilon(A)$ and the supercommutator

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - (-1)^{\varepsilon_A \varepsilon_B} \hat{B}\hat{A} . \tag{2.3}$$

Technically,the problem is to find the structure constants of this Lie superalgebra (G). Its generators T_A are expressed through the powers of the operators $\hat{Z}_A$ in the Weyl (symmetric) normal form, and one must compute $[T_A, T_B] = F^C_{AB} T_C$. Such problems are most conveniently solved, by the method of symbols (*Beresin (1966)*). Let the Weyl symbol of the gauge field $\omega_\mu(\hat{Z})$ corresponding to the superalgebra G be $\omega_\mu(Z)$. Then for the operator curvature one has

$$R_{\mu\nu}(\hat{Z}) = \partial_{[\mu}\omega_{\nu]}(\hat{Z}) + [\omega_\mu(\hat{Z}), \omega_\nu(\hat{Z})] \tag{2.4}$$

and the corresponding symbol is of the form

$$R_{\mu\nu}(Z) = \partial_{[\mu}\omega_{\nu]}(Z) + [\omega_\mu(Z), \omega_\nu(Z)\}^*, \tag{2.5}$$

$$[A, B\}^* = A * B - (-1)^{\varepsilon(A)\varepsilon(B)} B * A \tag{2.6}$$

where $*$ denotes the associative multiplication of Weyl symbols:

$$A * B = A(Z) \exp(\overleftrightarrow{\Delta})\, B(Z), \tag{2.7}$$

$$\overleftrightarrow{\Delta} = \frac{1}{2}\frac{\overleftarrow{\partial}}{\partial Z_A} C_{A,B} \frac{\overrightarrow{\partial}}{\partial Z_B}, \; [Z_A, Z_B\}^* = C_{A,B} \,. \tag{2.8}$$

Now let us consider the higher spin extension of AdS supersymmetry (where $g = osp(N \mid 4)$). First we choose an appropriate operator realization of $osp(N \mid 4)$.

Let $q_\alpha, r_{\dot\beta}$ and ψ_i be Heisenberg-Clifford generating elements with non-zero supercommutators

$$[q_\alpha, q_\beta] = 2i\varepsilon_{\alpha\beta}, \; [r_{\dot\alpha}, r_{\dot\beta}] = 2i\varepsilon_{\dot\alpha\dot\beta}, \tag{2.9a}$$

$$\{\psi_i, \psi_j\} = 2\delta_{ij}, \tag{2.9b}$$

where $\dot\alpha, \dot\beta = 1,2$ and $\alpha, \beta = 1,2$ are dotted and undotted two-component $sl(2;C)$ multispinorial indices, $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \varepsilon_{\dot\alpha\dot\beta} = -\varepsilon_{\dot\beta\dot\alpha}$ are the symplectic metrics; the Hermitian conjugation is defined as follows

$$(q_\alpha)^+ = r_{\dot\alpha}, \; (r_{\dot\alpha})^+ = q_\alpha, \; (\psi_i)^+ = \psi_i, \tag{2.9c}$$

and $i = 1, \cdots, N$ are SO(N) internal indices. Then the second-order polynomials on the generating elements q, r and ψ with the basis

$$M_{\alpha\beta} = \frac{1}{2} q_{(\alpha} q_{\beta)}, \; \bar M_{\dot\alpha\dot\beta} = \frac{1}{2} r_{(\dot\alpha} r_{\dot\beta)}, \; P_{\alpha\dot\beta} = q_\alpha r_{\dot\beta}$$
$$T_{ij} = \frac{1}{2}\psi_{[i}\psi_{j]}, \; Q_{i\alpha} = q_\alpha \psi_i, \; \bar Q_{i\dot\alpha} = r_{\dot\alpha}\psi_i \tag{2.10a}$$

form a Lie superalgebra isomorphic to $osp(N \mid 4)$ ($M_{\alpha\beta} = M_{\beta\alpha}$, $\bar M_{\dot\alpha\dot\beta} = \bar M_{\dot\beta\dot\alpha}$ are the $sl(2;C)$ Lorentz generators, $P_{\alpha\dot\beta}$ - AdS_4 translations, $T_{ij} = -T_{ji}$ – $so(N)$-generators, and $Q, \bar Q$-supersymmetry generators).

An idea to generalize this construction to the case of all higher spins consists in introducing into the game polynomials of all even orders on the generating elements q, r and ψ (the restriction to only even polynomials here is for convenience; it does not lead to any loss of the information, see *(Vasiliev (1988))*.) According to this idea, the gauge fields of all higher spins are generated from the single object $\omega_\mu(q, r, \psi \mid x)$ as follows:

$$\omega_\mu(q,r,\psi \mid x) = \sum_{n,m=0}^{\infty} \sum_{k=0}^{N} \frac{1}{n!m!k!}\, \omega_{\mu,\alpha_1\cdots\alpha_n,\dot\beta_1\cdots\dot\beta_m,i_1\cdots i_k}$$
$$\times\ q^{(\alpha_1}\cdots q^{\alpha_n)}\, r^{(\beta_1}\cdots r^{\beta_m)}\, \psi^{[i_1}\cdots\psi^{i_k]},\ \ n+m+k \text{ is even}\,. \tag{2.11a}$$

(The round (square) brackets mean complete symmetrization (anti-symmetrization), that corresponds to the Weyl ordering of the bosonic $q_\alpha, r_{\dot\beta}$ and fermionic ψ_i operators.) The terms with $n+m+k=2$ in the above expansion corresponds to the N-extended AdS_4 supergravity gauge fields, while the collection of fields with $n+m+k=2(s_m\text{-}1)$, form $osp(N \mid 4)$-irreducible higher spin supermultiplets with the maximal spin s_m (take into account additional unity due to the vector index μ of ω_μ). Such an infinite-dimensional Lie superalgebra formed by the generators

$$T_{\alpha(n),\dot\beta(m),i(k)} = \frac{1}{n!m!k!}\, q_{(\alpha_1}\cdots q_{\alpha_n)}\, r_{(\dot\beta_1}\cdots r_{\dot\beta_m)}\, \psi_{[i_1}\cdots\psi_{i_k]} \tag{2.10b}$$

was denoted *(Vasiliev (1988))* as $shs^E(N \mid 4)$ (shs means super higher spin). The curvatures of the AdS_4 higher spin gauge theory based on shs-algebras are defined as usual

$$R_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu,\omega_\nu\}\,. \tag{2.11b}$$

To calculate the supercommutator of two gauge fields $\omega_\mu(q,r,\psi)$ and $\omega_\nu(q,r,\psi)$ it is helpful to employ the very convenient formalism of the Weyl symbols corresponding to quantum operators. As a result one arrives at the final expression for the curvature

(Vasiliev (1988)).

$$
\begin{aligned}
R_{\mu\nu,i(f),\alpha(n),\dot\beta(m)} &= \partial_\nu \omega_{\mu,i(f),\alpha(n),\dot\beta(m)} - \partial_\mu \omega_{\nu,i(f),\alpha(n),\dot\beta(m)} \\
&+ \sum_{p,q,s,k,l,t,u,v,r=0}^{\infty} i^{s+t-1}(-1)^{r(r-1)/2} \frac{f!n!m!}{p!q!s!k!l!t!u!v!r!} \\
&\times (\hbar)^{r+s+t-1} \lambda^{1+(|n-m|-|p+s-k-t|-|q+s-l-t|)/2} \\
&\times \delta(f-u-v)\,\delta(n-p-q)\,\delta(m-k-l) \\
&\times \delta(|\,(p+k)(q+l)+(p+k)(s+t) \\
&\quad + (q+l)(s+t)+u\cdot v+u\cdot r+v\cdot r+1\,|_2) \\
&\times \omega_{\nu,i(u)j(r),\alpha(p)\gamma(s),\dot\beta(k)\dot\delta(t)}\, \omega_{\mu,}{}^{j(r)}{}_{i(v),\alpha(q)}{}^{\gamma(s)}{}_{,\dot\beta(l)}{}^{\dot\delta(t)} .
\end{aligned}
\tag{2.12}
$$

In the case when $N=1$, (2.12) coincides with (2.2).

In the structure constants of the AdS_4 higher spin superalgebra and, consequently, in the expressions for the curvatures there is a dimensional parameter λ. All the dependence on λ is uniquely fixed by the AdS_4 commutation relations $[P_\mu, P_\nu] = -\lambda^2 M_{\mu\nu}$ (that is $-\lambda^2 \sim \Lambda$ is the AdS cosmological constant) and the requirement on the gauge higher spin fields to have correct physical dimensions to describe free dynamics. The crucial point here is a non-analyticity of the structure constants and curvatures in the AdS cosmological constant $\Lambda \sim -\lambda^2$. Indeed it is easy to see that in the above expression for the curvatures both positive and negative powers of λ are involved. This non-analyticity results in the non-analyticity of the higher-spin-gravitational interaction and does not permit one to pass to the naive flat background $\lambda \to 0$.

