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Emittance Growth Issues

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An invited talk given at the *Workshop on Vibration Control and Dynamic Alignment* at the SSC, Dallas, Texas, February 11-14, 1992.

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EMITTANCE GROWTH ISSUES†

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I. INTRODUCTION

A particle beam in a circular storage ring will experience kicks turn by turn due to fluctuations of field strengths in the lattice elements. These fluctuations can come from the movement of the lattice elements due to ground motions, intentionally perturbed field current such as Jostlein's beam centering scheme,[1] as well as unintentional perturbations such as field current and voltage noises. All these fluctuations may eventually lead to the growth of transverse emittances[2] as well as longitudinal emittance. The growth is caused by smearing, mainly due to momentum spread plus chromaticity and by the nonlinearity of the forces in both the transverse and longitudinal phase spaces.

Here, we are going to emphasize on perturbations with *long correlation time* or *nonrandom* and low frequencies. Jostlein's beam centering scheme is a sinusoidal perturbation on the current in some dipoles and therefore falls into such a category. The ground motions at tunnel depth at the SSC site due to a crossing train above and quarry blasts 9 miles away had been measured by Hennon and Hennon.[3] We see from Fig. 1 that the displacement waveforms are quite periodic and from Fig. 2 that their spectra consist of contributions from some definite frequencies. These motions are perturbations with long correlation time and low frequencies.

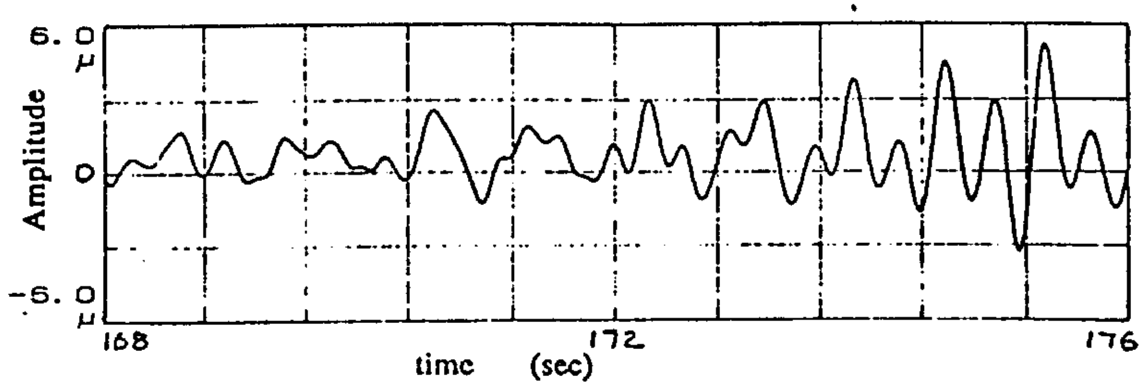
II. EFFECTS OF CHROMATICITY AND MOMENTUM OFFSET

II.1 Single-particle Motion in Transverse Phase Space

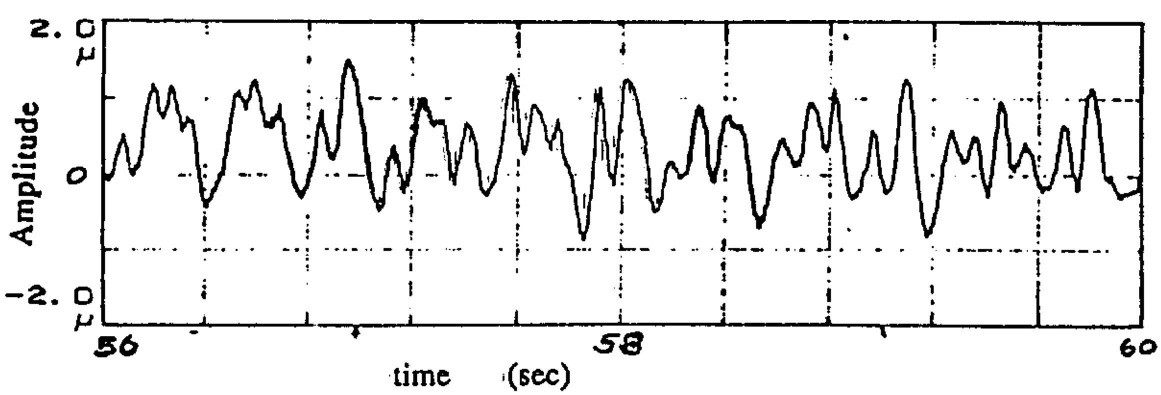
Assume that the kicks occur at only one point along the ring where the momentum dispersion vanishes. Let us look at the position of the particle just before that point in a transverse plane. The normalized coordinates $(X, \beta X' + \alpha X)$ written as a complex number are used, so that betatron motion will be a circle in phase space. Here, β and α are the Twiss parameters at the point of observation, and X and X' are the transverse displacement and the transverse angular displacement.

If the particle is originally at $\vec{a}_0$ and receives a kick of

$$\delta \vec{X}_1 = (0, \theta_1 \beta) \tag{2.1}$$



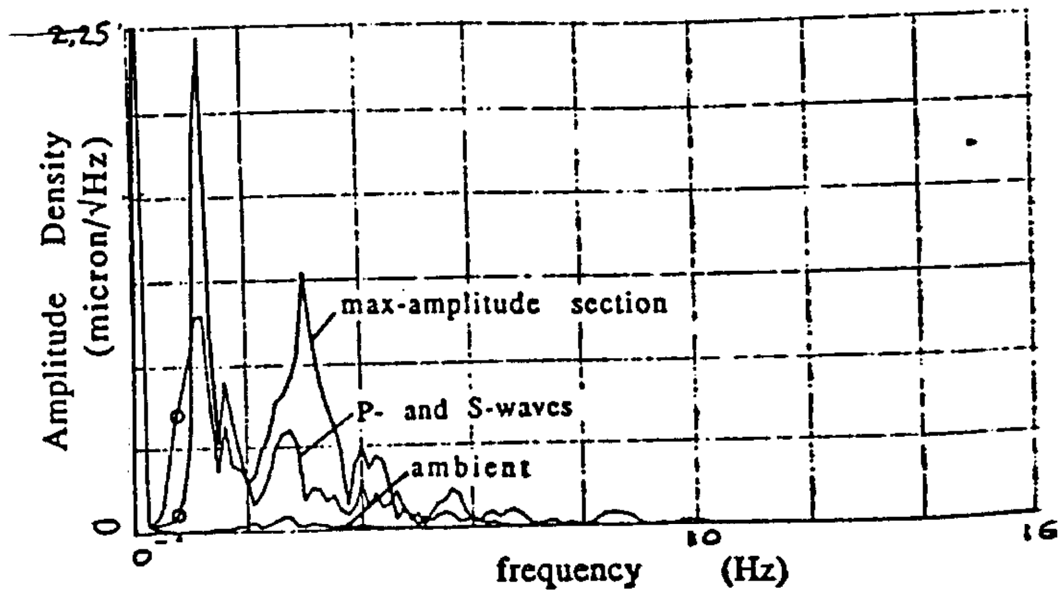
(a)



(b)

Figure 1: The approximate displacement waveforms obtained by integrating the geophone signals of (a) a quarry blast and (b) a train crossing above. The average geophone calibration constant has been applied.

