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Beam Breakup in Dielectric-Lined Waveguide

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ABSTRACT

We examine beam-breakup effects of a 100 nC source bunch with 1 mm rms length inside a cylindrical dielectric waveguide, with dielectric $\epsilon = 2.65$ filling the radius between 7.5 and 9.0 mm. We find that only $\sim 78\%$ of the bunch with an initial transverse (or radial) offset of 0.3 mm survives the passage of the 3.75 m waveguide. The loss is mainly due to the slowing down of some particles to nearly zero velocity. As a result, quadrupole focussing of any sort will not help. However, if the waveguide is shortened to 3.3 m, the loss reduces to only 5.5%.

I. INTRODUCTION

Dielectric-lined waveguides which generate wakes through Cerenkov radiation are very promising candidates of wakefield accelerators. This is because their structure is rather simple and, for the model to be built at the Argonne Advanced Accelerator Test Facility (AATF),[1] an acceleration gradient as high as ~ 50 MeV/m can be expected. In order to generate the required wake, the source bunch is designed to be of rms length $\sigma = 1$ mm carrying a total charge of $q = 100$ nC. The vacuum part of the waveguide has a radius of $b = 7.5$ mm. In a circular ring, the head and tail of a bunch are interchanged due to synchrotron motion. Here, however, just as in a linac, such an interchange does not take place. The wake force from the head of the bunch will cause its own tail to continuously lose energy and be deflected transversely, leading to eventual particle loss or beam breakup. In this paper, we analyse the stability of the source bunch, estimate the growth of transverse (or radial) deflection, and finally compute the deflection and beam loss numerically.

II. WAKE FORCES

Consider a particle of charge q traveling along a cylindrical dielectric-lined waveguide of inside and outside radii b and a, respectively. The region between b and a is filled with a dielectric having a dielectric constant ϵ. The Cerenkov wake forces produced have been computed.[2,3] When the charged particle is at a radial offset r_0, the longitudinal and transverse (or radial) wake fields acting on a point s behind at radial offset r can be written, respectively, in terms of reduced quantities as[2]

$$\bar{F}_{zm\lambda}(r,s;r_0) = -\frac{eq}{a^2}\left[\frac{r_0}{a}\right]^m \left[\frac{r}{a}\right]^m \hat{F}_{zm\lambda}\cos\frac{x_{m\lambda}s}{s\sqrt{\epsilon-1}}\ , \tag{2.1}$$

$$\bar{F}_{rm\lambda}(r,s;r_0) = \frac{eq}{a^2}\left[\frac{r_0}{a}\right]^m \left[\frac{r}{a}\right]^{m-1} \hat{F}_{rm\lambda}\sin\frac{x_{m\lambda}s}{s\sqrt{\epsilon-1}}\ , \tag{2.2}$$

where m and λ designate the azimuthal and radial mode numbers. The characteristic frequency $f_{m\lambda}$ for mode (m,λ) is related to the reduced frequency $x_{m\lambda}$ by

$$f_{m\lambda} = \frac{x_{m\lambda}}{2\pi a\sqrt{\epsilon-1}}\ . \tag{2.3}$$

The reduced longitudinal and transverse (or radial) wake forces for mode m, λ are defined as

$$\hat{F}_{zm\lambda} = 8x_{m\lambda}\left(\frac{a}{b}\right)^{2m} \frac{p_m(x_{m\lambda})r_m(x_{m\lambda})}{\mathcal{D}'_m(x_{m\lambda})} , \tag{2.4}$$

$$\hat{F}_{rm\lambda} = \hat{F}_{zm\lambda}\frac{m\sqrt{\epsilon-1}}{x_{m\lambda}} , \tag{2.5}$$

where p_m, r_m, and $\mathcal{D}_m$ are some polynomial of products of Bessel functions, and are defined in Ref. 2. As an example, the design of AATF consists of a waveguide with inside and outside radii $b = 7.5$ mm and $a = 9.0$ mm, the dielectric having $\epsilon = 2.65$ filling the radius between b and a. The first longitudinal mode is $x_{01} = 5.516$ corresponding to a frequency of $f_{01} = 27.34$ GHz, and the reduced longitudinal wake force is $\hat{F}_{z01} = 4.11$. The reduced frequencies and wake forces for this and other low-lying modes are listed in Table I.