It should be mentioned that we have presented here only the simplest versions of the AdS_4 higher-spin superalgebras to illustrate the very idea of construction. In fact, in practice some more complications are needed to obtain also the necessary sets of auxiliary fields and to meet the requirement of unitarity.

It is known that in supergravity, without auxiliary fields, the numbers of boson and fermion off-shell degrees of freedom are not equal to each other, and the local gauge algebra is open. The introduction of auxiliary fields closes the algebra off shell. This

essentially facilitates both the construction of the full Lagrangian and the quantization procedure. Such a superalgebra is obtained *(Fradkin & Vasiliev (1988))* which, in addition to the higher-spin gauge fields, gives rise to auxiliary fields generalizing the "new minimal" formulation of $N = 1$ supergravity *(Sohnius & West (1981))*. This superalgebra $shs\alpha(1)$ ("α" means auxiliary) admits a natural operator realization generalizing the previous construction. Namely, in addition to the spinor operators (2.11a) one introduces the operators $\hat{Q}$ and $\hat{R}$ such that

$$\begin{aligned}
&\hat{Q}^2 = 1, \ \hat{R}^2 = 1, \ [\hat{Q}, \hat{R}] = 0, \ \{\hat{Q}, \hat{q}_\alpha\} = 0, \\
&[\hat{Q}, \hat{\tau}_{\dot{\beta}}] = 0, \ \{\hat{R}, \hat{\tau}_{\dot{\beta}}\} = 0, \ [\hat{R}, \hat{q}_\alpha] = 0, \\
&\hat{R}^+ = \hat{Q}, \ \hat{Q}^+ = \hat{R}\,.
\end{aligned} \tag{2.13}$$

The gauge fields of $shs\alpha(1)$ are of the form

$$\omega_\mu = \sum_{A,B=0,1} (\hat{Q})^A \, (\hat{R})^B \, \omega_\mu^{AB}(q, \tau), \tag{2.14}$$

and for the curvatures one finds *(Fradkin & Vasiliev (1988))*

$$\begin{aligned}
R^{FG}_{\nu\mu,\alpha(n),\dot{\beta}(m)} &= \partial_\nu \omega^{FG}_{\mu,\alpha(n),\dot{\beta}(m)} - \partial_\mu \omega^{FG}_{\nu,\alpha(n),\dot{\beta}(m)} \\
&+ \sum_{p,q,s,k,l,t;A,B,C,D} \delta(|\, F + A + C \,|_2)\, \delta(|\, G + B + D \,|_2)\, \delta(n - p - q) \\
&\times \delta(m - k - l)\, i^{s+t-1} (-1)^{C(p+s)+D(t+k)} \times \frac{n!m!}{p!q!s!k!l!t!} \\
&\times \delta(|\, C(p+s) + D(k+t) + A(q+s) + B(l+t) \\
&+ s + t + (p+k+s+t)\,(q+l+s+t) + 1 \,|_2) \\
&\times \omega^{AB}_{\nu,\alpha(p)\gamma(s),\dot{\beta}(k)\dot{\delta}(t)} \, \omega^{CD}_{\mu,\alpha(q)}{}^{\gamma(s)}{}_{,\dot{\beta}(l)}{}^{\dot{\delta}(t)}\,.
\end{aligned} \tag{2.15}$$

The fields ω^{AB} with $A = B = 0, 1$ describe higher spins and ω^{AB} with $A = 0$, $B = 1$ or $A = 1$, $B = 0$ are auxiliary fields as one can see after the localization of this algebra.

In the superalgebra $shs\alpha(1)$ all spins are contained twice. That such a doubling should indeed take place is seen from the following considerations. A necessary condition for

the existence of a full theory of massless higher-spin fields is the existence of massless unitary particle-like representations of the global superalgebra describing the higher-spin symmetry, which have the same spectrum as gauge fields have. The existence of such representations requires the doubling of the fields of higher spins *(Konstein & Vasiliev (1989), (1990))*. This fact can be illustrated as follows. The anti-de Sitter group 0(3,2) possesses two elementary representations (singletons) discovered by Dirac*. They are denoted by Di and Rac.

The supersingleton $S = Di \oplus Rac$ forms a representation of $osp(1 \mid 4)$ which can be continued to a representation of the higher-spin superalgebra. The square of the supersingleton equals

$$\begin{aligned} S \otimes S &= (Di \otimes Di) \oplus (Rac \otimes Rac) \oplus \\ &\oplus (Di \otimes Rac) \oplus (Rac \otimes Di) \end{aligned} \tag{2.16}$$

and the spin content of these representations is as follows

$$\begin{aligned} (Rac \otimes Rac) &= (0^1, 1, 2, \cdots), \\ (Di \otimes Di) &= (0^2, 1, 2, \cdots), \\ (Di \otimes Rac) &= \left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\right) \end{aligned} \tag{2.17}$$

where 0^1 and 0^2 are masless representations with zero spin: (1,0) and (2,0) respectively. (It is important that the unitary representations contain all lower-order spins including spin 0.) The square $S \otimes S$ contains all massless unitary representations of 0(3,2) twice and can be continued to a unitary representation of the full superalgebra $shs^f(1)$ which contains all higher spins twice and is a subalgebra of $shs\alpha(1)$. On the other hand, the superalgebra $shs(1 \mid 4)$ which contains all spins only once has no unitary representations with the required properties.

The fact of the doubling of all spins raises the question of their interpretation, particularly of spin 2. One possibility is that only some specific combinations of the spin-2

*See also: *Flato M.*, and *Fronsdal C.*, Phys. Lett. **97B** (1980) 236; *Fronsdal C.*, Phys. Rev. **D26** (1982) 1988.

fields will remain massless in a physically relevant phase with spontaneously broken higher spin symmetries, and this combination will be the only candidate for the role of the gravitational field.

To summarize, higher spin symmetries constitute a novel class of fundamental symmetries generalizing the space-time symmetries, somewhat like supersymmetries. The discovery of supersymmetry introduced in physics spin 1/2 supersymmetry generators and spin 3/2 gauge fields. Similar to this, higher spin symmetries introduce into the game an infinite tower of higher spin generators and gauge fields, which are also obligatory for finiteness of the local theory.

2.2. AdS$_4$ Higher Spin Gauge Theory.

Having the infinite-dimensional higher-spin superalgebra, corresponding gauge fields and curvatures at one's disposal and basing on the supergravity experience, one can try to construct an interacting gauge theory for higher spins. One of the elegant approaches to build up an action starting from the known curvatures is the higher spin extension of the $R_A \wedge R_B Q^{AB}$ approach*. This $R \wedge R$ approach was applied first to the AdS$_4$ higher-spin gauge theory based on the higher-spin superalgebras $shs(N \mid 4)$ *(Fradkin & Vasiliev (1987a))*. It turned out that there exists a unique real, P-invariant action which, when supplemented with some suitable curvature constraints, currently describes the free fields of higher spins in AdS$_4$ as well as their cubic interaction. This action is of the form

$$S = \frac{\beta}{2} \sum_{n,m=0}^{\infty} \frac{i^{n+m+1}}{n!m!} \lambda^{-|n-m|} \varepsilon(n-m) \times \\ \times \int d^4x \, \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu,\alpha(n),\dot\beta(m)} R_{\rho\sigma,}{}^{\alpha(n)}{}_{,}{}^{\dot\beta(m)} \tag{2.18}$$

and the additional constraints read

$$\varepsilon^{\nu\mu\rho\sigma} R_{\mu\nu,\alpha(n),\dot\beta(m-1)\dot\delta} h_{\rho,\alpha}{}^{\dot\delta} = 0 \text{ for } n \geq m > 0, \tag{2.19}$$

*This approach was introduced in gravity by *MacDowell S.* and *Mansouri F.*, (1977).

$$\varepsilon^{\nu\mu\rho\sigma}\, R_{\nu\mu,\alpha(n-1)\gamma,\dot\beta(m)}\; h^{\gamma}_{\rho,\dot\beta} \;=\; 0 \text{ for } m \geq n > 0. \tag{2.20}$$

Here $\varepsilon(n-m)$ equals 1 for $n > 0, 0$ for $n = m$ and -1 for $n < m$. The action S contains two independent parameters – the cosmological constant λ and the overall factor β. To arrive at the correctly normalized Einstein (spin-2) action – $(1/4\kappa^2)\int d^4x\sqrt{-g}\; R$ the constant β should be $\beta = -1/16\kappa^2$.