(a)



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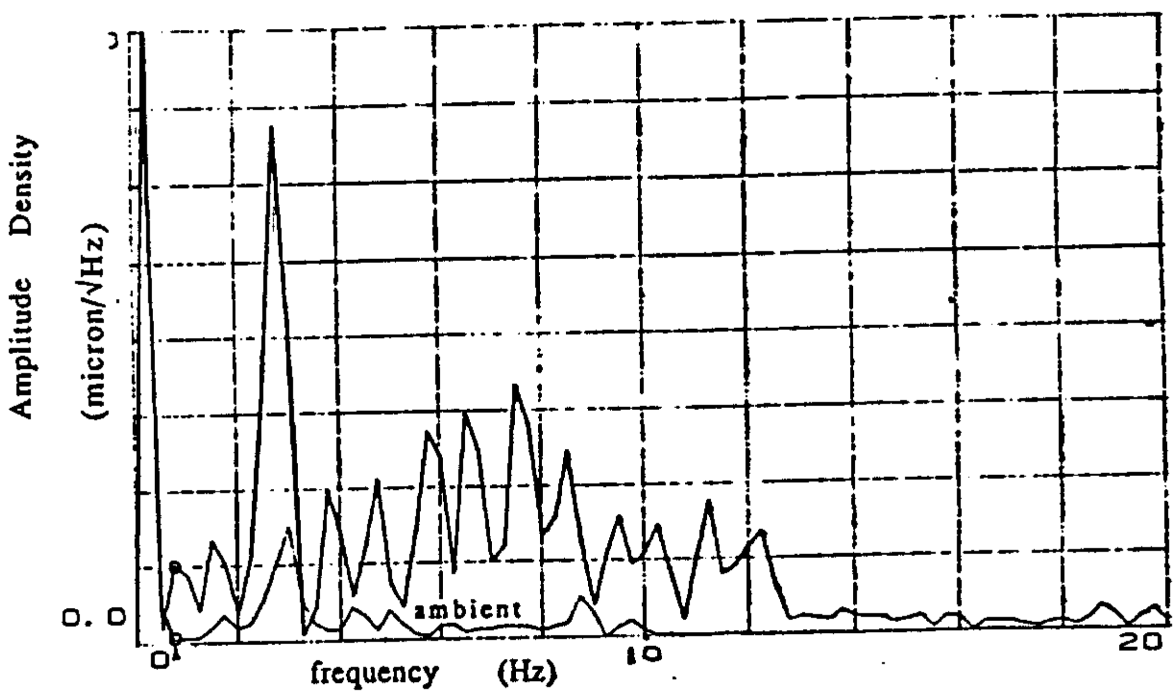


Figure 2: The vibration amplitude spectra obtained by a fast Fourier transform of the signals of (a) a quarry blast and (b) a train crossing above.

at the first turn, when it comes back at the next turn, the transverse position is

$$\vec{X}_1 = \left(\vec{a}_0 + \delta\vec{X}_1\right) e^{-i2\pi\nu_1} \ , \tag{2.2}$$

where ν_1 is the average betatron tune of the particle in the first turn. Now, there is another kick

$$\delta\vec{X}_2 = (0, \theta_2\beta) \ . \tag{2.3}$$

When the particle comes back at the next turn, the position is

$$\vec{X}_2 = \left(\left(\vec{a}_0 + \delta\vec{X}_1\right) e^{-i2\pi\nu_1} + \delta\vec{X}_2\right) e^{-i2\pi\nu_2} \ . \tag{2.4}$$

Therefore, at the N-th turn, just before the $(N+1)$-th kick, the position is

$$\vec{X}_N = \left(\ldots\left(\left(\vec{a}_0 + \delta\vec{X}_1\right) e^{-i2\pi\nu_1} + \delta\vec{X}_2\right) e^{-i2\pi\nu_2} \ldots\right) e^{-i\pi\nu_N} \ , \tag{2.5}$$

or

$$\vec{X}_N = \vec{a}_0 e^{-i2\pi\sum_{k=1}^N \nu_k} + \sum_{j=1}^N \delta\vec{X}_j e^{-i2\pi\sum_{k=j}^N \nu_k} \ . \tag{2.6}$$

Thus, the position at the N-th turn is just the linear superposition of the evolution of first-turn kick of the particle from the origin, the evolution of the second-turn kick from the origin, etc., plus the evolution of the original particle position $\vec{a}_0$.

II.2 Effects of Synchrotron Motion

Without synchrotron motion the nominal betatron tune is ν_0. With synchrotron motion turned on, the betatron tune at the k-th turn is modulated according to

$$\nu_k = \nu_0 + \Delta\nu \sin(2\pi\nu_s k + \varphi_s) \ , \tag{2.7}$$

where ν_s is the synchrotron tune, φ_s is the initial phase of the particle in the longitudinal phase space and $\Delta\nu$ is the tune modulation amplitude which is related to the chromaticity ξ and the fractional momentum offset δ_p by

$$\Delta\nu = \xi\delta_p \ . \tag{2.8}$$

The position of the particle at the N-th turn in the transverse phase space is therefore, according to Eq.(2.6),

$$\begin{aligned} \vec{X}_N &= \vec{a}_0 e^{-i2\pi\left[N\nu_0 + \Delta\nu\sum_{k=1}^N \sin(2\pi k\nu_s + \varphi_s)\right]} \\ &\quad + \sum_{j=1}^N \delta\vec{X}_j e^{-i2\pi\left[\nu_0(N-j+1) + \sum_{k=j}^N \Delta\nu \sin(2\pi k\nu_s + \varphi_s)\right]} \ . \end{aligned} \tag{2.9}$$

The bunch rotates in the transverse phase space and is kicked every turn. Since we want to observe here only the spread of the bunch due to synchrotron motion, we need to subtract away the transverse position of the corresponding longitudinally synchronized particle ($\Delta\nu = 0$), which is denoted by

$$\vec{X}_N^c = \vec{a}_0 e^{-i2\pi\nu_0 N} + \sum_{j=1}^{N} \delta\vec{X}_j e^{-i2\pi\nu_0(N-j+1)} \ . \tag{2.10}$$

Therefore, the spread of the vector

$$\vec{\tilde{X}}_N = \vec{X}_N - \vec{X}_N^c \tag{2.11}$$

for a particle having maximum momentum excursion but with different initial phase φ_s will give the growth of bunch emittance at the N-th turn.