III. THE EQUATIONS OF MOTION

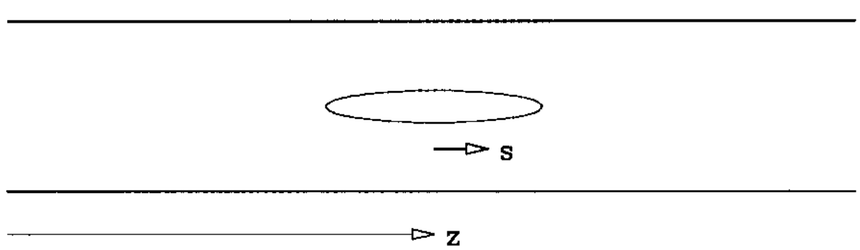

Fig. 1. A source bunch moving down the waveguide.

Consider a particle inside the source bunch at a distance s from some reference point of the bunch, which has moved a distance z down the waveguide (Fig. 1). The longitudinal momentum p_z and radial momentum p_r propagate with time according to

$$\frac{dp_z(s,z)}{dt} = F_z(s,z) , \tag{3.1}$$

$$\frac{dp_r(s,z)}{dt} + F_Q(z)r(s,z) = F_r(s,z) , \tag{3.2}$$

where F_Q denotes the focussing force of the quadrupoles placed along the waveguide. Note that the wake forces F_z and F_r also depend on the instantaneous radial position

Mode		Reduced Freq	Reduced Long. Force	Reduced Trans. Force
λ	m	$x_{m\lambda}$	$\hat{F}_{zm\lambda}$	$\hat{F}_{rm\lambda}$
1	0	5.5	4.12	
2	0	20.7	0.88	
3	0	38.7	0.28	
4	0	57.2	0.13	
5	0	75.9	0.08	
6	0	94.6	0.04	
1	1	5.2	7.39	1.83
2	1	10.6	3.41	0.41
3	1	20.7	2.41	0.15
4	1	28.7	0.56	0.03
5	1	38.7	0.80	0.03
6	1	47.4	0.21	0.01
1	2	6.2	8.67	3.57
2	2	11.3	6.66	1.51
3	2	21.5	5.08	0.61
4	2	29.0	1.41	0.12
5	2	39.2	1.92	0.13
6	2	47.6	0.54	0.03

Table I: Reduced frequencies and wake forces for the low-lying modes.

$r(s,z)$ of the particle under consideration. Therefore, Eqs. (3.1) and (3.2) are coupled for all modes except $m=0$. For example, the longitudinal force is given, according to Eq. (2.1) by

$$F_z(s,z) = -\frac{eq}{a^2}\sum_{m=0}^{\infty}\sum_{\lambda=1}^{\infty}\int_s^{\infty} ds'\rho(s',z)\hat{F}_{z0\lambda}\left[\frac{r_0(s',z)}{a}\right]^m\left[\frac{r(s,z)}{a}\right]^m \cos\frac{x_{m\lambda}(s'-s)}{a\sqrt{\epsilon-1}} \ , \quad (3.3)$$

where ρ is the particle distribution inside the bunch.

IV. THE MONOPOLE MODE

When only the monopole mode $m=0$ is included, the transverse wake force F_r of Eq. (3.2) vanishes and there is only the longitudinal wake force. We change the time variable t for the center of the bunch to its distance along the waveguide z. As will be shown below, the center of the bunch will remain ultra-relativistic,. It is accurate enough to assume $z=ct$, where c is the velocity of light. We can then rewrite Eq. (3.1) as

$$mc^2\frac{\partial(\gamma\beta)}{\partial z} = F_z(s) \ , \quad (4.1)$$

where m is the electron mass and βc the particle velocity. Note that the longitudinal wake force is now a function of only s, the position inside the bunch. For a Gaussian bunch shape