To analyze the action S one may expand it to the point where all fields, except the gravitational field, vanish and

$$h_{\mu\alpha\dot\beta} \;=\; \overset{\circ}{\omega}_{\mu,\alpha,\dot\beta},\; W_{\mu,\alpha(2)} \;=\; \overset{\circ}{\omega}_{\mu,\alpha(2)},\; \bar W_{\mu,\dot\beta(2)} \;=\; \overset{\circ}{\omega}_{\mu,\dot\beta(2)} \tag{2.21}$$

where h and $W, \bar W$ are respectively the vierbein and Lorentz connection of the background AdS_4 space. It can be shown that the fields $\omega(n,m)$ with $\mid n-m \mid > 2$ (the so-called "extra" fields) fall out of the action (more precisely, they enter the total-derivative terms). However, these fields enter the full curvatures of $shs(1 \mid 4)$. This is the reason why the action should be supplemented with the constraints (2.19), (2.20). The constraints and equations of motion make it possible to express the curvature as follows *(Vasiliev (1987))*.

$$\begin{aligned} R^{l}_{\mu\nu,\alpha(n),\dot\beta(m)} &= \delta(n)\; h_{\mu,\gamma}{}^{\dot\delta} h_{\nu,}{}^{\gamma\dot\delta}\; \bar C_{\dot\beta(m)\dot\delta(2)} + \\ &+ \delta(m)\; h^{\gamma}_{\mu\dot\delta}\; h^{\gamma\dot\delta}_{\nu}\; C_{\alpha(n)\gamma(2)} \end{aligned} \tag{2.22}$$

where C and $\bar C$ generalize the gravitational Weyl tensor ($C_{\alpha(4)}$ and $\bar C_{\dot\beta(4)}$) to the case of higher spins. (2.22) generalizes the linearized Einstein equations and expresses all curvatures through the higher-spin Weyl tensors. The Weyl tensors of the AdS_4-space satisfy the equations

$$h^{\rho\gamma}_{\dot\beta}\; \mathcal{D}^{L}_{\rho}\; C_{\alpha(n-1)\gamma} \;=\; 0, \tag{2.23}$$

$$h^{\rho\dot\delta}_{\alpha}\; \mathcal{D}^{L}_{\rho}\; \bar C_{\dot\beta(m-1)\dot\delta} \;=\; 0 \tag{2.24}$$

which follow from (2.22) and the linarized Bianchi identities for the curvatures of $shs(1 \mid 4)$. In (2.23) and (2.24), $\mathcal{D}^L$ is the Lorentz-covariant derivative with the connection $W, \bar W$.

The gauge invariance of the action in the trilinear approximation can be verified as follows. The variation of the action in this approximation has the structure $\int R^l R^l \varepsilon$. By virtue of the linearized equation (2.22), the expression contains only terms of the type CC and $\bar{C}\bar{C}$ (the terms $C\bar{C}$ do not enter δS owing to the identity $\varepsilon^{\mu\nu\rho\sigma} h_{\mu\alpha\dot{\delta}} h_{\nu\alpha}{}^{\dot{\delta}} h_{\rho\gamma\dot{\beta}} h^{\gamma}{}_{\sigma\dot{\beta}} \equiv 0$). The condition for the cancellation of these terms leads to an equation for the bilinear form Q^{AB} in the action $\int R_A \wedge R_B Q^{AB}$, which has the unique solution consistent with the free theory (see *Vasiliev (1987))* of higher spins in AdS_4. This unique solution is given by (2.18). So the above action and constraints provide a consistent description of the interacting collection of massless spins $3/2 \leq s \leq \infty$, in the cubic approximation. Taking into account that the no-go theorem also dealt with just the cubic approximation, one comes to the conclusion that <u>the no-go theorem does not hold well on the non-trivial AdS background</u>. The key point here is the celebrated non-analyticity in Λ, and the existence of non-minimal higher-derivative interaction vertices.

Actually this is a quite non-perturbative result with respect to the flat background, because in this case cosmological terms cannot be treated as perturbations over the trivial flat vacuum. In fact, the anti-de Sitter vacuum and the flat vacuum may be related only by some non-trivial phase transition, rather than a naive flat limit $\Lambda \to 0$, when massless higher spins are presented in the AdS phase.

It is worthwhile to emphasize now that in the AdS higher spin gauge theory there are only two independent coupling constants. These are gravitational κ and cosmological $\Lambda \sim -\lambda^2$ constants. In the N-extended theories in the action (or cubic order) there is also an additional Yang-Mills term integration with a constant e but it is in fact related to the first two $e^2 = 4\lambda^2\kappa^2$ *(Fradkin & Vasiliev (1987b))*.

The same situation with coupling constants takes place also in the AdS_4 extended supergravity and, therefore, the introducing of an infinite tower of higher spin fields does not bring any new essential independent physical parameters in the theory. All higher spin interaction coupling constants turn out to be related with the only fundamental

ones Λ, κ and e by the very powerful infinite-dimensional higher spin symmetry.

This situation is quite similar to string theory with the string tension $T = (2\pi\alpha')^{-1}$ on the place of the cosmological constant Λ. Moreover, the connection of e^2 and κ^2 is quite similar to the heterotic superstring case with the accuracy of the interchange $\Lambda \leftrightarrow T$. So to say, we come back again to our parallel between the AdS radius and the string "length".

Above we have considered the higher spin dynamics in the cubic approximation. The way beyond this approximation requires knowledge of the complete constraints up to all orders, along with the very Lagrangian. It produces some serious difficulties which have not been overcome in this approach up to now.

However, a very important progress has been recently achieved in quite another approach *(Vasiliev (1990), (1991))* where on the basis of superalgebra $shs\alpha(1)$ (2.13) - (2.15) and its conformal extensions consistent equations of motion desribing interacting massless fields of all spins, from zero to infinity, have been constructed in the framework of the so-called free differential algebra.

This system of equations describes higher spin interactions up to all orders and it is gauge invariant under an infinite-dimensional algebra. They have a rather specific form, like some zero-curvature conditions for an enlarged infinite algebra, supplemented with some additional constraints. The problem of corresponding complete Lagrangian is now under investigation.

3. Conformal Higher Spin Theory

Let us turn to the second possibility for a higher spin gauge theory, which is a higher spin ($s > 2$) generalization of usual conformal supergravity.

In a series of papers *(Fradkin & Linetsky (1989a,b) (1990a) (1991a))* a superconformal higher spin gauge theory has been proposed in dimensions $D \leq 4$ (but only for lower dimensions ($D < 4$) the action is constructed exactly).

Below I adduce more details of the main results of our attempt to construct Superconformal Higher Spin Theory in three and four dimensions.

3.1. Conformal Higher Spin Gauge Theory in $D = 2+1$.

For the construction of a conformal higher spin theory in $D = 2+1$ first of all we choose a suitable operatorial realization of the conformal superalgebra in $D = 2+1$, $osp(N \mid 4)$. Let $\hat{a}_\alpha, \hat{b}_\alpha$ and $\hat{\psi}_i$ be the generating elements with non-zero commutators

$$[\hat{a}_\alpha, \hat{b}_\beta] = 2i\varepsilon_{\alpha\beta} \quad , \quad \hat{a}_\alpha^+ = \hat{a}_\alpha \quad , \quad \hat{b}_\alpha^+ = \hat{b}_\alpha \, , \tag{3.1a}$$

$$\{\hat{\psi}_i, \hat{\psi}_j\} = 2\,\delta_{ij} \quad , \quad \hat{\psi}_i^+ = \hat{\psi}_i \tag{3.1b}$$

where the Greek letters $\alpha, \beta, \cdots = 1,2$ are the $so(2,1)$-spinorial indices and $i,j = 1, \cdots, N$ are the internal indices. Then the $osp(N \mid 4)$ generators (translation, $SO(2,1)$, conformal boost, dilatation, $SO(N)$, supersymmetry and special conformal supersymmetry generators) are $P_{\alpha(2)} = \frac{1}{4i}\hat{a}_\alpha\hat{a}_\alpha$, $M_{\alpha(2)} = \frac{1}{4i}(\hat{a}_\alpha\hat{b}_\alpha + \hat{b}_\alpha\hat{a}_\alpha)$, $K_{\alpha(2)} = \frac{1}{4i}\hat{b}_\alpha\hat{b}_\alpha$, $D = \frac{1}{8i}(\hat{a}_\alpha\hat{b}^\alpha + \hat{b}^\alpha\hat{a}_\alpha)$, $T_{ij} = \frac{1}{4}\hat{\psi}_i\hat{\psi}_j$, $Q_{i\alpha} = \frac{1}{2}\hat{a}_\alpha\hat{\psi}_i$ and $S_{i\alpha} = \frac{1}{2}\hat{b}_\alpha\hat{\psi}_i$, respectively. Generators of the infinite-dimensional generalization of the conformal superalgebra $osp(N \mid 4)$ can be chosen in the form of all order polynomials that commute with "particle number" operator $N = a^\alpha b_\alpha$

$$\begin{aligned} T^{(s,c)}_{i(k),\alpha(2l)} &= \left[\frac{(2l+1)!}{(s-l)!(l+c)!(l-c)!(s+l+1)!}\right]^{1/2} \\ &\times\ \psi_{i_1}\cdots\psi_{i_k}\ \underbrace{a_\alpha\cdots a_\alpha}_{l-c}\ \underbrace{b_\alpha\cdots b_\alpha}_{l+c}\,(a^\alpha b_\alpha)^{s-l}\,, \end{aligned} \tag{3.2}$$

where (and below) we have passed to the Weyl symbols of the operators and a, b, ψ are the symbols of the generating elements in (3.1). In (3.2) the indices have the following meaning: s determines the $SO(3,2)$-irreducible representation space with the dimension $d(s) = (2s+3)(2s+1)(s+1)/3$; c is a conformal weight of the generators ($[D, T^c] = cT^c$); l is the $SO(2,1)$-signature; k is a number of the internal indices and

$$s = 0, \frac{1}{2}, 1, \cdots; \quad c = -s, \cdots, s; \quad l = |\,c\,|, \cdots, s; \tag{3.3}$$

$k = 0, 1, \cdots N$ and $k + 2s$ is even.