II.3 Evaluation of spread for small chromaticity

The summation of the sine in the exponent of Eq. (2.9) can be performed to give

$$\begin{aligned} \vec{X}_N &= e^{-i2\pi\nu_0(N+1)} e^{i\Delta'\cos[\pi\nu_s(2N+1)+\varphi_s]} \times \\ &\times \sum_{j=1}^{N} \delta\vec{X}_j e^{i2\pi\nu_0 j - i\Delta'\cos[\pi\nu_s(2j-1)+\varphi_s]} \ , \end{aligned} \tag{2.12}$$

where

$$\Delta' = \frac{\pi\Delta\nu}{\sin\pi\nu_s} = 0.259 \ , \tag{2.13}$$

if we take chromaticity $\xi = 5$, maximum fractional momentum spread $\delta_p = 6.0 \times 10^{-5}$, synchrotron frequency $f_s = 4$ Hz, and revolution frequency $f_0 = 3440$ Hz. We try to expand Eq. (2.12) and keep only the first order term in Δ'. Since Eq. (2.12) resembles the integral representation of Bessel functions with argument Δ', the expansion should turn out to be better than expected, although Δ' is not too small. In the above, the term involving the initial point $\vec{a}_0$ has been dropped for convenience, because as is evident from Eq. (2.6) or (2.9), this term does not contribute to the emittance growth.

Let us consider a sinusoidal kick

$$\delta\vec{X}_j = ia_m\beta\sin 2\pi\nu_m j \tag{2.14}$$

for the j-th turn with frequency equal to $\nu_m f_0$ and angular amplitude equal to a_m. Keeping only up to first order in Δ' and subtracting away the position of the synchronous particle, we obtain

$$\vec{\hat{X}}_N = e^{-i2\pi\nu_0(N+1)} i a_m \Delta' \left\{ S_0 \cos[\pi\nu_s(2N+1)+\varphi_s] - S_1 \right\} \ , \tag{2.15}$$

where, with $\nu_\pm = \nu_0 \pm m$,

$$S_0 = \sum_{j=1}^{N} \sin 2\pi\nu_m j e^{i2\pi\nu_0 j} = \frac{\sin \pi\nu_+ N}{2i \sin \pi\nu_+} e^{i\pi\nu_+(N+1)} - \frac{\sin \pi\nu_- N}{2i \sin \pi\nu_-} e^{i\pi\nu_-(N+1)} \ , \tag{2.16}$$

and

$$\begin{aligned} S_1 \quad &= \sum_{j=1}^{N} \sin 2\pi\nu_m j \cos(2\pi\nu_s j + \varphi_s - \pi\nu_s) e^{i2\pi\nu_0 j} \\ &= \frac{\sin \pi(\nu_+ + \nu_s)N}{4i \sin \pi(\nu_+ + \nu_s)} e^{i\pi(\nu_+ + \nu_s)(N+1) + i(\varphi_s - \pi\nu_s)} \\ &\quad + \frac{\sin \pi(\nu_+ - \nu_s)N}{4i \sin \pi(\nu_+ - \nu_s)} e^{i\pi(\nu_+ - \nu_s)(N+1) - i(\varphi_s - \pi\nu_s)} \\ &\quad - (\nu_+ \to \nu_-) \ . \end{aligned} \tag{2.17}$$

We learn from Section II.1 that this expression is in fact a superposition of the individual evolution due to synchrotron oscillation of a particle being kicked at each turn. Since we start out as one point in the transverse phase space with the longitudinally synchronous particle, all nonsynchronous particles should spread out and return to one point in N turns, where N is nearest to an integral number of synchrotron periods, or

$$N = n/\nu_s + \varepsilon \ , \tag{2.18}$$

where n is an integer and $|\varepsilon| < 1/2$. This can be checked by substituting Eq. (2.18) into Eq. (2.17), giving

$$S_1 = S_0 \cos\varphi_s \ , \tag{2.19}$$

where ν_s and $\varepsilon\nu_s$ have been dropped as compared to unity. This approximation is justified since the typical value of ν_s is $f_s/f_0 = 1.16 \times 10^{-3}$. It is then easy to see from Eq. (2.15) that the spread does contract back to a point after an integral number of synchrotron periods.

We believe the maximum spread will occur at half a synchrotron period and therefore evaluate Eq. (2.15) when N is equal to the closest integer of $(2n+1)/2\nu_s$. After that we again set $\nu_s = 0$. The result is

$$\begin{aligned}\vec{\hat{X}}_N &= \Delta' \cos\varphi_s e^{-i2\pi\nu_0(N+1)} \times \\ &\times \left\{ -\frac{\sin\pi\nu_+ N}{2\sin\pi\nu_+} e^{i\pi\nu_+(N+1)} - \frac{i\cos\pi\nu_+ N}{2\sin\pi\nu_+} e^{i\pi\nu_+(N+1)} - (\nu_+ \to \nu_-) \right\} . \quad (2.20)\end{aligned}$$

Since the kicking frequency is typically less than 20 Hz ($\nu_m \lesssim 5.81 \times 10^{-3}$) and the betatron tune ν_0 should never be too close to an integer, we can further simplify Eq. (2.20) by expanding in terms of ν_m, giving finally

$$\vec{\hat{X}}_N = \frac{\pi\nu_m a_m \Delta' \cos\varphi_s}{i\sin\pi\nu_+ \sin\pi\nu_-} e^{-i2\pi\nu_0(N+1)} . \qquad (2.21)$$

When the initial position $\vec{a}_0$ is included, we get

$$\vec{\hat{X}}_N = e^{-i2\pi\nu_0 N} \cos\varphi_s \Delta' \left[-2i\vec{a}_0 + \frac{i\pi\nu_m}{\sin\pi\nu_+ \sin\pi\nu_-} e^{-i2\pi\nu_0} \right] . \qquad (2.22)$$

The above describes the breathing oscillation of the bunch as illustrated in Fig. 3. We can see that the first exponential describes just the N-turn betatron oscillation of the synchronous particle and can therefore be dropped. The first term in the squared brackets is always perpendicular to radial vector $\vec{a}_0$ from the bunch center. It describes the synchronous oscillatory spread in the azimuthal direction and therefore does not contribute to the expansion or contraction of the bunch in the transverse phase space. The second term can give a spread radially and therefore contribute to the emittance growth. Equation (2.21) shows that $\vec{\hat{X}}_N = 0$ for particles with initial synchronous phase $\varphi_s = \pi/2$. This does not imply that these particles do not spread out. In fact, these particles spread out to a maximum at a quarter synchrotron period and contract back to the initial point at half a synchrotron period. These expansions and contractions are illustrated for some special cases in Fig. 4.

