ANALYTICAL METHOD FOR TREATMENT OF NONLINEAR RESONANCES IN ACCELERATORS *

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I. Introduction

We have developed an analytical method for investigating the nonlinear effects on the dynamical systems such as circular accelerators including the perturbation of tune; emittance growth; Hamiltonian resonance strength; generating function resonance strengths; fixed points; Chirikov criteria; island width, etc. We have incorporated some of our results [1-3] into an algorithm which we used to study the structure resonances in the Booster [2], RHIC, and the SSC [3]. A brief theoretical discussion is given in Section II. In Section III, using program NONLIN the structure resonances of the SSC CDR clustered lattice is analyzed and some of the results are tabulated. Using the alternate program HARMON we obtained some of the results for the SSC (given in [4]). Comparison of the results obtained from the programs NONLIN and HARMON is given in Reference [5].

II. Theory

A circular accelerator can be described by the following Hamiltonian (in terms of the action angle variables):

$$H_1 = \frac{J_x}{\beta_x} + \frac{J_z}{\beta_z} + V\,(J_x,\ J_z,\ \phi_x,\ \phi_z,\ s) \qquad (1)$$

where J_x, J_z are the actions; ϕ_x, ϕ_z are the angles; β_x, β_z are the betatron functions and s the length particle travels, is the "time" variable of the Hamiltonian. The nonlinear effects such as those from sextupoles, octupoles, etc. are included in $V(J_x,\ J_z,\ \phi_x,\ \phi_z,\ s)$ of Eq. (1).

In order to study the Hamiltonian in equation (1) we seek a canonical transformation with generating function (F) that eliminates the nonlinear terms in $V\,(J_x,\ J_z,\ \phi_x,\ \phi_z,\ s)$

$$F\,(K_x,\ K_z,\ \phi_x,\ \phi_z,\ s) = K_x\phi_x + K_z\phi_z + \sum_k \frac{R_k\,(K_x,\ K_z,\ s)}{\mathrm{Sin}\ \pi\left(n_{x_k}\nu_x + n_{z_k}\nu_z\right)} \mathrm{Cos}(n_{x_k}\phi_x + n_{z_k}\phi_z + \xi_k) \qquad (2)$$

Where $R_k\,(K_x,\ K_z,\ s)$ are the generating function resresonant strengths whose magnitude shown to what extent J_x and J_z deviate from the invariants of the motion. The n_{x_k} and n_{z_k} are integers (defining a given resonant); K_x and K_z are the new action variables and ξ_k is the phase.

With $J_x = \frac{\partial F}{\partial \phi_x}$, $J_z = \frac{\partial F}{\partial \phi_z}$ and F the generating function (given by eq. (2)), the transformed Hamiltonian become:

$$H_2 = H_1 + \frac{\partial}{\partial s} F\,(K_x,\ K_z,\ \phi_x,\ \phi_z,\ s)$$

for example:

$$H_2 = \frac{K_x}{\beta_x} + \frac{K_z}{\beta_z} + a(s)\,K_x^2 + b(s)\,K_xK_z + c(s)\,K_z^2 + \ldots \qquad (3)$$

where a(s), b(s), c(s),... are the stabilizing coefficients given in Reference [1]. From this Hamiltonian perturbation to the betatron tunes can be found as

$$\nu_x = \nu_{0_x} + 2\alpha_{xx}K_x + 2\alpha_{xz}K_z + \ldots.$$
$$\nu_z = \nu_{0_z} + 2\alpha_{xz}K_x + 2\alpha_{zz}K_z + \ldots. \qquad (4)$$

where ν_{0_x} and ν_{0_z} are the unperturnbed tunes and $2K_x$, $2K_z$ are equal to the beam emittance divided by π just before the beam enters the accelerator.