The last restriction has been introduced for convenience. The generators (3.2) form a conformal basis (D is diagonal) in a conformal higher spin superalgebra which we call $shsc(N \mid 3)$. The structure constants of $shsc(N \mid 3)$ in the conformal basis associated with the reduction to a subalgebra $osp(N \mid 4) \to SO(3,2) \oplus SO(N) \to SO(2.1) \oplus SO(1,1) \oplus SO(N)$. Gauge fields and curvatures corresponding to $shsc(N \mid 3)$ have the form *(Fradkin & Linetsky (1989b), (1990a))*

$$\omega_\mu = \frac{1}{2i} \sum i^{-\varepsilon_s} \, \omega_\mu^{(s,c)i(k),\alpha(2l)} \, T^{(s,c)}_{i(k),\alpha(2l)} \,, \tag{3.4}$$

$$\begin{aligned} R^{(s,c)}_{\mu\nu,i(k),\alpha(2l)} &= \partial_\mu \, \omega^{(s,c)}_{\nu,i(k),\alpha(2l)} - (\mu \leftrightarrow \nu) \\ &+ \sum i^{s'+s''-s+r-|r|_2-1} \, \frac{k!}{u!v!r!} \, \delta(k-u-v) \\ &\times \delta(p - l' - l'' + l) \, \delta(q - l' + l'' - l) \, \delta(t - l'' + l' - l) \\ &\times \pi\big(4s's'' + s' + s'' - s + uv + r(u+v) + 1\big) \\ &\times \begin{pmatrix} s' & s'' & s \\ c' & c'' & c \\ l' & l'' & l \end{pmatrix} \omega^{(s',c')}_{\mu,i(u)j(r),\alpha(q)\gamma(p)} \\ &\times \omega^{(s'',c'')j(r)}_{\nu,i(v)} \,, \; {}^{\gamma(p)}_{\alpha(t)} \,, \end{aligned} \tag{3.5}$$

where
$$\begin{aligned} \begin{pmatrix} s & s' & s'' \\ c & c' & c'' \\ l & l' & l'' \end{pmatrix} &= \delta(c + c' - c'') \, \varepsilon(s, s', s'') \, \varepsilon(l, l', l'') \\ &\times \left[\frac{(2l+1)!(2l'+1)!(2l''+1)!}{(l+l'-l'')!(l-l'+l'')!(l'-l+l'')!(l+l'+l''+1)!} \right]^{1/2} \end{aligned}$$

$$\times \sum_{k,k',k''} (-1)^{1/2(s+s'-s''-k-k'+k'')} \begin{Bmatrix} \frac{s+k}{2}, & \frac{s-k}{2}, & l \\ \frac{s'+k'}{2}, & \frac{s'-k'}{2}, & l' \\ \frac{s''+k''}{2}, & \frac{s''-k''}{2}, & l'' \end{Bmatrix}$$

$$\times \frac{d^{l}_{c,k}\left(\frac{\pi}{2}\right) d^{l'}_{c',k'}\left(\frac{\pi}{2}\right) d^{l''}_{k'',c''}\left(-\frac{\pi}{2}\right)}{\left[\triangle\left(\frac{s+k}{2}, \frac{s'+k'}{2}, \frac{s''+k''}{2}\right) \triangle\left(\frac{s-k}{2}, \frac{s'-k'}{2}, \frac{s''-k''}{2}\right)\right]^{1/2}},$$

$$\varepsilon(a,b,c) = 1(0) \quad \text{for} \quad c \in \{|\, a-b\,|, \cdots a+b\}\ (c \notin \{|\, a-b\,|, \cdots, a+b\}),$$

$$\triangle(a,b,c) = \frac{(a+b-c)!(a-b+c)!(b-a+c)!}{(a+b+b+1)!},$$

$$k = -l, \cdots, l, \quad k' = -l', \cdots, l', \quad k'' = -l'', -l''+1, \cdots, l''$$

$$\pi(n) = 0(1) \quad \text{for } n \text{ even (odd)}. \tag{3.5a}$$

We have expressed the structure coefficients of the *-product in the conformal basis in terms of the 9 j-symbols and particularly values of the Wigner d-functions $d\left(+\frac{\pi}{2}\right)$.

The gauge field ω_μ is the element of the second-class Grassmann shell of $shsc(N \mid 3)$ ($\omega^s_\mu T^{s'} = (-1)^{4ss'} T^{s'} \omega^s_\mu$, Grassmann parity is $\varepsilon(T^s) = \varepsilon(\omega^s) = \varepsilon_s = 0(1)$ for s integer (half)) and the hermiticity condition is read

$$\omega^+_\mu = -\omega_\mu, \ \left(\omega^{(s,c)}_{\mu,i(k),\alpha(2l)}\right)^+ = (-1)^{\frac{k(k-1)}{2}} \omega^{(s,c)}_{\mu,i(k),\alpha(2l)}. \tag{3.6}$$

In three dimensions, the action invariant under the gauge transformations of the algebra $shsc(N \mid 3)$ has the Chern-Simons form *(Fradkin & Linetsky (1990a))*

$$S = \int tr\left(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right), \tag{3.7}$$

where $\omega \wedge \omega = \omega_\mu * \omega_\nu dx^\nu \wedge dx^\nu$ and the *-multiplication of Weyl symbols. This action generalizes the three-dimensional conformal supergravity action. The conformal supergravity fields in our notation are

$$\begin{aligned}&\left(e_{\mu\alpha(2)}, f_{\mu\alpha(2)}, \omega_{\mu\alpha(2)}, b_\mu, A_{\mu ij}, \phi_{\mu i\alpha}, \psi_{\mu i\alpha}\right)\\ &\sim \left(\omega^{(1,-1)}_{\mu\alpha(2)}, \omega^{(1,1)}_{\mu\alpha(2)}, \omega^{(1,0)}_{\mu\alpha(2)}, \omega^{(1,0)}_{\mu}, \omega^{(1,0)}_{\mu ij}, \omega^{(1/2,1/2)}_{\mu i\alpha}, \omega^{(1/2,-1/2)}_{\mu i\alpha}\right).\end{aligned} \tag{3.8}$$

The equations of motion of the conformal higher spin fields in $D = 2+1$ have the form

$$R^{(s,c)}_{\mu\nu,i(k),\alpha(2l)} = 0 . \tag{3.9}$$

These equations generalize the $D = 2+1$ conformal supergravity equations.

For the quantization of Chern-Simons theory (the rang-one theory) we introduce a ghost and antighost fields $C = \sum_A C_A T^A, \bar{C} = \sum_A \bar{C}^A T_A, \varepsilon(C_A) = \varepsilon(\omega_{\mu,A}) + 1$ and a gauge fermion $\Psi = \sum_A \int d^3x \bar{C}^A \Psi_A$ with gauge conditions Ψ_A (A is the collective index in the algebra). Then the exactly generating functional for the three-dimensional superconformal higher spin quantum theory reads as follows

$$\mathcal{Z} = \int \mathcal{D}\omega \mathcal{D}C \mathcal{D}\bar{C} \mathcal{D}\pi \, \exp\{iS_{eff}\} , \tag{3.10}$$

$$\begin{aligned} S_{eff} = S + \int d^3 x tr \Big(& \frac{\delta r \Psi}{\delta \omega_\mu} * \mathcal{D}_\mu C \\ & + \frac{\delta r \Psi}{\delta C} * C * C + \frac{\delta r \Psi}{\delta \bar{C}} * \pi + \mathcal{J}^\mu * \omega_\mu \Big) , \end{aligned} \tag{3.11}$$

where S is the origin Chern-Simons action, π is the Lagrangian multiplier for the gauge conditions, $\frac{\delta \Psi}{\delta \mathcal{Z}} = \sum_A \frac{\delta \Psi}{\delta \mathcal{Z}_A} T_A$ for $\mathcal{Z} = (\omega, C, \bar{C}, \pi)$ and $\mathcal{D}_\mu$ is the covariant derivative.

