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STABILITY OF PHASE OSCILLATIONS UNDER TWO APPLIED FREQUENCIES

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Abstract

Stability criteria are developed for the stability of motions of particles in a bucket while subjected to an acceleration voltage at small frequency difference $\Delta\nu$. The motion is stable if $\Delta\nu \geqslant 10\nu_s$ where ν_s is the phase oscillation frequency. This is essentially the same criterion as that the buckets not touch.

I. Introduction

We would like to investigate the stability of particle motion in nearly stationary rf buckets when subjected to a neighboring rf frequency. Intuitively we surmise that the particles in a bucket are subjected to, primarily, the difference frequency of the two rf systems, and that since the perturbing voltage appears at all phases, resonances will appear in the phase oscillations when $\Delta\nu = n\nu_s$, where $\Delta\nu$ is the difference frequency, ν_s is the phase oscillation frequency and n = 1, 2, ---. Clearly n = 1 corresponds to the excitation of coherent phase oscillations, n = 2 to bunch shape type oscillations (or half integral resonance), n = 3 and higher to the type of resonances which determine limits of stability of motion. We note further that the frequency of phase oscillations <u>decreases</u> with amplitude, so that if a stability criterion is found for small amplitude motion, larger amplitude will be <u>more</u> stable, for a given frequency difference. The purpose of this note is to estimate the stopband widths associated with these resonances.

The existence of such stability can allow the simultaneous acceleration of many adjacent "unstacked" beams in an accelerating storage ring or slow cycled synchrotron. This causes an amelioration of the high rf power requirements normally associated with such schemes. We note in passing only that if beams are combined before acceleration, the required voltage is proportional to the square of the number of bunches stacked, but if the "adjacent bucket" scheme is used, the voltage is proportional to the first power of the number of stacks.

II. Hamiltonian

Following the fundamental work[1] on rf acceleration schemes, the Hamiltonian for motion of particles under two rf frequencies, ν, and $\nu_1 = \nu + \Delta\nu$ for particles whose revolution frequency $f \approx \nu/h$ is

$$H = 2\pi E(W) - \frac{2\pi\nu W}{h} + \frac{V}{h} \cos(h\Theta) + \frac{V_1}{h} \cos(h\Theta + 2\pi\Delta\nu t + \psi)$$

The particle revolution frequency is a function of energy $f(E)$, and $W = \int \frac{dE}{f(E)}$ is canonically conjugate to Θ, the angle measured in a frame rotating with frequency ν/h. V is the voltage of the rf system near the particle frequency, V_1 is the voltage of the neighboring frequency, and ψ is an arbitrary phase. We postulate no other "accidental" harmonic relationships which perturb the local W domain, and assume short accelerating gaps so that "voltages", or energy gains, are independent of frequency applied.

III. Small Amplitude Motion

Setting $V_1 = 0$ for the moment, we expand $E(W)$ in a Taylor series about W_s where $f(W_s) = \nu/h$. Then we have, dropping small and constant terms,

$$H_o = 2\pi f_s' \frac{W^{*2}}{2} + \frac{V}{h} \cos h\Theta, \text{ where } W^* = W - W_s,\ f_s' = \frac{df}{dW}(W_s).$$

(This can be done by a canonical transformation $S = \Theta(W-W_s)$). Now expanding the cosine term we find

$$H_o = 2\pi f_s' \frac{W^{*2}}{2} + hV \frac{\Theta^2}{2}$$

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1. K.R. Symon and A.M. Sessler. Proc. CERN Symp. on Particle Accelerators, 1956, p. 44. Also MURA Report 106 Unpublished.

Then the frequencies of small amplitude oscillation are

$$\nu_s = \frac{\omega_s}{2\pi} = \frac{1}{2\pi}\sqrt{2\pi f_s' hV} = \sqrt{\frac{f_s' hV}{2\pi}}$$

Next we perform a time independent transformation[2] to, essentially, angle action variables ρ,γ

$$\Theta = \sqrt{2\rho\sqrt{\frac{2\pi f_s'}{hV}}}\ \sin\gamma,\ W^* = \sqrt{2\rho\sqrt{\frac{hV}{2\pi f_s'}}}\ \cos\gamma, \text{ whence}$$

$$H_o = \sqrt{2\pi f_s' hv} \qquad \rho = \omega_s\,\rho, \text{ so that}$$

$$\rho = \text{const},\ \dot{\gamma} = \omega_s\ .$$

IV. <u>The Perturbation Term</u>

$$H_1 = \frac{V_1}{h}\cos(h\Theta - \Delta\omega t + \psi).$$

This may be expanded to yield

$$H_1 = \frac{V_1}{h}\sum_{n=o}^{\infty}\frac{(h\theta)^n}{n!}\cos\left(\Delta\omega t + \psi + \frac{n\pi}{2}\right)$$

Under the transformation above, this becomes

$$H_1 \equiv \sum_{n=o}^{\infty} V_1\ \frac{h^{n-1}}{n!}\left(\frac{2\omega_s}{hV}\right)^{\frac{n}{2}}\rho^{\frac{n}{2}}\sin^n\gamma\cos\left(\Delta\omega t + \psi + \frac{n\pi}{2}\right)$$