Additionally, we find the emittance growth to be

$$E_x \leq 2\pi K_x + \sum_k n_{x_k} \frac{R_k\,(K_x, K_z, s)}{\mathrm{Sin}\pi(n_{x_k}\nu_x + n_{z_k}\nu_z)}$$
$$E_z \leq 2\pi K_z + \sum_k n_{z_k} \frac{R_k\,(K_x, K_z, s)}{\mathrm{Sin}\pi(n_{x_k}\nu_x + n_{z_k}\nu_z)} \qquad (5)$$

These estimate the upper limit that emittance grow to as long as the tunes are far from any resonances.

To study the nonlinear effects when near a resonance we isolate the resonance as we find the fixed points of the dynamical system [that is, the distance from resonance at which there is no motion in a special reference frame where the Hamiltonian is an invariant,] by defining the bandwidth $\delta \equiv n_x\nu_x + n_z\nu_z - P$ that determines how far the tunes ν_x and ν_z are from the resonance (defined by the integers n_x, n_z and P). Then the system will be on a fixed point if

$$\delta = \pm \frac{1}{2\pi\, h(K_x,K_z)} \left[n_x \frac{\partial h(K_x,K_z)}{\partial K_x} + n_z \frac{\partial h(K_x,K_z)}{\partial K_z} \right]$$

at actions equal to K_x and K_z, where $h(K_x,K_z)$ is the Hamiltonian resonance strength.

Finally, we can determine whether a nearby resonance is important to the dynamics of the system if we are on a resonance (e.g. Chirikov Criterion). That is if the bandwidth δ of a nearby resonance satisfies

$$\delta >> \frac{4}{C} \sqrt{2\pi\left(n_x \frac{\partial T(K_x,K_z)}{\partial K_x} + n_z \frac{\partial T(K_x,K_z)}{\partial K_z}\right) h(K_x,K_z)}$$

(where T is the term in the Hamiltonian that causes the perturbation of tune) then this resonance can be neglected. Otherwise, to describe the behavior of the system properly the resonance (can not be neglected and) must be included. Detailed treatment of the stopbandwidth [1,6] and island widths are given in Reference [1].

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IV. SSC CDR CLUSTER LATTICE

The SSC has one superperiod with an operating point (O) at $\nu_x = 78.265$ and $\nu_z = 78.280$. This operating point is near at least 4 resonances of up to the 4th order (see Fig. 1 and Tables);

$$4\nu_z = 313 \qquad (2A)$$
$$2\nu_x - 2\nu_z = 0 \qquad (7B)$$
$$2\nu_x + 2\nu_z = 313 \qquad (9A)$$
$$4\nu_x = 313 \qquad (13A)$$

plus $\nu_x + 3\nu_z = 313$ and $3\nu_x + \nu_z = 313$ (due to Skew terms). In addition to these resonances (2A, 7B, 9A, 13A calculated using NONLIN) there are higher order resonances that may be important (some of which we have deduced and can analyze using other methods such as KAM theorem [8] and are shown in Figs. (2-4). For example, 11th order resonance $5\nu_x + 6\nu_z=861$ (shown in Fig. 2), 12th order resonance $8\nu_x - 4\nu_z=313$ (shown in Fig. 3), etc.

To study these resonances we have used the SSC CDR cluster lattice with only chromaticity sextupoles [9]. We calculated the resonance strengths; the fixed points; stop bandwidths; island widths and Chirikov criterion. The perturbation to tune and the emittance growth are also calculated. Some of our results are shown in the following Tables.

Table I gives the emittance growth and the perturbation to the betatron tune at the operating point of the SSC. These values were calculated for the average beam emittance of $E_O=2.49\times10^{-7}\pi$ m-rad, (corresponding to the expected beam size of 9.1 mm), which is larger than the expected beam emittance of $9.37\times10^{-10}\pi$ m-rad [that we obtained $EO = \frac{E_N}{\beta\gamma}$ from the normalized emittance, $E_N=10^{-6}\pi$ m-rad, at injection given in SSC-Design Manual]. This value of initial emittance was chosen because it was close to the estimated value of dynamic aperture given in the SSC-Design Manual. Additionally, the bandwidths from these resonances (e) are calculated and some are given in Table III.