To conclude this section let us consider the compactification of Chern-Simons theory. Introducing a notation $\phi = \omega_2$ and expanding the gauge fields by coordinate $x^2, \omega_\mu = \sum_{n=-\infty}^{\infty} \omega_\mu(n) e^{in\, r^2/r_c}$, where r_c is a compactification radius, we have an action in $D = 1+1$

$$S = \sum_{n=-\infty}^{\infty} \int tr \left(\phi(n) * R(-n) + \frac{in}{r_c} \omega(-n) \wedge \omega(n) \right) . \tag{3.12}$$

In addition to usual action in $D = 1+1$, the action (3.12) also contains a "topological mass term" $tr \sum_n \frac{in}{r_c} \omega(-n) \wedge \omega(n)$. The corresponding ghost Lagrangian has the form

$$\begin{aligned} \mathcal{L}_{gh} = \sum_{n=-\infty}^{\infty} tr \Big(& \frac{\delta r \Psi}{\delta \omega_\mu(n)} * (\mathcal{D}_\mu C)(-n) \\ & + \frac{in}{r_c} \frac{\delta r \Psi}{\delta \phi(-n)} * C(n) + \frac{\delta r \Psi}{\delta C(n)} * \sum_{m+k=-n} C(m) * C(k) \\ & + \frac{\delta r \Psi}{\delta \phi(-n)} * [\phi, C](n) \Big) . \end{aligned} \tag{3.13}$$

3.2. Conformal Higher Spin Theory in $D = 3 + 1$.

Now let us turn to the second possibility for a higher spin gauge theory in four dimensions as a higher spin extension of conformal supergravity. We start with the conformal supergravity which is a supersymmetry generalization of the Weyl gravity with the quadratical Weyl tensor action.

Similar to the AdS_4 supergravity, conformal supergravity is constructed as a gauge theory for a conformal superalgebra $SU(2,2 \mid N)$ [see rev. of conformal supergravity *(Fradkin & Tzeytlin (1985a))*].

Since we are going to construct higher spin generalizations of the conformal supergravity, a first thing to do is to find a higher spin generalization of the four-dimensional conformal supersymmetry. It was done *(Fradkin & Linetsky (1989a), (1990a))* by the method of the operator realization.

A conformal higher spin superalgebra can be simply constructed by employing quantum (super)twistors: ($\hat{\alpha}$ and $\hat{\alpha}^+$ are Fermi-operators)

$$Z_A = (a^\alpha, a_{\dot{\beta}}, \hat{\alpha}), \;\; \bar{Z}^B = (\bar{a}_\alpha, \bar{a}^{\dot{\beta}}, \hat{\alpha}^+); \; [Z_A, \bar{Z}^B\} = 2\,\delta_A^B . \tag{3.14}$$

Then the conformal superalgebra $SU(2,2 \mid N)$ is realized as an algebera of second-order polynomials that commute with the "particle number" operator $T = \bar{Z}^A Z_A$, i.e. the $SU(2,2 \mid N)$ generators are:

$$T_A^B = \frac{1}{2}\left(\bar{Z}^A Z_B + (-1)^{P_A P_B} Z_B \bar{Z}^A\right) - \frac{1}{4-T}\,\delta_A^B\, T . \tag{3.15}$$

Then an infinite-dimensional conformal higher spin superalgebra $shc^\infty(4 \mid 1)$ is defined as an algebra at all order polynomials that commute with T, excluding powers of T itself.

To construct a gauge theory it is necessary to introduce the basis on $so(4,2)$-irrepses connected with the decomposition $so(4,2) \oplus so(3,1)$, where $so(3,1)$ is the Lorentz subalgebra and $so(1,1)$ generated by dilatation generator D. In this basis all generators and the gauge fields will have the defined Weyl weight and the manifest four-dimensional index Lorentz structure. Such a superconformal basis was constructed by standard group

theory methods and the structure constants of $shsc^{\infty}(4 \mid 1)$ in this basis was obtained (see *Fradkin & Linetsky (1990a), (1991a)*).

As a result, the gauge field corresponding to $shsc^{\infty}(4 \mid 1)$ is a function of supertwistors $(\omega_\mu(z,\bar z \mid x);\ [T,\omega_\mu(z,\bar z \mid x)] = 0)$ and has the form*

$$\omega_\mu = \sum_{N=1}^{\infty} \sum_{s=1}^{N} \sum_{(s,c,u,l,j)} i^{-\varepsilon_s}\, \omega^{(N,s,s,c,u)}_{\mu,\alpha(2l),\dot\beta(2j)} \times T^{(N,s,s,c,u)\,\alpha(2l),\dot\beta(2j)} \tag{3.16}$$

$\varepsilon_s = 0(1)$ for s integer (half-integer).

Here indices of the fields and generators have the following meaning. The index $N = 1,2,\cdots$ is a number of the level $L(N)$. The index $s = 1,2,\cdots,N$ defines $su(2,2 \mid 1)$-irreps $V(s)$ (the gauge fields with the fixes s form the $su(2,2 \mid 1)$ -supermultiplet with maximal spin $s+1$; 1 due to the additional vector index μ). Thus on the N-th level there are n conformal supermultiplets with the maximal spins from $N+1$ to 2. The index $s = s-1,\ s-1/2,\ s$ defines the conformal multiplet (the gauge fields describe the spin $s+1$). The index u is a chiral weight and $u = 0$ for Bosons (s = integer) and $u = \pm 1/2$ for Fermions (s = halfinteger). The index $c = -s,\ -s+1,\cdots,s$ is a conformal weight of the generators and the indices $l,j = 0,1/2,1,\cdots$ define a Lorentz

*Under the $su(2,2 \mid 1)$ representation each level decomposes into the direct sum of $su(2,2 \mid 1)$ irreducible representation spaces (irrepses) $L(N) = \bigoplus_{s=1}^{N} V(s)$. In its turn the irreps $V(s)$ decomposes into the sum of $so(4,2)$ irrepses, $V(s) = D(s,s,0) \oplus D\left(s-\frac{1}{2}, s-\frac{1}{2}, \frac{1}{2}\right) \oplus D\left(s-\frac{1}{2}, s-\frac{1}{2}, -\frac{1}{2}\right) \oplus D\left(s-1, s-1, 0\right)$, where $D(n_1,n_2,n_3)$ is the $so(4,2)$ irreps with the highest weight (n_1,n_2,n_3) under the Cartan subalgebra of $so(4,2)$. Here n_1 is the maximal conformal weight in the representation and $(n_2-n_3)/2$ and $(n_2+n_3)/2$ define a Lorentz signature ((n_2-n_3) and (n_2+n_3) are the numbers of dotted and undotted indices, respectively) of the vector with highest conformal weight. Note that the representations $\left(s-\frac{1}{2}, s-\frac{1}{2}, \pm\frac{1}{2}\right)$ are mutually conjugated under the Weyl reflection (these are usually called chirally or complex conjugated representations). The dimension of the $so(4,2)$ irreps $D(s,s,u)$ is equal to $d(s,s,u) = \frac{1}{12}(2s+3)(s+u+1)(s+u+2)(s-u+1)(s-u+2)$. The first level consists of only one $su(2,2 \mid 1)$ irreps $V(1)$ which is the adjoint representation, and under $so(4,2)$ we have $V(1) = D(1,1,0) \oplus D\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \oplus D\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) \oplus D(0,0,0)$. The basis in these irrepses can be chosen as follows: $D(1,1,0) : \{P,K,M,D\}$, $D\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) : \{S_\alpha, Q_\beta\}$, $D\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) : \{S_\beta, Q_\alpha\}$, $D(0,0,0) : \{U\}$.

signature ($2l$ and $2j$ are the numbers of dotted and undotted indices) and the following restrictions take place

$$l+j \leq s, \; l \geq \left| \frac{c-u}{2} \right|, \; j \geq \left| \frac{c+u}{2} \right|, \tag{3.17}$$

with $l+\frac{c-u}{2}$ and $j+\frac{c+u}{2}$ integers.

It should be mentioned that the structure of the gauge field (2.13) is analogous to the structure of the string field $\Phi[X]$ in the string field theory. There is an infinite tower of levels and there are fields with all spins from maximal ($s = N+1$) to minimal ($s = 1$) on each N-th level. For effective work with the infinite tower of levels one can use the following approximation procedure. The number N should be limited by N_{max} and then $N_{max} \to \infty$.