$$\rho(s) = \frac{1}{\sqrt{2\pi}\sigma}e^{-s^2/2\sigma^2} \ , \quad (4.2)$$

where σ is the rms bunch length, $F_z(s)$ can be computed exactly in terms of the complex error function $w(z)$. Then Eq. (4.1) can be integrated easily to give

$$\gamma\beta(s,z) = (\gamma\beta)_i - \frac{eqz}{2a^2mc^2}\sum_{\lambda=0}^{\infty}\hat{F}_{z0\lambda}e^{-s^2/2\sigma^2}\,\mathcal{R}e\,w\left(\frac{\sigma x_{0\lambda}}{a\sqrt{2(\epsilon-1)}} + i\frac{s}{\sqrt{2}\sigma}\right) \ . \quad (4.3)$$

For initial $\gamma_i = 300$ and $\sigma = 1$ mm, this is plotted in Fig. 2 at the end of a waveguide of length $L = 3.75$ m, with 6 radial modes included. We see that γ drops down to a minimum of ~ 100 at 1σ, showing that the beam particles are relativistic for the total passage, so that longitudinal mixing of particles does not occur. This justifies the assumption of Eq. (4.2) that the particle distribution is independent of z. This behavior will be modified to a large extent when higher azimuthal modes are included, because those longitudinal wake forces depend strongly on transverse displacements. We will see in Sect. VII that some particles will actually be slowed to $\gamma\beta < 0$. Nevertheless, the above estimation does tell us that the lowest $\gamma\beta$ occurs around 1σ.

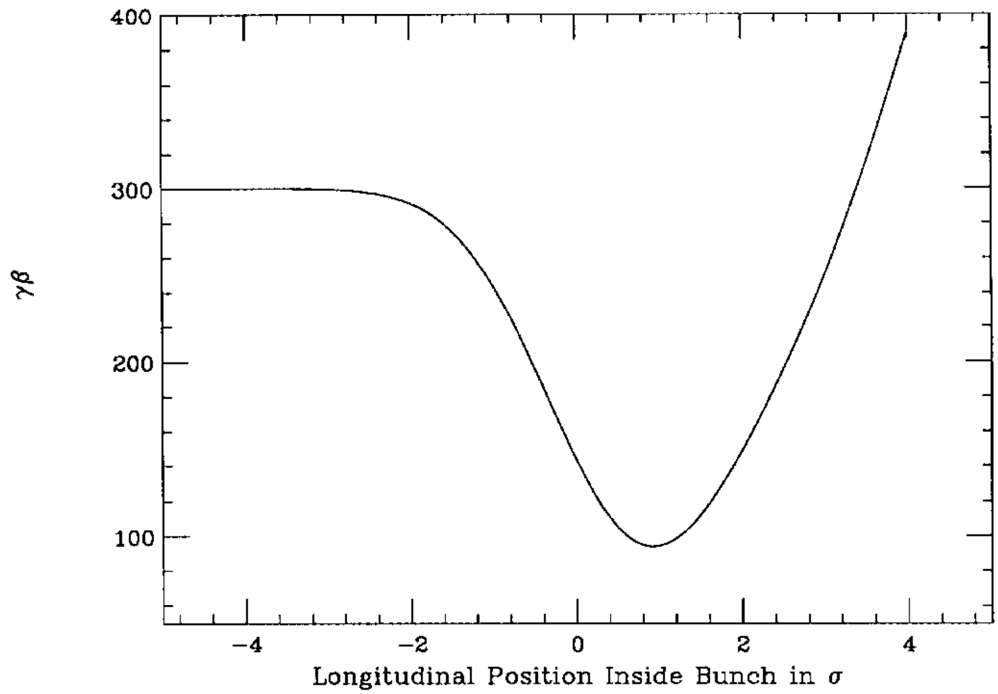

Fig. 2. $\gamma\beta$ distribution inside a bunch at the exit of a 3.75 m waveguide. Here, only monopole modes are included.