Thus a point bunch in the transverse phase space will develop into a circle of radius

$$\left| \vec{\hat{X}}_N \right| = a_m \beta \frac{\pi\nu_m \xi \delta_p}{\nu_s \sin^2 \pi\nu_0} , \qquad (2.23)$$

and contract back to a point after an integral number of synchrotron periods. Here, Eqs. (2.8) and (2.13) have been used. However if the betatron tune is dependent on

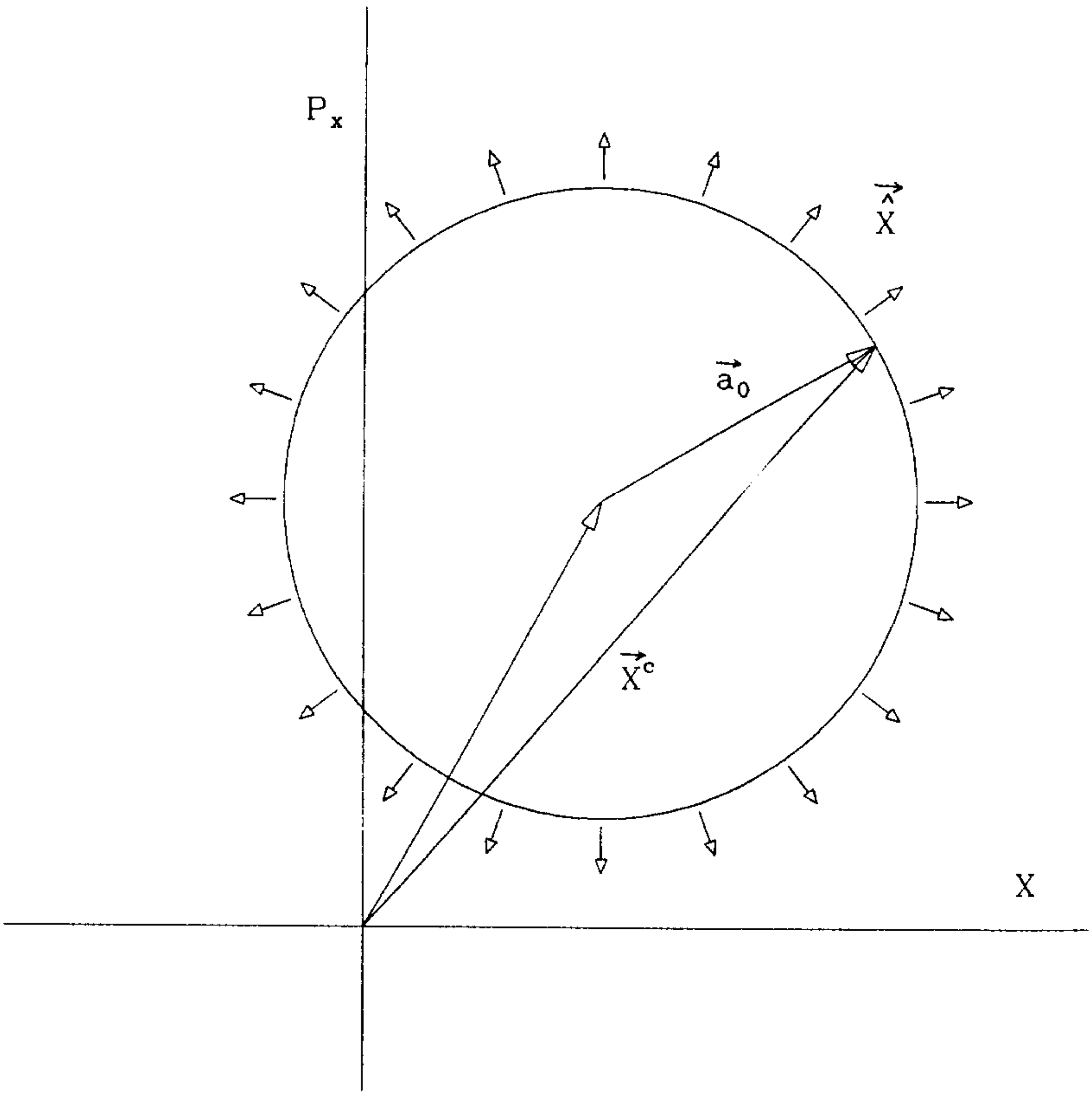


Figure 3: The circle is a bunch in the transverse phase space being kicked off-centered. The breathing motion is illustrated by the arrows.

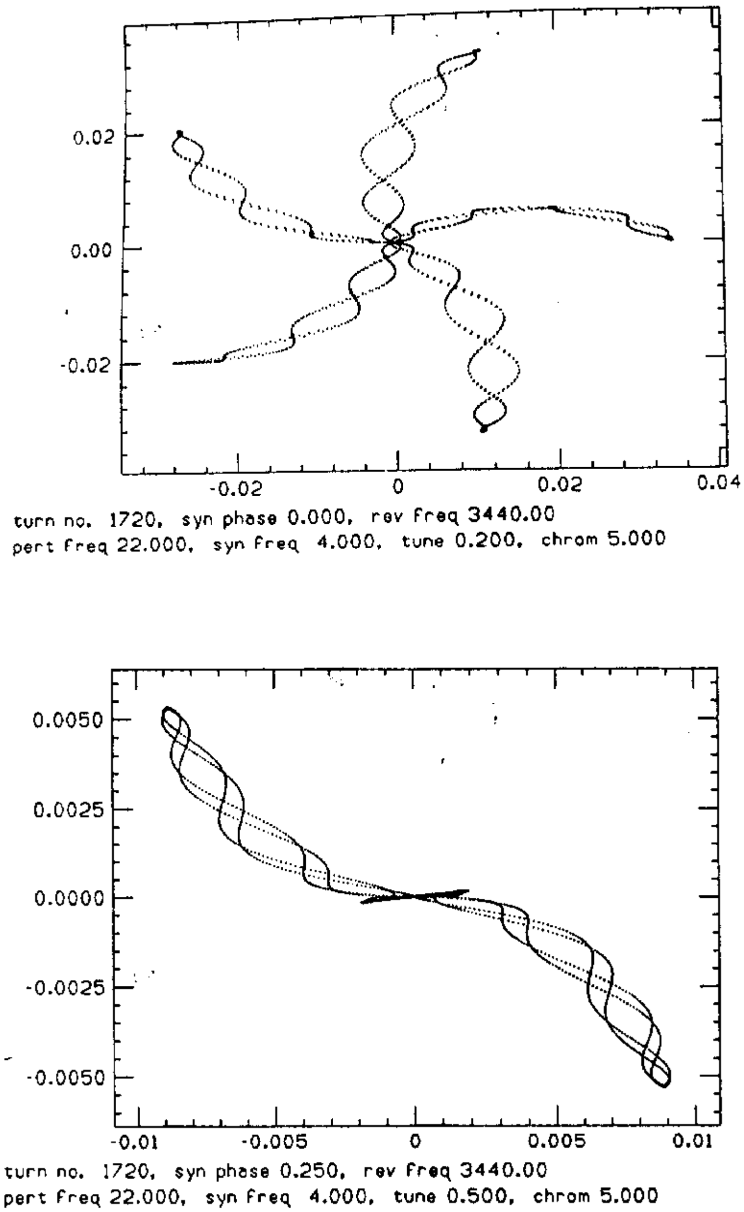


Figure 4: Simulations of the expansion from and contraction to a point in the transverse phase space.

amplitude and if the tunespread is large enough, the contraction will not come back to a point, resulting in a growth of emittance. Note that the growth is proportional to the incremental kick per turn as expected physically. If we start out with a bunch occupying a circle of radius $|\vec{a}_0|$, $|\vec{X}_N|$ describes the breathing of a displaced bunch. The displaced bunch itself will be treated in Section III.