Due to the space limitation the full result for this lattice ($\beta^*=.5$) as well as the result obtained for the SSC-CDR clustered lattice $\beta^*=6$ is given in Reference [3].

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Table I

CDR Lattice for SSC $\beta^* = .5$

Perturbation of tunes

$\nu_x = \nu_x0 + -2659.335 * Ex + -9080.386 * Ez$
$\nu_y = \nu_z0 + -9080.386 * Ex + -1773.600 * Ez$

where the unperturbed betatron tunes are $\nu_x0 = 78.26497$ and $\nu_z0 = 78.27995$ with circumference = 82944.00 and periodicity = 1

Given a beam with emittances $E_x = 2.4939999E-07$ and $E_z = 2.4939999E-07$ (π m-rad), the perturbed tunes become

$\nu_x = 78.26204$ and $\nu_z = 78.27724$

the emittance can grow to
$E_xmax = 3.5259643E-07$ at the 331th element and
$E_zmax = 3.4869160E-07$ at the 331th element.

The resonances are numbered as follows:

No.	Resonance	Fix pts.	Strength Width	Stop Bandwidth Chirikov cr.
2A	$0\ \nu_x+4\ \nu_z=313$	5.9411E-07	1.7570E-14 3.9343E-10	1.1272E-06 6.7660E-09
7B	$2\ \nu_x-2\ \nu_z=0$	4.8323E-03	3.3390E-10 7.7979E-08	5.3553E-03 1.2975E-06
9A	$2\ \nu_x+2\ \nu_z=313$	8.3701E-07	2.6094E-14 5.3733E-10	8.3701E-07 1.4715E-08
13A	$4\ \nu_x+0\ \nu_z=313$	6.2142E-07	1.1546E-14 2.6046E-10	7.4075E-07 6.7162E-09

Table II

CDR Lattice for SSC $\beta^* = .5$

The generating function resonance strength for the resonances:

Elements	Resonance Numbers		
	1	2	3
331	2.59284E-11	4.83069E-12	1.94224E-09
	4	5	6
	5.27804E-09	4.99578E-10	1.43731E-11
	7	8	9
	4.16080E-09	5.07310E-11	3.12842E-12
	10	11	12
	8.33345E-10	1.90890E-10	2.51736E-12
	13	14	15
	1.36763E-12	5.30889E-12	1.69158E-10

** The resonance numbers in the above Table correspond to the resonance numbers given in the Hamiltonian Table (Table I).

Table III

The resonance lines closest to the operating tunes (see Fig. 1).

	Resonance	e
2A	$4\nu_z=313.11996$	0.11996
7B	$2\nu_x-2\nu_z=-0.03006$	-0.03006
9A	$2\nu_x+2\nu_z=313.0899$	0.0899
13A	$4\nu_x=313.05984$	0.05984
*	$8\nu_x-4\nu_z=312.99972$	-0.00028

The effect of $8\nu_x-4\nu_z=313$ resonance also should be considered although (since it is a difference resonance) its effect may not lead to instability.

Fig. 1 SSC-CDR Lattice O(78.265, 78.280)