V. DIPOLE MODE

For the azimuthal mode $m = 1$, Eq. (3.2) can be written as

$$m\gamma c^2 \frac{\partial^2 r}{\partial z^2} + mc^2 \frac{\partial \gamma}{\partial z}\frac{\partial r}{\partial z} + F_Q(z) r = F_r(s, z) \ , \tag{5.1}$$

where the dipole transverse or radial wake force is

$$F_r(s, z) = \frac{eq}{a^2} \sum_{\lambda=1}^{\infty} \int_s^{\infty} ds' \rho(s', z) \hat{F}_{r1\lambda} \frac{r_0(s, z)}{a} \sin \frac{x_{1\lambda}(s-s')}{a\sqrt{\epsilon - 1}} \ . \tag{5.2}$$

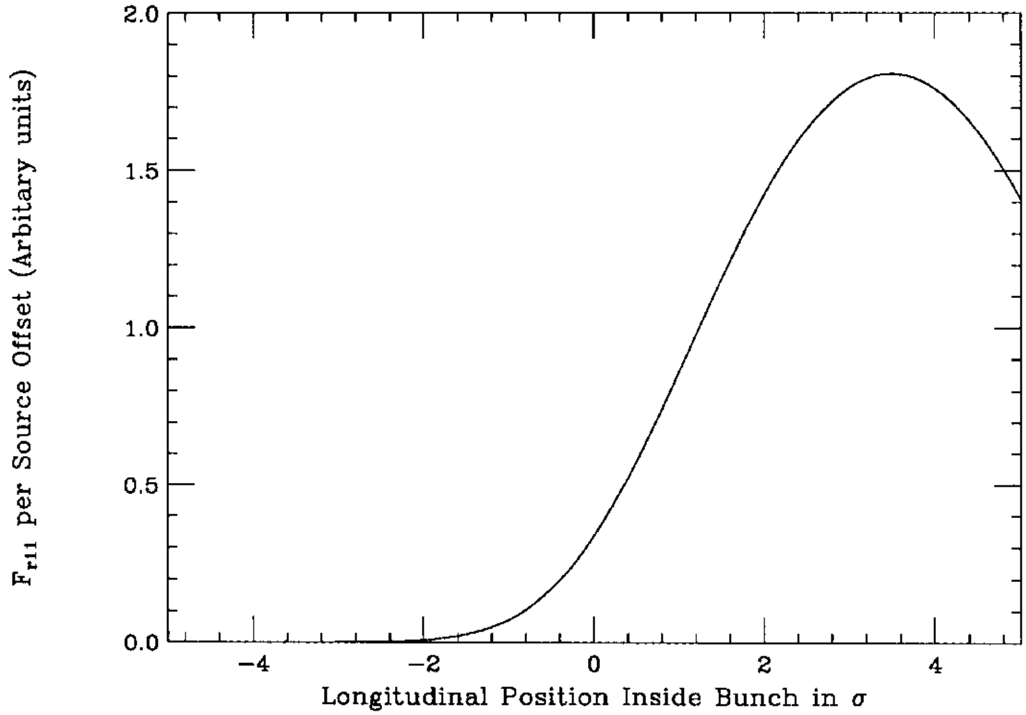

Fig. 3. The transverse wake force per unit displacement of the source bunch under consideration. Here, only the mode $m = 1$, $\lambda = 1$ has been taken into account.

The radial wake force per unit radial displacement of the source is plot in Fig. 3. We see that it only affects the tail of the bunch and peaks at $\sim 3\sigma$. We can therefore approximate the bunch by two macro-particles, with the 'front particle' acting as the source which sends a wake to the 'tail particle.' Thus, we have

$$F_r(s, z) \sim \frac{eq}{a^2} \hat{F}_{r11} \frac{r_0}{a} \frac{1}{2} , \tag{5.3}$$

where we have approximated the particle density ρ by $\frac{1}{2}$. Assuming a smooth focussing force F_Q, we can introduce the betatron-oscillation wave number k_β. The equation of motion of the 'tail particle' is then