Note that the gauge fields of usual conformal supergravity ($N = s = 1$) are, in our notations*,

$$\begin{aligned}
&\left(e_{\mu\alpha\dot\beta}, \omega_{\mu\alpha(2)}, \omega_{\mu\dot\beta(2)}, b_\mu, f_{\mu\alpha\dot\beta}, A_\mu, \psi_{\mu\alpha}, \psi_{\mu\dot\beta}, \psi_{\mu\alpha}, \psi_{\mu\dot\beta}\right) \\
&\sim \Big[\omega^{(1,1,1,-1,0)}_{\mu,\alpha,\dot\beta}, \omega^{(1,1,1,0,0)}_{\mu,\alpha(2)}, \omega^{(1,1,1,0,0)}_{\mu,\dot\beta(2)}, \omega^{(1,1,1,0,0)}_{\mu}, \omega^{(1,1,1,1,0)}_{\mu,\alpha,\dot\beta}, \omega^{(1,1,0,0,0)}_{\mu}, \\
&\omega^{(1,1,1/2,-1/2,1/2)}_{\mu,\alpha}, \omega^{(1,1,1/2,-1/2,-1/2)}_{\mu,\dot\beta}, \omega^{(1,1,1/2,1/2,-1/2)}_{\mu,\alpha}, \omega^{(1,1,1/2,1/2,1/2)}_{\mu,\dot\beta}\Big] .
\end{aligned} \tag{3.18}$$

Now we can give explicit formulae for the curvatures of $hsc^\infty(4 \mid 1)$ *(Fradkin & Linetsky (1990a), (1991a))*.

$$\begin{aligned}
R_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu\}, \\
R^A_{\mu\nu} &= \partial_{[\mu} \omega^A_{\nu]} + \int^A_{BC} \omega^B_\mu \, \omega^C_\nu ,
\end{aligned} \tag{3.19}$$

*The physical spin-$(s+1)$-fields in $\omega^{(N\,s\,s\,c\,u)}_{\mu,\alpha,\dot\beta}$ are: $\omega^{(N,s,s,-s0)}_{\mu,\alpha(s),\dot\beta(s)}$ (for integer s); or $\omega^{(N,s,s,-s,1/2)}_{\mu,\alpha(s+1/2),\dot\beta(s-1/2)}$ and $\omega^{(N,s,s,-s,-1/2)}_{\mu,\alpha(s-1/2),\dot\beta(s+1/2)}$ (for half-integers s). These fields generalize the conformal supergravity ones $e_{\mu\alpha\dot\beta}$, $\psi_{\mu\alpha}$, $\psi_{\mu\dot\beta}$, A_μ. The constraints allow to express all other (auxiliary) fields through the physical ones up to a pure gauge part.

$$
\begin{aligned}
R^{(N,s,\mathfrak{s},c,u)}_{\mu\nu,\alpha(2l),\dot\beta(2j)} &= \partial_\mu \omega^{(N,s,\mathfrak{s},c,u)}_{\nu,\alpha(2l),\dot\beta(2l)} - (\mu \leftrightarrow \nu) \\
&+ \sum \delta(c'+c''-c)\,\delta(u'+u''-u)\,\delta(m-l'-l''+l)\,\delta(r-l''+l'-l) \\
&\times \delta(t-l'+l''-l)\,\delta(p-j'-j''+j)\,\delta(q-j''-j+j') \\
&\times \delta(k-j-j'+j'')\,\pi(N+N'+N'')\,i^{j'+j''-j-l} \\
&\times \begin{Bmatrix} N' & s' & \mathfrak{s}' & c' & u' & l' & j' \\ N'' & s'' & \mathfrak{s}'' & c'' & u'' & l'' & j'' \\ N & s & \mathfrak{s} & c & u & l & j \end{Bmatrix} \\
&\times \omega^{(N',s',\mathfrak{s}',c',u')}_{\mu,\alpha(l)\gamma(m),\dot\beta(k)\dot\delta(p)}\ \omega^{(N'',s'',\mathfrak{s}'',c'',u'')\gamma(m)}_{\nu,\alpha(r)\ \ ,\dot\beta(q)\ \ \dot\delta(p)} .
\end{aligned}
\tag{3.20}
$$

Here the parity function $\pi(n) = 0(1)$ for n even (odd). They are expressed through the group-theoretical factors which are well known from angular momentum theory (Clebsch-Gordan coefficients, $9j$-symbols etc.). Some simple symmetry properties of the structure coefficients provide the existence of invariant bilinear form on $shc^\infty(4 \mid 1)$ $(A, B) = tr(A * B)$, where the trace is defined by $tr(A(z,\bar z)) = A(0,0)$ and $A * B$ is the Weyl product of the symbols A and B (see in detail: *Fradkin & Linetsky (1991a)*).

3.3. The Cubic Invariant Action in Superconformal Higher-Spin Theory.

The action based on the conformal higher-spin superalgebra $shsc^\infty(4 \mid 1)$ of the type (3.20) can be written in the form *(Fradkin & Linetsky (1991a))*

$$
\begin{aligned}
A_0 = \frac{-1}{8\alpha^2} \sum_{N=1}^{\infty} \sum_{s=1}^{N} \sum_{s,c,u,n,m} (-1)^{N-s}\, j^{n+m+1}\, \mathcal{E}(n-m) \\
\times \int R^{(N,s,\mathfrak{s},c,u)}_{\alpha(n),\dot\beta(m)} \wedge R^{(N,s,\mathfrak{s},-c,-u)\alpha(n),\dot\beta(m)}
\end{aligned}
\tag{3.21}
$$

and the Yang-Mills term for vector fields $\omega^{(N,1,0,0,0)}_\mu$ reads

$$
A_{YM} = \frac{-1}{4\alpha^2} \sum_{N=1}^{\infty} (-1)^{N-1} \int d^4x\sqrt{-g}\, g^{\mu\rho}\, g^{\nu\sigma}\, R^{(N,1,0,0,0)}_{\mu\nu}\, R^{(N,1,0,0,0)}_{\rho\sigma}. \tag{3.22}
$$

and the following constraints*

$$KR = 0 . \tag{3.22a}$$

The action $A = A_0 + A_{YM}$, does not contain any dimensionful parameters. The metric in (3.22) is defined as

$$\begin{aligned} g_{\mu\nu} &= \frac{1}{2}\, \omega^{(1,1,1,-1,0)}_{\mu,\alpha,\dot\beta}\, \omega^{(1,1,1,-1,0)\alpha,\dot\beta}_{\nu}, \\ g^{\mu\nu} &= \left(g_{\mu\nu}\right)^{-1}, \quad g = \det\left(g_{\mu\nu}\right) . \end{aligned} \tag{3.23}$$

First it was demonstrated *(Fradkin & Linetsky (1991a))* that (3.21)-(3.22a) determined the linearized theory completely. The linearized curvatures $R^l = d\omega + \mathcal{P}\omega$ are constructed with the help of the nilpotent $\hat{P}$ operator, $\mathcal{P}\omega = [P^{\alpha\beta}, \sigma_{\alpha\beta} \wedge \omega]$, acting on the differential forms taking their values in $shs^{\infty}(4 \mid 1)$. With the help of the generalization $\circledast = \star \circ \mathcal{R}$ of the Hodge star $\star$ including Weyl reflection $\mathcal{R}$ (in the $so(4,2)$ representations it changes the sings of c and u), the nilpotent operator $\mathcal{K} = \circledast \mathcal{P} \circledast$ conjugated with $\mathcal{P}$ under some natural scalar product $\int tr(A \wedge \circledast B)$ was introduced. The operators $\mathcal{P}$ and $\mathcal{K}$ converted the sequence of linear spaces of q-forms into the conformal cohomological complex which is analogous to the de Rham complex on the Riemann manifold. The linearized constraints $KR^l = 0$ allow us to express all auxiliary fields through the physical ones up to a pure gauge part. The general solution of these constraints in terms of the curvatures was obtained. It turned out that all curvatures can be expressed through (derivatives on) the Weyl multispinors representing the Weyl tensors and spinor-tensors. The Weyl tensors in this context are non-trivial cohomological classes (harmonic forms) for the conformal cohomological complex. The linearized actions quadratic on the curvatures $R^l \wedge R^l$ both for integer and half-integer spins were brought to the free C^2-form as in the tensor formalism. In this way the equivalence of our geometrical formulation and the usual formulation for free higher spins in the symmetric tensor formalism was established (see *Fradkin & Linetsky (1991a))*.

*Two nilpotent operators P and K, which are very useful on conformal higher spin theory, was introduced by us. The operator P increases the rank and decreases the conformal weight of an arbitrary differential form at unity and anticommutes with the usual exterior differential. The operator K decreases the rank and increases the conformal weight of a differential form at unity.

The (3.21) is a special form of our general extension of MacDowell-Mansouri actions $i\int[R^A\wedge R^B G_{AB} - R^{\bar{A}}\wedge R^{\bar{B}} G_{\bar{A}\bar{B}}$ for higher spins, where $R^A(R^{\bar{A}})$ is the set of curvatures $R^{\Omega}_{\alpha(n),\dot{\beta}(m)}$ with $n > m(n < m)$ (the curvatures R^a with $n = m$ do not enter the action), and $G_{AB}, G_{\bar{A}\bar{B}}$ are the blocks of the invariant bilinear form in the algebra. The cubic invariant action for conformal higher-spin theory (Bose case) then is chosen in the above form. The proof of cubic gauge invariance has the following steps. The general structure of the gauge variation in cubic order is $R^l \wedge R^l \mathcal{E}$. Firstly, due to the symmetry properites of G_{AB} and $G_{\bar{A}\bar{B}}$, the terms $R^A \wedge R^B \mathcal{E}$ and the complex conjugates $R^{\bar{A}} \wedge R^{\bar{B}} \mathcal{E}$ are cancelled separately. Secondly, taking into account the self-duality of $R^{l\bar{A}}$ and the antiself-duality of R^{lA} as follows from the linearized conventional constraints, the terms $R^{lA} \wedge R^{l\bar{B}} \mathcal{E}$ vanish identically. Finally, among the remaining terms $R^{la} \wedge R^A \mathcal{E}$ and $R^{la} \wedge R^{\bar{A}} \mathcal{E}$ only the terms with $R^{la} = R^{l(N,s,s)}_{\alpha(s),\dot{\beta}(s)}$ are non-zero due to the constraints. They must be compensated for by some deformations $\Delta\omega$ in the gauge transformation law for auxiliary fields. Obtaining $\Delta\omega$ from the requirement $\delta_g A + \Delta\omega = 0$, one should verify that these deformations are compatible with the conventional constraints. In this way we find the second-order constraints for the part of auxiliary fields (only for those which get the deformations $\Delta\omega$). The action found in such a way is unique.

