CHANGE IN THE PARTICLE ACTION DUE TO MAGNETIC FIELD PERTURBATIONS IN THE SSC

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Introduction

The motion of a particle in the linear lattice of the SSC is stable, that is, the Courant–Synder[1] invariant (W_0) remains exactly constant and the betatron phase (ψ) increase by ($2\pi\nu$) for every revolution around the circumference. However, when magnetic field perturbations are present around the ring, this mode of describing the motion is no longer valid. One can then still speak of action–angle variables (W, ϕ) which in the limit of zero perturbation reduce to the linear betatron invariant and the betatron phase. The change in phase ($\phi - \psi$) is then directly related to the perturbed betatron tune, while the change in action ($W - W_0$) is closely related to the idea of "smear" or "change in the linear invariant", as employed by the Central Design Group.[2,3] In this paper, consistent with these concepts, we develop the theory of particle motion for field perturbations around the machine. We confine ourselves to motion in one plane only and treat the field perturbations to first order in their strength. Based on the assumed random multipoles in the SSC dipole magnets, we then obtain an estimate of the "linear aperture" of the machine.

Theory of Perturbed Motion

The Hamiltonian for the perturbed motion of a particle is

$$H(W,\phi) = W + \sum_{n=1} \frac{2P_n}{n+1} W^{(n+1)/2} (\cos\phi)^{n+1}, \qquad (1)$$

where the perturbation function P_n around the ring is described as a function of the magnetic multipole normal coefficient b_n, the unperturbed machine beta function β, and the radius of curvature ρ. Thus for the location specified by the unperturbed phase ψ, we have

$$P_n(\psi) = \frac{b_n}{\rho} \beta^{(n+3)/2}. \qquad (2)$$

Applying Hamilton's equations

$$\frac{\partial H}{\partial W} = \dot{\phi}, \qquad \frac{\partial H}{\partial \phi} = -\dot{W}, \qquad (3)$$

yields the equations of motion

$$\dot{\phi} = \frac{d\phi}{d\psi} = 1 + \sum_{n=1} P_n W^{(n-1)/2} (\cos\phi)^{n+1}, \qquad (4)$$

and

$$\dot{W} = \frac{dW}{d\psi} = \sum_{n=1} 2P_n W^{(n+1)/2} (\cos\phi)^n \sin\phi. \qquad (5)$$

We note, as emphasized in Eqs. (4) and (5), that the independent variable is the betatron phase ψ of the unperturbed motion. The solutions of these last equations can now be put in integral form

$$\phi = \psi + \int_0^{\psi} d\tau \sum_{n=1} P_n(\tau) W^{(n-1)/2}(\tau) [\cos\phi(\tau)]^{n+1}, \qquad (6)$$

and

$$W = W_0 + \int_0^{\psi} d\tau \sum_{n=1} 2P_n(\tau) W^{(n+1)/2}(\tau) \times [\cos\phi(\tau)]^n \sin\phi(\tau). \qquad (7)$$

In this paper we shall fix our attention on the particle action, Eq. (7), and treat it to first order in the perturbing functions P_n. Accordingly the angle variable in the integral becomes the unperturbed phase and the action becomes the unperturbed action. Consequently we can write for the fractional change in the action

$$\frac{\Delta W}{W_0} = \frac{W - W_0}{W_0} = \sum_{n=1} W_0^{(n-1)/2} \int_0^{\psi} d\tau 2P_n(\tau) (\cos\tau)^n \sin\tau. \qquad (8)$$

Considering the motion over ℓ revolutions of the particle, we now have

$$\frac{\Delta W}{W_0} = \sum_{n=1} W_0^{(n-1)/2} \int_0^{2\pi\nu\ell} d\tau 2P_n(\tau) (\cos\tau)^n \sin\tau, \qquad (9)$$

or alternatively

$$\frac{\Delta W}{W_0} = \sum_{n=1} W_0^{(n-1)/2} \sum_{j=1}^{\ell} \int_{2\pi\nu(j-1)}^{2\pi\nu j} d\tau \times 2P_n(\tau) (\cos\tau)^n \sin\tau. \qquad (10)$$