$$r'' + k_\beta^2 r = \frac{eq}{2a^3} \frac{\hat{F}_{r11}}{\gamma mc^2} \hat{r}_0 \cos k_\beta z \ , \tag{5.4}$$

where the 'prime' denotes differentiation with respect to z and $\hat{r}_0$ is the betatron-oscillation amplitude of the 'front particle.' This is a resonance equation. For small variation of γ with z, the solution can be written as

$$r(z) = \hat{r}_0 \left[\cos k_\beta z + \eta_1 z \sin k_\beta z\right] \ , \tag{5.5}$$

with the transverse growth per unit distance given by

$$\eta_1 = \frac{eq\hat{F}_{r11}}{4a^3\gamma mc^2 k_\beta} \ . \tag{5.6}$$

The focussing force F_Q can be constructed as FODO cells, each of which is of length ℓ. We can then introduce an energy independent focussing strength G given by

$$G = \gamma\beta \frac{B'\bar{\ell}}{B\rho} = \frac{\beta F_Q \bar{\ell}}{mc^2} \ , \tag{5.7}$$

where $\bar{\ell}$ is the length of the quadrupole. For example, 5 kilogauss pads placed with alternating polarity every $\ell/2 = 10$ cm will give $G \simeq 1200$ m^{-1}. For FODO cells, the phase advance per cell μ is given by

$$\frac{\ell G}{\gamma\beta} = 4 \sin\frac{\mu}{2} \ , \tag{5.8}$$

so that the betatron wave number is

$$k_\beta = \frac{\mu}{\ell} = \frac{2}{\ell} \sin^{-1} \frac{\ell G}{4\gamma\beta} \ . \tag{5.9}$$

For the above focussing configuration and assuming $\gamma = 300$, $\ell G/4\gamma = 0.2$. Thus,

$$k_\beta \simeq \frac{G}{2\gamma\beta} \ . \tag{5.10}$$

The increase in transverse deflection for a waveguide of length L is therefore

$$\eta_1 L = \frac{eq\hat{F}_{r11} L\beta}{2a^3 mc^2 G} \ , \tag{5.11}$$

which turns out to be 6.9 for a total charge of $q = 100$ nC, waveguide length $L = 3.75$ m, and when only one mode $\hat{F}_{r11} = 1.83$ is included. However, Eq. (5.11) is incorrect because $\gamma\beta$ of the 'tail particle' does drop to a very low value at the exit end of the waveguide. In fact, when $\ell G/(4\gamma\beta) > 1$, the FODO focussing will become divergent instead. For this reason, the strength of the quadrupoles needs to be tapered towards the exit end of the waveguide. One possibility is to hold $G/\gamma\beta$ constant. Then the total growth can be obtained by integrating η_1 of Eq. (5.6) along the waveguide. The result is

$$\int_0^L \eta_1 dz = \frac{eq\hat{F}_{r11}L}{2a^3mc^2G}\frac{\gamma_i\beta_i}{\gamma_i-\gamma_f}\ln\frac{\gamma_i}{\gamma_f} , \tag{5.12}$$

where G is the initial focussing strength while the subscripts i and f denote initial and final values of the 'tail particle.' Note that, through the expansion of the $\sin^{-1}$ in Eq. (5.9), the integrated growth of Eq. (5.12) is insensitive to the cell length ℓ. Putting in $\gamma_i = 300$ and $\gamma_f = 100$, the total growth is 11.4 times. Assuming an initial transverse deviation of 0.3 mm, the final growth can exceed the 7.5 mm aperture of the waveguide if γ drops down to below 8.9. As will be seen in Sect. VII, this will indeed happen when higher azimuthal longitudinal modes are included.