In the superconformal case the Yang-Mills term for the vector fields must be added along with the above-considered action. The proof of invariance here is analogous to the purely bosonic case with some technical complications stipulated by the increased number of terms (more details see: *Fradkin & Linetsky (1991a)).*

Let us briefly sum up the main results and point out a number of problems that need further study in conformal higher spin theory.

We have shown that there exists a gauge invariant cubic interaction among bosonic and fermionic conformal higher-spin fields incorporating conformal supergravity. This result opens up the possibility of constructing a self-consistent interacting conformally invariant higher-spin theory. Together with previous results about higher-spin interaction in AdS_4, it gives hope to solve the longstanding higher-spin problem which would be

a considerable step towards a unified theory. It seems natural that in our construction there is an infinite number of fields of each spin. It is completely analogous to string field theory. Each level contains all spins from maximal to minimal (spin 1). Such a structure of levels also looks natural from the point of view of spontaneous symmetry breaking. Only the first level (spins ≤ 2) might remain massless; the other higher levels should become massive.

However regarding the infinite multiplicity of spins in the gauge invariant conformal higher-spin interacting theory one should keep the following circumstances in mind. Right from the start we have dealt with the superalgebra $shs^{\infty}(4 \mid 1)$. In principle it is not impossible for the invariant interaction to be based on another superalgebra containing each spin with a finite multiplicity. We constructed *(Fradkin & Linetsky (1990a))* a whole family of such superalgebras $shsc_{\rho}^{(n)}(4 \mid N)$, where $n = 1, 2, \cdots$ is the multiplicity of $SU(2,2 \mid N)$-supermultiplet with the fixed maximal spin in the algebra and $\rho \in \mathbb{R}$ some numerical parameter. They are factor-algebras of the original superalgebra $shsc^{\infty}(4 \mid 1)$. However all those superalgebras seemingly may not be localized. To build a cubic gauge invariant interaction it is necessary that some invariant bilinear form exists on the algebra. But $shsc_{\rho}^{(n)}(4 \mid 1)$ apparently does not possess any invariant bilinear form (the structure constants of factor-algebras have no simple symmetry properties which differ from $shsc^{\infty}(4 \mid 1)$). Meanwhile it cannot be excluded that any such superalgebras [e.g. certain factor-algebras of the universal enveloping algebra $U(so(4,2))$] with a finite multiplicity of spins exist.

There are a number of problems that require a further study. The first is to expand the construction presented here to all orders in the interaction; in particular, to find a non-linear version of the standard constraints. Another problem is to construct N-extended theories. These theories may be based on the N-extended conformal higher-spin superalgebras $shsc^{\infty}(4 \mid N)$ constructed by us *(Fradkin & Linetsky (1990a))*. To transfer the theory presented here to higher dimensions and the extensions with non-trivial (in general an operator) central terms as well as certain non-linear terms in the

right side of this supercommutator in the higher spin symmetry are also important tasks.

4. Conclusion

Now it is time to summarize everything. Thus, there are just three principal possibilities.

- Supergravity (without cosmological term) and an infinite tower of massivly higher spins (superstrings, or string-like models);
- AdS supergravity and an infinite tower of massless higher spin fields with an infinite-dimensional higher spin gauge symmetry (AdS higher spin theory);
- Conformal supergravity and an infinite tower of conformal higher spin fields with infinite-dimensional conformal higher spin symmetry (conformal higher spin theory).

These three seem drastically different. However, a synthesis could be conjectured, when with the increase of temperature there is a phase transition in strings. Then a higher spin symmetry is restored, higher spin excitations become massless and a cosmological constant of the Planck order is induced.

However, there is still a dimensionful parameter, a cosmological constant, in the theory. Meanwhile any quantum theory in the ultra high energies domain may be described by some scale-free conformally invariant theory. Conformal higher spin theory might play a role of such an asymptotical theory.

Summing-up the above expounded arguments, the following scenario may be suggested. In the ultra high-energy domain a unified theory is effectively described as a conformal higher-spin theory generalizing Weyl gravity. The spontaneous conformal symmetry breaking leads to the massless higher-spin theory in the anti-de Sitter universe generalizing the AdS supergravity. Further the AdS higher-spin symmetry

breaking leads to the string-like phases in AdS transitory asymptotically to the massivly higher-spin string phase coupled to the Einstein gravity on the flat background with zero cosmological constant. The above scenario is schematically illustrated in *fig. 1.*

It should be mentioned that one can look at the above scenario in two different ways. Firstly, it may be treated in a straightforward way as a scenario for the fundamental unified theory, i.e. the unified hypothetical lagrangian, or its spontaneously broken versions,

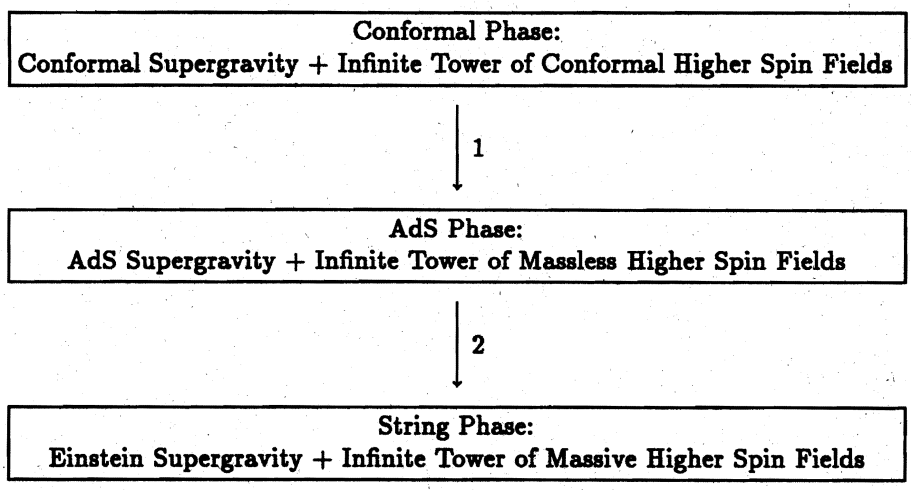

FIGURE 1: Phases of Unified Theory. The arrows 1,2 denote the symmetry breakings (see in the text).

actually describes this picture at all energies. Then, apparently, the scenario should be played in high dimensions. Secondly, it may be considered as a hierarchy of effective theories, each working effectively only in its own energy domain; and their lagrangians, generally speaking, are not connected in a straightforward way with each other. Subsequently it also might be considered in four dimensions.

To have such a scenario of phase transitions it is necessary first to demonstrate a) that a critical AdS string (superstring) model does exist not only in 26(10) dimensions; b) that an AdS higher spin theory exists in the same space-time dimension with the same degrees of freedom as in AdS string model. Such a family of critical strings in $D < 26(D < 10)$ described by anti-de Sitter non-compact caset models $SO(D -$

$1,2)/SO(D-1,1)$ (euclidian version $SO(D,1)/SO(D)$) and their (Kazama-Suzuki) superextension has been recently introduced *(Bars & Nemeshansky (1991); Fradkin & Linetsky (1991c))*.

It was demonstrated that these models are free of conformal anomaly in any $D < 26$ and the anomaly cancellation condition actually fixes the value of the "effective radius of the universe" at a given D. The crucial features of these vacuum solutions in string theory are: **1)** the space-time is non-flat and **2)** there is a space-time dependence of the background fields (metric, dilaton and, generally, also antisymmetric tensor field). For $D = 2$ such a calculation has been performed by *(Elitzur, Forge & Rabinovici (1990); Witten (1991); Dijkgraaf, H. Verlinde & E. Verlinde (1991); Tseytlin (1991))*. For $D = 3$ by *(Fradkin & Linetsky (1991); Crescimanno (1991); Bars & Sfetsos (1991); Chamseddine (1991))*. A preliminary cosmological interpretation of these results is proposed. Roughly, string theory seems to admit non-trivial cosmological (time-dependent) background solution where the universe created as an anti-de Sitter universe (after the phase transition from the higher spin phase to the string phase in AdS) and then under the influence of the fundamental fields, which acquires non-zero background value (dilaton, metric, Higgs particle, etc.) increasing in time, it begins to expand and the higher spin gauge fields become (generally step by step) massive. Let us note, that non-perturbative effects leading to the appearance of a non-trivial dilaton potential (with a mass term) are very essential.