Since the perturbation functions $P_n(\tau)$ are periodic with period ($2\pi\nu$), one can, after some manipulation, express this last relationship as

$$\frac{\Delta W}{W_0} = \sum_{n=1} \frac{W_0^{(n-1)/2}}{2^{n-1}} \sum_{k=0}^{n} {}^nC_k \frac{\sin(n+1-2k)\pi\nu\ell}{\sin(n+1-2k)\pi\nu} \times \int_0^{2\pi\nu} d\tau P_n(\tau) \sin[(n+1-2k)(\tau - \pi\nu + \pi\nu\ell)]. \qquad (11)$$

Here ${}^nC_k$ is the binomial expansion coefficient.

Equation (11) is the central result of this paper. It shows the dependence of the change in action over ℓ revolutions on the tune ν and on the driving terms of the perturbations. These latter are the integrals:

$$S_n(n+1-2k) = \int_0^{2\pi\nu} d\tau P_n(\tau) \times \sin[(n+1-2k)(\tau - \pi\nu)], \qquad (12a)$$

and

$$C_n(n+1-2k) = \int_0^{2\pi\nu} d\tau P_n(\tau) \times \cos[(n+1-2k)(\tau - \pi\nu)]. \qquad (12b)$$

'or critical tune values (ν values for which the denominators of 'q. (11) vanish), the fractional change in action is linear with evolution number ℓ, and the rate of this linear change depends n the initial action W_0 as well as on the driving terms. We ave described such a behavior in Ref. [4]. Away from the resonances", the action variable oscillates up and down with evolution number and has a variation which depends on the riving integrals, S_n and C_n.

Effect of Multipoles

Equation (11) is exact to first-order in the strength of the agnetic multipoles and it exhibits their combined effect on ie action variable. If we consider the term for $n = 1$, we see ie contribution of a quadrupole distribution. Thus

$$\left(\frac{\Delta W}{W_0}\right)_1 = \frac{1}{2}\frac{\sin 4\pi\nu\ell}{\sin 2\pi\nu}\int_0^{2\pi\nu} d\tau P_1(\tau)\sin 2(\tau - \pi\nu)$$
$$= \frac{1}{2}\frac{\sin 4\pi\nu\ell}{\sin 2\pi\nu} S_1(2). \qquad (13)$$

In the interest of simplifying the writing, we have assumed that the perturbing quadrupole only has a "sine" distribution, $S_1(2)$, though, according to Eq. (11), both sine and cosine distributions must, in general, be taken into account. Similarly for the sextupole term ($n = 2$), we write

$$\left(\frac{\Delta W}{W_0}\right)_2 = \frac{W_0^{1/2}}{4}\left\{\frac{\sin 6\pi\nu\ell}{\sin 3\pi\nu} S_2(3) + \frac{\sin 2\pi\nu\ell}{\sin \pi\nu} S_2(1)\right\}. \qquad (14)$$

In contrast to the quadrupole term which, as expected, is independent of the initial action variable, the sextupole ($n = 2$) contribution depends on the 1/2 power of the initial action. In addition there are now two driving integrals, depending on the third harmonic and on the fundamental of the sextupole distribution around the ring. For the sake of completeness we present the general contributions for the other multipoles. For $n = 3, 5, \ldots$ (odd), we have

$$\left(\frac{\Delta W}{W_0}\right)_n = \frac{W_0^{(n-1)/2}}{2^n}\left\{\frac{\sin(n+1)2\pi\nu\ell}{\sin(n+1)\pi\nu} S_n(n+1) + \sum_{k=1}^{(n-1)/2}\left({}^nC_k - {}^nC_{n+1-k}\right)\frac{\sin(n+1-2k)2\pi\nu\ell}{\sin(n+1-2k)\pi\nu} S_n(n+1-2k)\right\}; \qquad (15)$$

hile for $n = 2, 4, 6, \ldots$ (even), we have

$$\left(\frac{\Delta W}{W_0}\right)_n = \frac{W_0^{(n-1)/2}}{2^n}\left\{\frac{\sin(n+1)2\pi\nu\ell}{\sin(n+1)\pi\nu} S_n(n+1) + \sum_{k=1}^{n/2}\left({}^nC_k - {}^nC_{n+1-k}\right)\frac{\sin(n+1-2k)2\pi\nu\ell}{\sin(n+1-2k)\pi\nu} S_n(n+1-2k)\right\}. \qquad (16)$$