VI. QUADRUPOLE WAKE

If we apply the two-particle model to the quadrupole wake, the equation of motion of the 'tail particle' is

$$r'' + k_\beta^2 r = \frac{1}{\gamma mc^2}\frac{eq}{2a^2}\hat{F}_{r21}\left(\frac{r_0}{a}\right)^2 r , \tag{6.1}$$

where only the $\lambda = 1$ mode has been included. Again for r_0 we substitute $\hat{r}_0 \cos k_\beta s$. Unlike Eq. (5.4), this equation cannot be solved exactly. For an initial offset of $\hat{r}_0 =$ 1 mm, $(\hat{r}_0/a)^2 \simeq 0.012$. Therefore, we can try to solve the equation perturbatively. The lowest-order solution is

$$r(z) = \hat{r}_0 \left[\cos k_\beta z + \eta_2 z \sin k_\beta z\right] , \tag{6.2}$$

where the total growth for a length L is

$$\eta_2 L = \frac{eqL}{\gamma mc^2a^2}\frac{3\hat{F}_{r21}}{16k_\beta a}\left(\frac{\hat{r}_0}{a}\right)^2 \simeq 0.0356 , \tag{6.3}$$

for a focussing strength of $G = 1200\ \mathrm{m}^{-1}$. The smallness of the growth implies that the perturbation treatment is justified and also that the effect of quadrupole modes can be neglected. For higher azimuthal m, the growth will be proportional to $(\hat{r}_0/a)^{2m-2}$. Therefore, all the higher azimuthal modes can be dropped.

VII. NUMERICAL SOLUTION

We now attempt to solve Eqs. (4.1) and (5.1) numerically. The bunch is made to propagate through the waveguide by small steps of size Δz. The derivatives of the radial deflection with respect to z can be approximated by differences:

$$r'(s,z) = \frac{r(s,z+\Delta z) - r(s,z-\Delta z)}{2\Delta z} , \tag{7.1}$$

$$r''(s,z) = \frac{r(s,z+\Delta z) + r(s,z-\Delta z) - 2r(s,z)}{\Delta z^2} . \tag{7.2}$$

Substituting into Eq. (5.1) and eliminating $(\gamma\beta)'$ by Eq. (4.1), we obtain

$$\begin{aligned} r(s,z+\Delta z) = \left[1 + \frac{\Delta z}{2}\frac{\beta(s,z)F_z(s,z)}{mc^2\gamma(s,z)}\right]^{-1} \Bigg\{ \frac{\Delta z^2 F_r(s,z)}{mc^2\gamma(s,z)} + r(s,z)\left[2 - \frac{\Delta z^2 F_Q(z)}{mc^2\gamma(s,z)}\right] \\ -r(s,z-\Delta z)\left[1 - \frac{\Delta z}{2}\frac{\beta(s,z)F_z(s,z)}{mc^2\gamma(s,z)}\right]\Bigg\} . \end{aligned} \tag{7.3}$$

Knowing $r(s,z)$, we can compute $F_z(s,z)$, and $F_r(s,z)$ from Eqs. (3.1) and (5.2), and solve for $\gamma\beta(s,z)$ from Eq. (4.1). Then the deflection at the next step, $r(s,z+\Delta z)$ can be computed using Eq. (7.3).

The bunch shape is taken to be Gaussian truncated at $\pm 3\sigma$, and is divided longitudinally into 30 bins of size 0.2 mm. When the particles in a particular bin reach ± 7.5 mm they are considered lost by hitting the dielectric and are removed. When $\gamma\beta$ of a bin becomes less than $\sqrt{8}$, the particles there are assumed to have too small a velocity and trail behind and get lost. The bin that becomes empty is denoted by an asterisk in the plots.

The quadrupoles are placed at the center of each section ($\ell/2 = 10$ cm) of the waveguide with interchanging polarity starting with an F quadrupole. The strength of the quadrupoles are tapered according to

$$F_Q(z) = F_Q(0)(1 - \xi\frac{z}{L}) , \tag{7.4}$$

where ξ is the tapering factor.

One of the best solutions is shown in Fig. 4. The initial focussing strength is $G = 1200$ m^{-1} and the tapering factor is 0.75. The initial transverse offsets of all the particles are set to 0.3 mm. The various plots show the radial deflections of $\pm 3\sigma$ of the bunch along the waveguide at the position of every quadrupole. The plots before $z = 1.1$ m are not of interest and are therefore omitted. In every plot, the distribution of $\gamma\beta$ inside the bunch is also shown as a dotted curve. We see that the bunch starts to lose particles at $z = 2.1$ m. At exit all particles $> 0.8\sigma$ or 22% have been lost.