II.4 Applications

1. Jostlein's Beam Centering

In Jostlein's beam-centering scheme,[1] one beam is rotated about the other at an interaction point, and the resulting variation in luminosity serves to measure the amount and direction by which the two beam centers miss each other. In this situation, each beam acts on the other like a moving quadrupole to first order. With the head-on beam-beam tuneshift $\Delta\nu_{bb} = 0.004$, and the betatron function at the interaction point $\beta_0 = 0.5$ m, the corresponding focal length of the equivalent quadrupole is $f_Q = \beta_0/4\pi\Delta\nu_{bb} = 10$ m. Note that in this consideration only the linear beam-beam effect has been taken into account. When one beam rotates in a circle (in the vertical plane) with a radius b_m and a frequency $f_m = \nu_m f_0$, the other beam is equivalent to being kicked periodically by an amplitude $a_m = b_m/f_Q$ according to Eq. (2.14). With a fractional betatron tune of 0.5 and a rotating frequency of 20 Hz, the maximum radially displacement is

$$\left|\vec{X}_N\right| = \frac{b_m\beta_0}{f_Q}\frac{\pi\nu_m\xi\delta}{\nu_s\sin^2\pi\nu_0} = 2.36\times10^4 b_m \ . \tag{2.24}$$

Since the perturbed amplitude b_m should be much less than the rms transverse size of the beam ($\sigma_0 = 5\ \mu$), this breathing of emittance is negligibly small.

2. Ground Motions

For the quarry approximately 9 miles away from the SSC rings, the vertical ground displacements due to its blasts measured at tunnel depth are shown in Fig. 2. The spectrum shows essentially a narrow resonance at 1 Hz and a relatively broader one at 3 Hz. Integrations were made around the two peaks to give the vertical ground displacements at the two frequencies. The result is

Frequency f_m	Displacement b_m
1 Hz	1.4 μ
3 Hz	0.1 μ

Each SSC collider ring consists of 90° cells of length $L = 228.5$ m. The focal lengths of the quadrupoles are therefore $f_q = L/4\sin 45° = 80.8$ m and $\beta = 390$ m at the F quad. If the quadrupole is displaced by b_m, the kick amplitude on the beam is $a_m = b_m/f_q$. We obtain by adding up the effect of ~1000 quadrupoles randomly,

$$|\vec{\hat{X}}_N| = \begin{cases} 0.040\ b_m & 1\ \mathrm{Hz} \\ 0.119\ b_m & 3\ \mathrm{Hz}\ , \end{cases} \tag{2.25}$$

which is pretty small compared with the rms bunch size of $\sigma_0\sqrt{\beta_x/\beta_0} = 63.6\ \mu$, especially the horizontal ground displacement b_m is believed to be about one-tenth of the vertical displacement.

For a train crossing the ring, the measured ground displacements at tunnel depth are shown in Fig. 2. Integration around the two peaks gives:

Frequency f_m	Displacement b_m
3 Hz	0.51 μ
7 Hz	0.58 μ

Due to the attenuation of the ground signal, we assume only 10 neighboring quadrupoles are affected and they contribute equally and constructively. Then

$$|\vec{\hat{X}}_N| = \begin{cases} 0.009\ b_m & 3\ \mathrm{Hz} \\ 0.017\ b_m & 7\ \mathrm{Hz}\ , \end{cases} \tag{2.26}$$

which will only lead to negligible growth in emittance.

III. TUNE DEPENDENCE ON AMPLITUDE

III.1 Hamiltonian

In this section, we turn off the synchrotron motion. Let us describe motion of a single particle in the transverse phase space by the Hamiltonian[4]

$$H = \frac{1}{2}p^2 + \frac{1}{2}\nu_0^2 x^2 - xF(\psi) , \tag{3.1}$$

where $x = X/\sqrt{\beta}$ is the normalized Floquet transverse displacement, p the canonical momentum, and $\psi = \int ds/\nu_0\beta$ the Floquet phase advance as well as the independent 'time' variable. The periodic perturbing force is

$$F(\psi) = \int d\nu_m \tilde{a}_m \sqrt{\beta}\nu_0 \sin \nu_m\psi \sum_{n=0}^{\infty} \delta(\psi - 2\pi n) , \tag{3.2}$$

where $\tilde{a}_m$ is the angular kick on the beam per unit ν_m. We next perform a canonical transformation to the action-angle variables (J, ϕ) with the aid of the generating function

$$F_1(x, \phi) = -\frac{1}{2}\nu_0 x^2 \tan\phi . \tag{3.3}$$

The transformed Hamiltonian is

$$H = \nu_0 J - \alpha\frac{4J^2}{\nu_0} - \sqrt{\frac{2J}{\nu_0}} \cos\phi F(\psi) , \tag{3.4}$$

The second term added to Eq. (3.4) introduces tune dependence on amplitude, which is, in the absence of the perturbing force $F(\psi)$,

$$\nu_\beta = \left.\frac{\partial H}{\partial J}\right|_{F(\psi)=0} = \nu_0 - \alpha A^2 , \tag{3.5}$$

where the betatron amplitude is given by $A = \sqrt{2J/\nu_0}$ and α is the detuning.

III.2 New Invariant Curves

In the absence of the perturbing force $F(\psi)$, the Hamiltonian is an invariant, implying that the particle stays on invariant curves in the x-p phase space. These curves are, in fact, circles of constant radii J. With the perturbing force having an amplitude very much less than the betatron amplitude, or

$$\int d\nu_m \tilde{a}_m \sqrt{\beta} \ll \sqrt{\frac{2J}{\nu_0}} \;, \tag{3.6}$$

we assume that the system remains integrable, at least approximately. We try to solve for the new invariant curves for each perturbing frequency ν_m. For this, we define the *integrated* perturbing amplitude for this particular frequency as

$$a_m \equiv \tilde{a}_m d\nu_m \;, \tag{3.7}$$

which is exactly the same a_m as in Eq. (2.14). The equation of motion for J is

$$\frac{dJ}{d\psi} = -\frac{\partial H}{\partial \phi} = -\sqrt{\frac{2J}{\nu_0}} \sin\phi F(\psi) \;. \tag{3.8}$$