Effect of Random Multipoles

We shall employ the previous development to estimate the fect of the random multipoles present in the SSC dipole magts. In this case the *rms* driving term takes the form

$$(S_n)_{rms} = \frac{2^{1/2}\pi}{M_D^{1/2}}(b_n)_{rms}\beta_{AV}^{(n+1)/2}, \qquad (17)$$

here M_D is the number of dipoles in the ring, β_{AV} is the erage betatron function at the operating tune, and $(b_n)_{rms}$ is e expected random multipole coefficient. To obtain an upper ound to the change in the action, we have set the value of the ine factor" involving the number of particle turns (ℓ) to be ual to one. Equation (13) then yields for the effect of the adrupole random perturbation

$$\left(\frac{\Delta W}{W_0}\right)_{1rms} = \frac{\pi}{2^{1/2}M_D^{1/2}}\frac{1}{|\sin 2\pi\nu|}(b_1)_{rms}\beta_{AV}. \qquad (18)$$

the case of a random sextupole distribution, Eq. (14) yields

$$\left(\frac{\Delta W}{W_0}\right)_{2rms} = \frac{W_0^{1/2}\pi}{2^{3/2}M_D^{1/2}}(b_2)_{rms}\beta_{AV}^{3/2} \times\left\{\left(\frac{1}{\sin 3\pi\nu}\right)^2 + \left(\frac{1}{\sin \pi\nu}\right)^2\right\}^{1/2}. \qquad (19)$$

r the higher multipoles, one can use Eqs. (15) and (16) to rive analogous expressions.

Table I is a summary of our numerical results. To calculate e third column we have chosen the SSC parameters: $M_D = 3840, \nu = 78.27, \beta_{AV} = 168.66m$, and the random errors as given in Ref. [5]. The last column, marked $a_{max}(0.1)$, is the initial amplitude of a particle ($a_{max} = \sqrt{W_0\beta_{max}}$) which results in a fractional change in the action $(\Delta W/W_0)_{rms}$ equal to 10%. In this calculation the maximum beta function is 332 m. It is to be noted that the "linear aperture", defined by this initial amplitude, is dominated by the sextupole random error. Actually when the effects of all the random multipole errors are considered in unison, the aperture only changes from 9.7 mm to 9.6 mm. Any attempt at increasing the aperture by modifying the driving integrals with correction coils around the ring would therefore initially concentrate on the third and first harmonics of the sextupole distribution. However, the overall success of this approach would then be limited by the other higher magnetic multipoles.

References

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Table 1. Aperture for 10% Change in Action

Multipole n		Random Multipole $(b_n)_{rms}$; $[m^{-n}]$	$\left(\frac{\Delta W}{W_0}\right)_n$	Initial Amplitude $a_{max}(0.1)$; (m)
Quad	1	0.72×10^{-2}	0.044	—
Sex	2	2.15	$W_0^{1/2}(187.66)$	9.7×10^{-3}
Oct	3	30	$W_0(34.40 \times 10^3)$	30.6×10^{-3}
Dec	4	7000	$W_0^{3/2}(70.35 \times 10^6)$	20.5×10^{-3}
12 pole	5	10^5	$W_0^2(24.21 \times 10^9)$	26.0×10^{-3}
14 pole	6	2×10^7	$W_0^{5/2}(25.78 \times 10^{12})$	23.9×10^{-3}
16 pole	7	2×10^4	$W_0^3(52.97 \times 10^{15})$	20.6×10^{-3}
18 pole	8	1×10^{11}	$W_0^{7/2}(20.90 \times 10^{18})$	22.8×10^{-3}