We try to vary the tapering factor from 0.63 to 1.0 and find the same loss of 22% after $z = 3.7$ m. We also vary the initial focussing strength from 900 m^{-1} to 1500 m^{-1}. The losses at exit are roughly the same. We even vary the initial offset of the bunch from 0.15 mm to 0.5 mm, the changes of the loss at exit are not significant.

The source of the loss becomes clear when we examine the $\gamma\beta$ curve carefully. The $\gamma\beta$ curve becomes negative in the region $\sim 0.8\sigma$ to $\sim 1.5\sigma$ when $z \gtrsim 3.5$ m. The drop of γ raises the F_r term and F_Q term of Eq. (7.3) tremendously but reduces the F_z term to zero. This implies that the radial deflection will be increased by very much. These particles, if driven off the transverse aperture will be removed automatically in the numerical computation. Another implication of low $\gamma\beta$ is that the particles in these bins will be slowed down by so much that they even travel in the opposite direction and leave the bunch longitudinally. Since this deceleration is the property of the wake forces, there is no way to improve the passage with the configuration of the focussing system. The only ways to reduce this 22% beam loss appear to be the shortening of the waveguide, the increase of the internal radius, the changing of the dielectric layer, or the reduction of the charge on the source bunch.

If the length of the waveguide is reduced to 3.3 m, the loss occurs only after 1.6σ or 5.5%; and for a length of 3.1 m, only after 2.2σ or 2.4%. These losses will depend more sensitively on the focussing configuration. For example, if the tapering factor is $\xi = 0.63$, the above two losses become 15.9% and 0.3%. If the tapering factor is 0.99, the losses are both 5.5%. If we keep the tapering factor fixed at 0.72, and change the initial focussing strength to $G = 900$ m^{-1}, the losses are both 8.1%, and for $G = 1500$ m^{-1}, the losses are, respectively, 15.9% and 1.4%.

We also try to change the cell length of the FODO focussing structure to $\ell = 30$ cm, but the change in the results is not significant, agreeing with the prediction of our analysis in Sect. V.

Gai[4] made a similar numerical solution recently. He actually tracked a Gaussian distribution of 400 macro-particles over a 3.75 m length of the waveguide. His results show only 2% loss, very different from ours. In his computation, the bunch occupies only $\pm 2\sigma$, and only two radial modes of $m = 0$ and 1 are included. We also try our solution with two radial modes only and find particle loss only after 1.8σ (3.6%). The $\gamma\beta$ curve here never dips down to below 20 explaining clearly why the loss has been small. However, this solution is not correct, because the higher modes cannot be neglected. They are of shorter wavelengths and will deflect and decelerate particles much closer to the front of the bunch. As a result, the tail of the bunch will be deflected more and the 1σ region will be decelerated more. Needless to say, the two-particle model can no longer apply.

VIII. DISCUSIONS

Our analysis shows that the contributions of higher-order modes are very significant, and that beam loss of 22% at the tail of the bunch cannot be avoided by positioning quadrupoles along the waveguide if the guide is 3.75 m long. The reason is that the wake forces will decelerate the particles in the 1σ region to $\gamma\beta < 0$, so that the particles will be (1) deflected outside the transverse aperture and (2) left trailing the bunch longitudinally. The only cure is to shorten the waveguide, reduce the charge density of the bunch, or change the configuration of the dielectric lining so that the wake forces can be reduced. If the waveguide is shortened to 3.3 m, however, the loss can be reduced to 5.5%, the main reason being that $\gamma\beta$ does not dip down to near zero.