Since only solution up to first order in a_m is required, we need to solve equation of motion for ϕ

$$\frac{d\phi}{d\psi} = \frac{\partial H}{\partial J} = \nu_0 + \mathcal{O}\left(\Delta\nu_\beta, \frac{a_m\sqrt{\beta}}{A}\right) \;, \tag{3.9}$$

to zeroth order only. At the same time, the nonlinear tunespread due to detuning $\Delta\nu_\beta$, which is assumed to be small compared with the nominal tune ν_0, is dropped also. Thus, approximately,

$$\phi = \phi_0 + \nu_0\psi \;. \tag{3.10}$$

Now Eq. (3.8) can be integrated easily to give

$$\begin{aligned} \sqrt{\frac{2J_N}{\nu_0}} - \sqrt{\frac{2J_0}{\nu_0}} &= -\sum_{n=0}^{N} a_m\sqrt{\beta}\sin(\phi_0 + 2\pi N\nu_0)\sin 2\pi N\nu_m \\ &= \frac{a_m\sqrt{\beta}}{2}\sum_{n=0}^{N}\left[\cos(\phi_0 + 2\pi n\nu_+) - \cos(\phi_0 + 2\pi n\nu_-)\right] \;, \end{aligned} \tag{3.11}$$

where J_0 and J_N are, respectively, the actions of the particle after the 0-th and N-th turn, and we have defined $\nu_\pm = \nu_0 \pm \nu_m$. To find the invariant curves, we should look at the position of the particle every perturbation period starting from the n_0-th turn. In other words, there is a set of invariant curves for every n_0. Therefore, we let

$$N = n_0 + \frac{\bar{n}}{\nu_m} \qquad \bar{n} \text{ an integer} \;. \tag{3.12}$$

Summing the cosine series, Eq. (3.11) becomes

$$\sqrt{\frac{2J_N}{\nu_0}} - \sqrt{\frac{2J_0}{\nu_0}} = \frac{a_m\sqrt{\beta}}{4\sin\pi\nu_+\sin\pi\nu_-}\left\{\cdots\right\} , \tag{3.13}$$

where

$$\begin{aligned}\left\{\cdots\right\} = & \sin 2\pi n_0\nu_m[\sin(\phi_0+2\pi N\nu_0)(\cos 2\pi\nu_0-\cos 2\pi\nu_m)+\cos(\phi_0+2\pi N\nu_0)\sin 2\pi\nu_0] \\ & -\cos 2\pi n_0\nu_m \sin 2\pi\nu_m \sin(\phi_0+2\pi N\nu_0) + \sin\phi_0 \sin 2\pi\nu_m \ . \end{aligned} \tag{3.14}$$

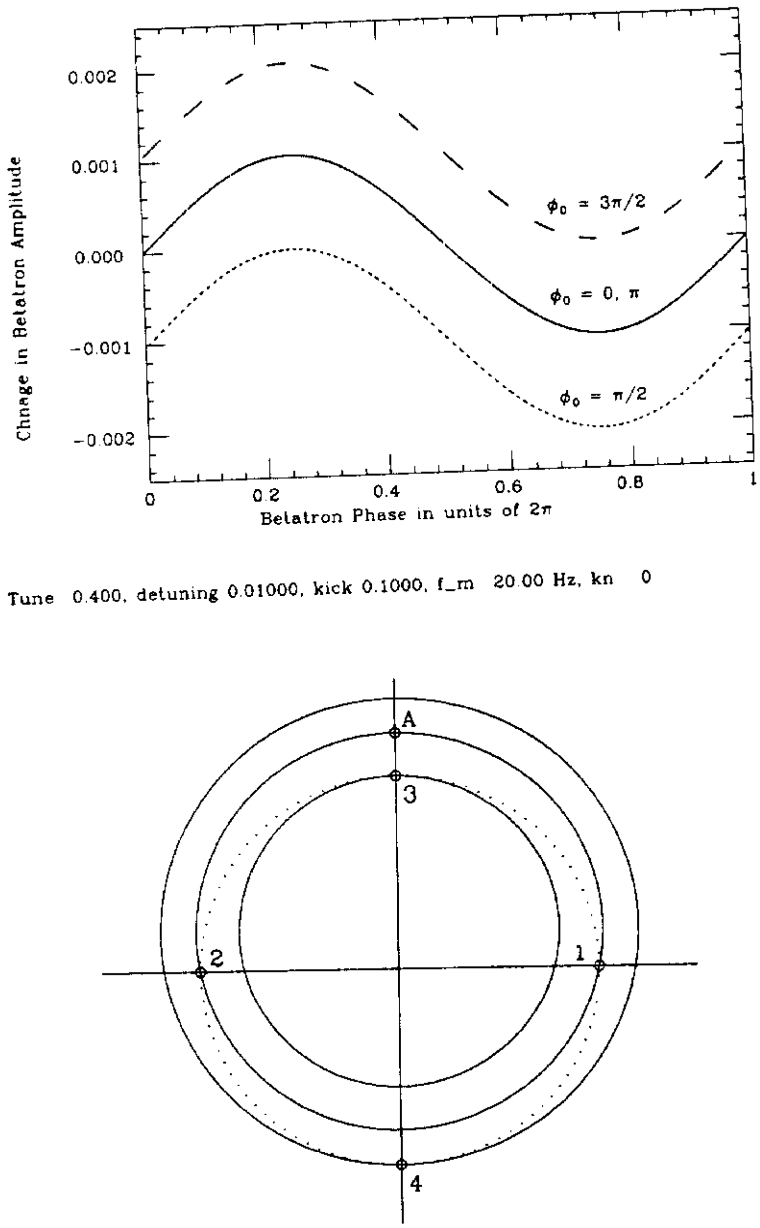
If we look at the invariant curves every $1/\nu_m$ turns starting from turn zero ($n_0 = 0$), Eq. (3.13) simplifies to

$$\sqrt{\frac{2J_N}{\nu_0}} - \sqrt{\frac{2J_0}{\nu_0}} = -\frac{a_m\sqrt{\beta}\sin 2\pi\nu_m}{4\sin\pi\nu_+\sin\pi\nu_-}[\sin\phi - \sin\phi_0] , \tag{3.15}$$

where we have substituted $\phi \approx \phi_0 + 2\pi N\nu_0$ according to Eq. (3.10).

To check the existence of invariant curves when the tunespread $\Delta\nu_\beta$ in Eq. (3.9) is included, we perform a turn-by-turn simulation according to Eq. (2.6) with the kick at the j-th turn given by Eq. (3.2). We followed four particles which were placed initially on the unit circle in the phase plane, at phases $\phi_0 = 0,\ \pi/2,\ \pi$, and $3\pi/2$. The nominal tune was $\nu_0 = 0.4$. The perturbing frequency was $f_m = 20$ Hz, or $\nu_m = 1/172$ for the SSC collider whose revolution frequency is $f_0 = 3.440$ kHz. The kick amplitude was taken as $a_m = 0.1$ unit and the detuning varied from 0.00 to 0.01 unit. The amplitude-phase plots are displayed in Fig. 5a after tracking for 5×10^4 turns. The plots demonstrate the existence of the invariance curves for the four particles even when the tunespread and higher-order of a_m are included. Also the results are as expected from Eq. (3.11). These invariant curves are plotted in the phase plane in Fig. 5b.