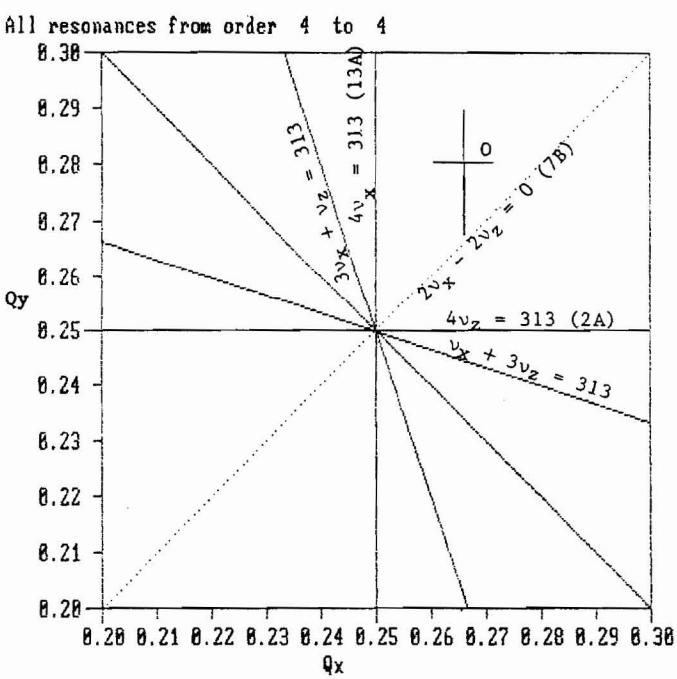

Fig. 2 SSC-CDR Clustered Lattice O(78.265, 78.280)

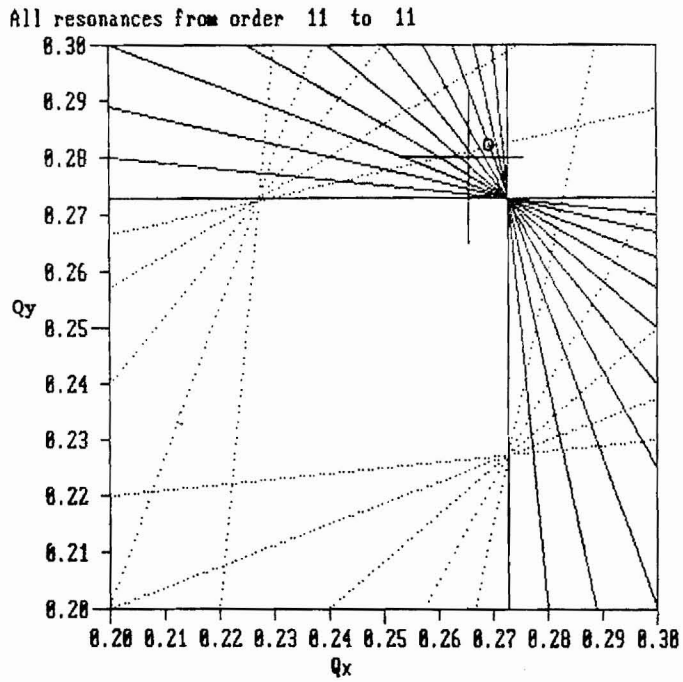
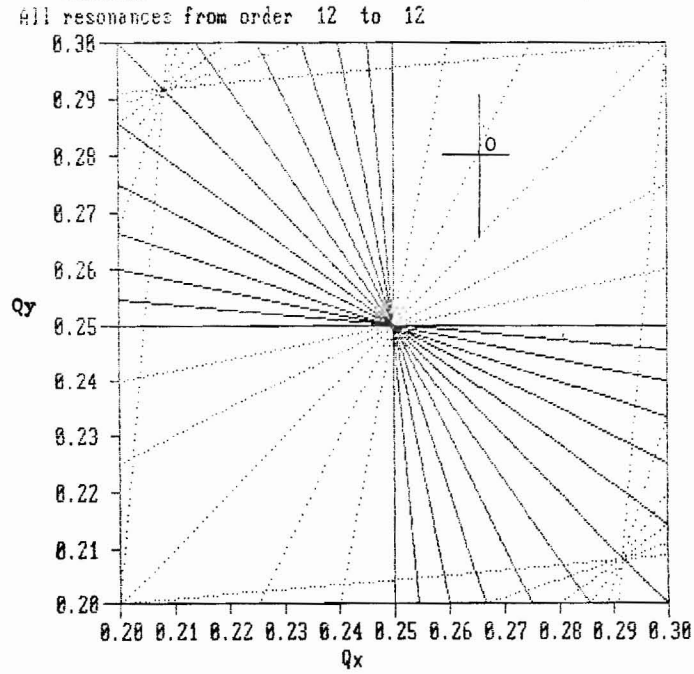
Fig. 3 SSC-CDR Clustered Lattice O(78.265, 78.280)