For each power of $\sqrt{\rho}$, there is a term $\sin^n\gamma\cos(\Delta\omega t + \psi + \frac{n\pi}{2})$ which contains terms up to $\sin(n\gamma - \Delta\omega t - \psi + \frac{n\pi}{2})$, plus terms of lower order periodicity in γ. The lower terms correspond to lower order resonances driven by new nonlinear terms. We need, at each order, to concentrate only on the term of highest periodicity. Each new term in the series corresponds to the introduction of higher order resonances. We

2. The reader may verify that it is canonical.

may consider these resonances separately, and infer the effects due to the introduction of the higher order term. Then we can investigate the behavior at each new resonance, which appears at $\Delta\omega = n\omega_s$. Consider the resonance of order n. We can perform a canonical transformation to a system rotating at the resonance frequency $\dot{\gamma} \approx n\omega_s$ via

$$S_n = \rho_n \left[\left(\gamma_n - \frac{\Delta\omega t + n + n\pi/2}{n}\right) \right]$$

Then, keeping only the slowly varying terms in H_1,

$$K_n = H_o + H_1 - \frac{\partial S_n}{\partial t} = \left(\omega_s - \frac{\Delta\omega}{n}\right) \rho_n + \underbrace{\frac{V_1 h^n}{hn!\,2^n} \left(\frac{2\omega_s}{hV}\right)^{\frac{n}{2}}}_{B_n} \rho_n^{\frac{n}{2}} \sin n\gamma_n$$

Stability limits are determined when ρ_n is of sufficient magnitude that fixed points occur, i.e., $\dot{\rho}_n = o$, $\dot{\gamma}_n = o$.

$$\dot{\rho}_n = n\,\rho^{n/2} \cos n\gamma_n = o \qquad n\gamma_n = \frac{\pi}{2}\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \text{etc.}$$

$$\text{or } \sin n\gamma_n = 1, -1, 1, -1, \text{ etc.}$$

$$\dot{\gamma}_n = \left(\omega_s - \frac{\Delta\omega}{n}\right) + \frac{n}{2} B_n \rho_n^{\left(\frac{n}{2} - 1\right)} \sin n\gamma_n = o \text{ when}$$

$$\left| \omega_s - \frac{\Delta\omega}{n} \right| = \frac{n}{2} B_n \rho_n^{\left(\frac{n}{2} - 1\right)}$$

To determine a useful value of $\Delta\omega$, we need to establish a practical value of ρ_n. We choose ρ_n to be that value for which $W^* = o$ and $\theta = \frac{\pi}{h}$. This is in any case the limit of the nonlinear phase stability, and is a more stringent requirement than may be necessary.

$$\frac{\pi}{h} = \sqrt{2\rho \sqrt{\frac{2\pi f s'}{hV}}} \quad \text{or } \rho = \frac{\pi^2}{2h^2} \frac{hV}{\omega s}. \quad \text{Then}$$

$$\left| \omega_s - \frac{\Delta\omega}{n} \right| = \frac{n}{2} \frac{B_n}{(2)^{\frac{n}{2} - 1}} \left(\frac{\pi}{h}\right)^{n-2} \left(\frac{hV}{\omega s}\right)^{\frac{n}{2} - 1} = \omega s \frac{1}{2^n (n-1)!} (\pi)^{n-2} \frac{V_1}{V} = \omega_s \xi_n$$

Then $\delta = 2\xi_n$ is the fractional stopband width of resonance n

$$\delta_n = \frac{\pi^{n-2} n}{2^n (n-1)!} \frac{V_1}{V} \quad . \quad \text{For successive resonances}$$

$$\frac{\delta_{n+1}}{\delta_n} = \frac{\pi(n+1)}{2(n-1)n} \quad .$$

Then the resonances decrease in width as 1/n.

For n = 4 the stopband width becomes $\delta_4 = 0.8$ (full width). A further effect, which has been ignored here, is the effect of the terms of lower multiplicity in γ. For $n \geq 4$ these terms cause detuning of the motion in such a way as to reduce the width of the resonance.[3,4] In fact, for $n > 4$ these resonances can usually be ignored. Then for $\Delta\nu \gtrsim (5\text{-}10)\nu_s$, the resonant effects can be ignored.

VI. Comparison to Bucket Width

It is interesting to compare the $\Delta\nu$ obtained by the resonance question to the $\Delta\nu$ in a natural bucket. We choose, $\Delta W = \frac{A}{2\pi}$ then,

$$\Delta\nu_{bucket} = hf_s{}'\Delta W = 8\nu_s .$$

Thus the separation due to resonances is simply related to bucket widths.

VII. Conclusion

The stability criterion has been found for neighboring buckets, depending on the frequency separation of the rf system in terms of the phase oscillation frequency. In a later report it will be seen that if phase space areas can be preserved in the AGS as well as they presently are, adjacent bucket acceleration can be used in a storage ring to enhance the stored beam at high energies.

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3. J. Moser. Stability of Systems of Canonical Differential Equations. Nachr. Akad. Wiss. Göttingen, IIa, pp 87-120 (1955).

4. R. Hagedorn. CERN 57-1.

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