Of course, the existence of new invariant curves does not necessarily imply the increase in emittance. It is the spread in betatron tune that leads to a smear and then the emittance increase. The eventual emittance will be given by the area of the closed invariant curve of particle 4 (see Fig. 5b). When particle 1 leads particle 4 by slightly more than $\pi/2$ arriving at point A with particle 4 remaining at the original position, the fractional increase in area will reach roughly half the maximum. The



(a)

(b)

Figure 5: Invariant curves for particles 1, 2, 3, and 4 are plotted amplitude-versus-phase in (a) and in the x-p phase plane in (b). The particles are marked on the dotted unit circles in (b) representing the initial invariant curve for all the 4 particles.

number of turns $N_{\frac{1}{2}}$ required is given by

$$\Delta\nu_\beta = \frac{d\nu_\beta}{dA}\Delta A_{\frac{1}{2}} N_{\frac{1}{2}} \sim \frac{1}{4} \ , \tag{3.16}$$

where $\Delta A_{\frac{1}{2}}$ is the amplitude difference between particles 1 and 4. With the aid of Eq. (3.5),

$$N_{\frac{1}{2}} \sim \frac{1}{8\alpha A_0 \Delta A_{\frac{1}{2}}} \ . \tag{3.17}$$

The maximum fractional increase in emittance (or area) can be obtained from either Eq. (3.15) or Eq. (3.13) for all n_0, by integrating ΔA^2 over $\frac{1}{2}\, d\phi$ and dividing by πA_0^2:

$$\frac{\Delta\epsilon}{\epsilon} \approx \frac{a_m \sin 2\pi\nu_m}{2A_0 \sin^2 \pi\nu_0} \ , \tag{3.18}$$

where $\nu_\pm$ in the denominator has been replaced by ν_0. These predictions had been confirmed by a simulation for 10^7 turns. We want to point out that the fractional growth in emittance, as depicted in Eq. (3.18), is in fact proportional to the incremental kick per turn $a_m \nu_m$.

The invariant curves for different values of n_0 are given by Eq. (3.14). Take for example the situation of largest kick, $n_0 = 1/4\nu_m$. The shift in amplitude becomes

$$\sqrt{\frac{2J_N}{\nu_0}} - \sqrt{\frac{2J_0}{\nu_0}} = \frac{a_m\sqrt{\beta}\cos(\phi + \pi\nu_0)}{2\sin\pi\nu_0} \ , \tag{3.19}$$

where we have replaced $\phi_0 + 2\pi N\nu_0$ by ϕ. The shift in amplitude in Eq. (3.15) when $n_0 = 0$ is of $\mathcal{O}(a_m\sqrt{\beta}\nu_m)$. However, when $n_0 = 1/4\nu_m$, the shift is of $\mathcal{O}(a_m\sqrt{\beta})$ which is $1/\nu_m$ times bigger.

III.3 Applications

1. Jostlein's Beam-centering Scheme

One beam rotates in a circle with radius b_m. The other beam has an initial amplitude $A_0 = X_{\max}/\sqrt{\beta_0}$ where $X_{\max}$ is the maximum transverse radius of the bunch. The fractional growth in emittance from Eq. (3.18) becomes

$$\frac{\Delta\epsilon}{\epsilon} = \frac{b_m}{X_{\max}} \frac{\beta_0}{2f_Q} \frac{\sin 2\pi\nu_m}{\sin^2\pi\nu_0} = 1.0 \times 10^{-3} \frac{b_m}{X_{\max}} \ , \tag{3.20}$$

which is indeed very small since we must choose $b_m \ll X_{\max}$ in practice. The parameters given in Section II.5 had been used. If the Jostlein's modulation is switched off abruptly, say, at turn number corresponding to $n_0 = 1/4\nu_m$ when the modulation amplitude is largest, the beam will eventually smear out due to nonlinear tunespread. The growth in emittance will therefore be derived from Eq. (3.19) instead, giving

$$\frac{\Delta\epsilon}{\epsilon} = \frac{b_m}{X_{\max}} \frac{\beta_0}{f_Q} \frac{1}{\sin \pi\nu_0} = 0.026 \frac{b_m}{X_{\max}} \; , \tag{3.21}$$

which is 26 times larger.

We can estimate the growth time. For the SSC, a typical value for nonlinear detuning is $\mu = 48.0\ \mathrm{m}^{-2}$ found by Yan[5] in simulations using a full spectrum of random errors. If we use $\beta = 390$ m, a value at the F quad, this translates into our detuning $\alpha = \mu\beta = 1.87 \times 10^4\ \mathrm{m}^{-1}$. At 20 TeV, the rms bunch size is about 0.12 mm at the F quad, thus $A_0 = 6.07 \times 10^{-6} \mathrm{m}^{\frac{1}{2}}$. However, this smearing time is extremely long. According to Eq. (3.14), it takes roughly $(\alpha A_0 \Delta A)^{-1} = 1.1 \times 10^8 \times (X_{\max}/b_m)$ turns or $9.0 \times (X_{\max}/b_m)$ hours. Therefore, the offset bunch can always be kicked back easily to the ideal closed orbit by an active kicker and no emittance growth due to nonlinear tunespread will occur.

2. Ground Motions

If we use the ground displacement due to quarry blasts as listed in Section II.5, the beam modulation amplitudes for the SSC are, respectively, for 1 Hz and 3 Hz,

$$a_m\sqrt{\beta} = \frac{b_m\sqrt{\beta}}{f_q} = \begin{cases} 3.50 \times 10^{-7}\ \mathrm{m}^{\frac{1}{2}} \\ 2.64 \times 10^{-7}\ \mathrm{m}^{\frac{1}{2}} \end{cases} \tag{3.22}$$

According to Eq. (3.18), the fractional growth in emittance is 0.0060, where a factor of $\sqrt{1000}$ has been included to account for the ~ 1000 quadrupoles in the collider ring. The time required to reach half maximum, estimated from Eq. (3.17), gives 1.2×10^8 turns or 9.7 hours. Both the ground-wave peaks at 1 Hz and 3 Hz have a full width of about 1 Hz, corresponding to a correlation time of $\tau \sim 2$ sec, for which the growth is extremely tiny. The quarry blast usually lasts for only 30 sec. The total growth is still negligibly small.