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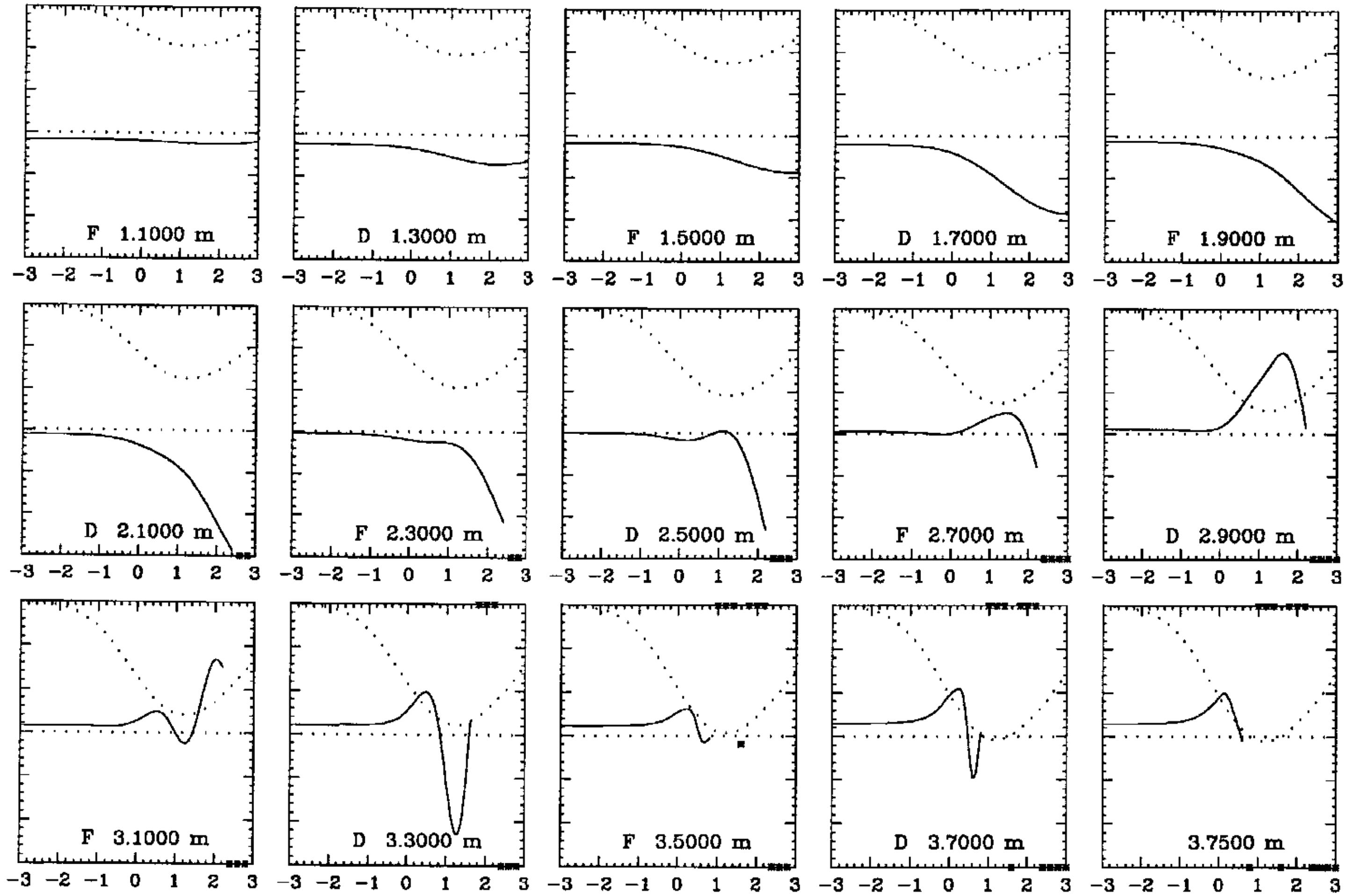

Fig. 4. Radial deflections (solid ±7.5mm) and $\gamma\beta$ (dots ±300) distributions in $\pm 3\sigma$ of a bunch at focussing quads, of strength $1200\mathrm{m}^{-1}$, 10cm apart, tapered by 0.72. Initial offset is 0.3mm. 6 eigenmodes are included for each azimuthal $m \leq 2$.