In the case, when a selfconsistence theory exists only in $D > 4$, it is natural to expect a Kaluza-Klein version, when in string phase by the non-static solutions of the background, only three-space dimensions have the effective expansion (inflation) by time, which brings us to an asymptotical flat four space-time. (The remaining dimensions can contract or oscillate with a radius of Planck order.)

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APPENDIX

We adopt here the notations and conventions. The Greek indices $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$ are the indices of components of differential forms. The flat Minkowsky metric is $\eta_{\mu\nu}(+,-,-,-)$. The two-component dotted and undotted spinorial indices $\dot{\alpha}, \dot{\beta}, \cdots$; $\alpha, \beta, \cdots$ take on the values 1 and 2. They are raised and lowered by means of the symplectic metric $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}, \varepsilon_{12} = \varepsilon^{12} = 1$ as

$$A^{\alpha} = \varepsilon^{\alpha\beta} A_{\beta}, \qquad A_{\alpha} = \varepsilon_{\beta\alpha} A^{\beta} \tag{A.1}$$

and analogously for dotted indices.

A symmetrization is implied separately for any set of upper or lower dotted or undotted spinorial indices denoted by the same letters. The ususal summation convention is understood for each pair of a lower and upper index denoted by the same letter. The number of indices is indicated in parantheses (except for a single index). After the symmetrization with the indices is carried out, the maximal possible number of upper and lower indices denoted by the same letter should be contracted. For instance,

$$A_{\alpha(n)} = \frac{1}{n!}\left(A_{\alpha_1\cdots\alpha_n} + (n!-1) \text{ permutations of } \alpha_1\cdots\alpha_n\right), \tag{A.2}$$

$$C_{\alpha(n-m)} = A_{\alpha(n)} B^{\alpha(m)} = A_{(\beta_1\cdots\beta_m\alpha_{m+1}\cdots\alpha_n)} B^{\beta_1\cdots\beta_m}, \; n \geq m, \tag{A.3}$$

where brackets denote full symmetrization. $A_{\mu} = \sigma^{\alpha\dot{\rho}} A_{\alpha\dot{\beta}}$; $A_{\alpha\dot{\beta}} = \sigma^{\mu}_{\alpha\dot{\beta}} A_{\mu}$.

The flat vierbein is

$$\sigma_{\mu}^{\alpha\dot{\beta}} = (I, \sigma_1, \sigma_2, \sigma_3)^{\alpha\dot{\beta}}, \tag{A.4}$$

where I is the unit matrix and $\sigma_{1,2,3}$ are the Pauli matrices. The flat vierbein satisfies the following properties:

$$\sigma_{\mu\alpha\dot{\beta}}\, \sigma^{\mu\gamma\dot{\delta}} = 2\delta^{\gamma}_{\alpha}\, \delta^{\dot{\delta}}_{\dot{\beta}}, \qquad \sigma_{\mu\alpha\dot{\beta}}\, \sigma^{\nu\alpha\dot{\beta}} = 2\delta^{\nu}_{\mu}\,. \tag{A.5),(A.6}$$

The vierbein one-form is

$$\sigma_{\alpha\dot{\beta}} = \sigma_{\mu\alpha\dot{\beta}}\, dx^{\mu}, \tag{A.7}$$

and for its exterior product we have

$$2\sigma^{\alpha\dot{\beta}} \wedge \sigma^{\gamma\dot{\delta}} = \varepsilon^{\alpha\gamma}\, \bar{\sigma}^{\dot{\beta}\dot{\delta}} + \varepsilon^{\dot{\beta}\dot{\delta}}\, \sigma^{\alpha\gamma}, \tag{A.8}$$

where

$$\sigma^{\alpha\gamma} = \sigma^{\alpha\gamma}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}, \qquad \bar{\sigma}^{\dot{\beta}\dot{\delta}} = \bar{\sigma}^{\dot{\beta}\dot{\delta}}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} \tag{A.9}$$

and

$$\sigma_{\mu\nu}{}^{\alpha\gamma} = \sigma_{\mu}{}^{\alpha}{}_{\dot{\beta}}\, \sigma_{\nu}{}^{\gamma\dot{\beta}}, \qquad \bar{\sigma}_{\mu\nu}{}^{\dot{\beta}\dot{\delta}} = \overline{(\sigma_{\mu\nu}{}^{\beta\delta})} = \sigma_{\mu\gamma}{}^{\dot{\beta}}\, \sigma_{\nu}{}^{\gamma\delta}. \tag{A.10}$$

The two-forms $\sigma^{\alpha(2)}$ and $\bar{\sigma}^{\beta(2)}$ ($\sigma^{\alpha\gamma} = \sigma^{\gamma\alpha}$, $\bar{\sigma}^{\dot{\beta}\dot{\delta}} = \bar{\sigma}^{\dot{\delta}\dot{\beta}}$) are antiself-dual and self-dual respectively,

$$\star\, \sigma^{\alpha(2)} = -i\, \sigma^{\alpha(2)}, \qquad \star\, \bar{\sigma}^{\dot{\beta}(2)} = \subset\, \bar{\sigma}^{\dot{\beta}(2)}, \tag{A.11}$$

or in the components

$$\frac{1}{2}\, \varepsilon^{\mu\nu\rho\sigma}\, \sigma_{\rho\sigma}{}^{\alpha(2)} = -i\, \sigma^{\mu\nu\alpha(2)}, \tag{A.12a}$$

$$\frac{1}{2}\, \varepsilon^{\mu\nu\rho\sigma}\, \bar{\sigma}_{\rho\sigma}{}^{\dot{\beta}(2)} = i\, \bar{\sigma}^{\mu\nu\dot{\beta}(2)}. \tag{A.12b}$$

The quantities $\sigma_{\mu\nu}{}^{\alpha(2)}$ and $\bar{\sigma}_{\mu\nu}{}^{\dot{\beta}(2)}$ are, as a matter of fact, projectors on the antiself-dual part and self-dual part of the arbitrary two-form R ('t Hooft tensors)

$$R_{\mu\nu} = \frac{1}{4}\, \sigma_{\mu\nu}{}^{\alpha(2)}\, R_{\alpha(2)} + \frac{1}{4}\, \sigma^{\dot{\beta}(2)}_{\mu\nu}\, \bar{R}_{\dot{\beta}(2)}, \tag{A.13}$$

$$R_{\alpha(2)} = \frac{1}{4}\, i\, \varepsilon^{\mu\nu\rho\sigma}\, \sigma_{\mu\nu\alpha(2)}\, R_{\rho\sigma}, \qquad \bar{R}_{\dot{\beta}(2)} = \overline{(R_{\beta(2)})}. \tag{A.14}$$

Evidently, the following relations hold:

$$\int R \wedge R = i \int d^4x \left(\bar{R}_{\dot{\beta}(2)}\, \bar{R}^{\dot{\beta}(2)} - R_{\alpha(2)}\, R^{\alpha(2)}\right), \tag{A.15}$$

$$\int R \wedge \star R = -\int d^4x \left(R_{\alpha(2)}\, R^{\alpha(2)} + \bar{R}_{\dot{\beta}(2)}\, \bar{R}^{\dot{\beta}(2)}\right), \tag{A.16}$$

due to the identities

$$\sigma^{\alpha(2)} \wedge \sigma_{\gamma(2)} = -16\, i\, \delta^{\alpha}_{\gamma}\, \delta^{\alpha}_{\gamma}\, d^4x, \tag{A.17a}$$

$$\bar{\sigma}^{\dot{\beta}(2)} \wedge \bar{\sigma}_{\dot{\delta}(2)} = 16\, i\, \delta^{\dot{\beta}}_{\dot{\delta}}\, \delta^{\dot{\beta}}_{\dot{\delta}}\, d^4x, \tag{A.18a}$$

$$\sigma_{\alpha(2)} \wedge \bar{\sigma}_{\dot{\beta}(2)} \equiv 0, \tag{A.19a}$$

or in the components

$$\varepsilon^{\mu\nu\rho\sigma}\, \sigma_{\mu\nu}{}^{\alpha(2)}\, \sigma_{\rho\sigma\gamma(2)} = -16\, i\, \delta^{\alpha}_{\gamma}\delta^{\alpha}_{\gamma}, \tag{A.17b}$$

$$\varepsilon^{\mu\nu\rho\sigma}\, \bar{\sigma}_{\mu\nu}{}^{\dot{\beta}(2)}\, \bar{\sigma}_{\rho\sigma\dot{\delta}(2)} = 16\, i\, \delta^{\dot{\beta}}_{\dot{\delta}}\delta^{\dot{\beta}}_{\dot{\delta}}, \tag{A.18b}$$

$$\varepsilon^{\mu\nu\rho\sigma}\, \sigma_{\mu\nu}{}^{\alpha(2)}\, \bar{\sigma}_{\rho\sigma}{}^{\dot{\beta}(2)} \equiv 0. \tag{A.19b}$$

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