However, at the end of a correlated wave, the beam can be kicked off-center, resulting in emittance growth due to nonlinear tunespread. If we average over the n_0 in Eq. (3.12), the average amount of off-center shift after the abrupt end of a correlated wave is

$$\langle \Delta A \rangle = \sum \frac{a_m \sqrt{\beta}}{\pi \sin \pi \nu_0} \times \sqrt{1000} = 6.50 \times 10^{-6} \text{ m}^{\frac{1}{2}} \ , \tag{3.23}$$

which is of the same order of magnitude as A_0, the original size of the bunch. The smearing time is found to be 4.8×10^5 turns or 140 sec. Thus, an active damper can always be used to kick the beam back to its ideal orbit avoiding any nonlinear smearing.

For the train, according to the measured displacements listed in Section II.5, the fractional growth in emittance is 0.0023, where a factor of 10 has been included to represent the assumption that 10 nearby quadrupoles are affected by the train and they contribute equally. It takes 3.12×10^8 turns or about 25 hours to reach half maximum. A one-mile train traveling at 30 mph will take about 120 sec to cross the ring. As a result, the growth should be negligibly small.

The peak at 1 Hz has a full width of 1 Hz and the one at 7 Hz has a full width of 12 Hz. The correlation time for the two frequencies are therefore 2 and 0.17 sec, respectively. Again abrupt stopping of a correlated wave will throw the beam off-center. But because of the small nonlinear tune spread, smearing can be avoided by an active damper.

IV. LONGITUDINAL EMITTANCE GROWTH

IV.1 Nonzero Momentum Dispersion

We now consider the situation when the kicked quadrupole is located at a place where the momentum dispersion is nonzero and see how the horizontal kick will affect the longitudinal phase space.[6] The periodic kick (amplitude b_m) on a quadrupole (focal length f_q) at the j-th turn produces the angular kick on the beam

$$\theta_j = \frac{b_m}{f_q} \sin 2\pi \nu_m j \ . \tag{4.1}$$

The incremental kick from the $(j-1)$-th to j-th turn

$$\Delta\theta_j = \theta_j - \theta_{j-1} \tag{4.2}$$

leads to a change in revolution orbit length of

$$\Delta C_j = \Delta\theta_j D \tag{4.3}$$

for the j-th turn, where D is the momentum dispersion at the quadrupole. The synchronous particle will arrive at the rf cavity late by the rf angle

$$\Delta\psi_j = 2\pi\frac{\Delta C_j}{\lambda_{\mathrm{rf}}} = \frac{2\pi h D}{C_0}\Delta\theta_j \ , \tag{4.4}$$

where λ_{rf} is the rf wavelength, h the rf harmonic, and C_0 is the ideal orbit length. If this late arrival accumulates, is it possible that the bunch area will increase and the bunch particles will eventually go out of the rf bucket?

The turn-by-turn equations of motion for the rf phase ϕ_j and fractional energy offset δ_{pj} at the j-th turn are given by

$$\begin{aligned} \frac{d\phi_j}{dn} &= 2\pi\eta h\delta_{pj} + \psi_j \ , \\ \frac{d\delta_{pj}}{dn} &= -\frac{eV}{E}\phi_j \ , \end{aligned} \tag{4.5}$$

where η is the phase-slip parameter, V the rf voltage (for a stationary bucket), and E the synchronous energy. In the above, the accumulated rf phase lag or mismatch due to the kick is

$$\psi_j = \sum_{i=1}^{j} \Delta\psi_j = \frac{2\pi h D}{C_0}\theta_j \ . \tag{4.6}$$

We want to point out that it is ψ_j and *not* $\Delta\psi_j$ that enters into the first equation of motion.

IV.2 Constant kick

Consider a misplaced quadrupole. It gives the beam a constant turn-by-turn kick θ. This just implies a new closed orbit with an extra length if $\Delta C = \theta D$. Thus, for every turn, there is an increase of phase lag

$$\psi = \frac{2\pi h D}{C_0}\theta \ . \tag{4.7}$$

But the focussing effect of the rf prevents the phase lag from accumulating. Instead, it shifts the synchronous center to a lower energy $\delta_{pc} = -\psi/2\pi\eta h$ so as to accommodate a shorter orbit length. This is illustrated in Fig. 6.

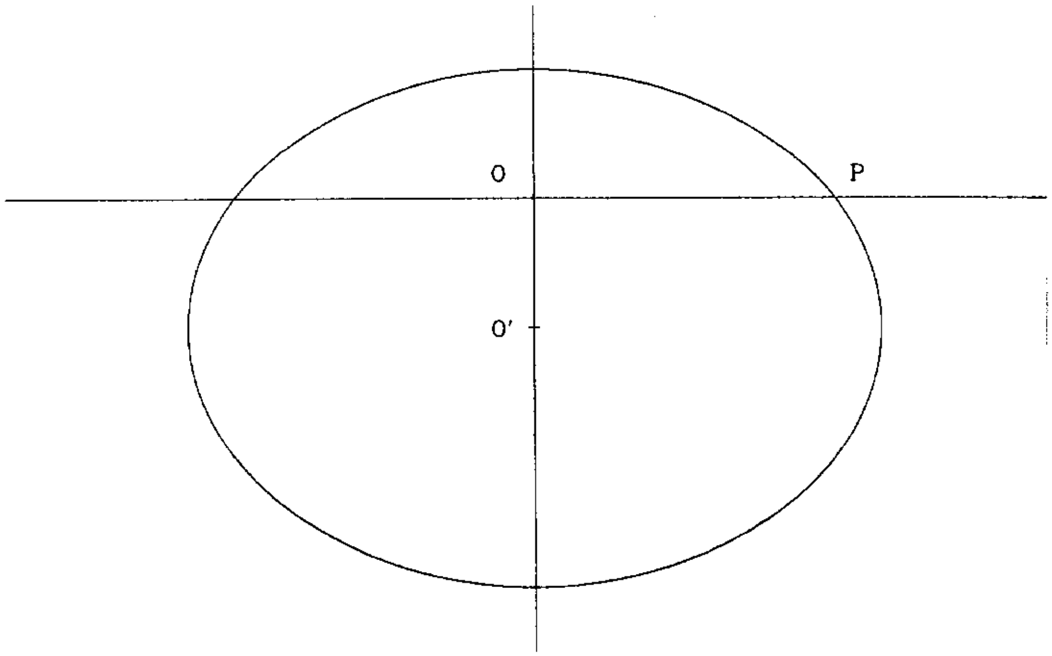


Figure 6: With a kick of the same size every turn, the previously synchronous particle is driven into synchrotron motion about a new synchronous center O' which is $\psi/2\pi\eta h$ below the original synchronous center O. The point P is at a constant phase lag ψ.

IV.3 Periodic kick

If we use time t as the independent variable, the equations of motion (4.5) becomes

$$\begin{aligned} \frac{d\phi}{dt} &= \eta h \omega_0 \delta_p + \frac{\omega_0 h D A}{C_0 f_q} \sin(\omega_m t) , \\ \frac{d\delta_p}{dt} &= -\frac{eV\omega_0}{2\pi E}\phi , \end{aligned} \tag{4.8}$$

with the initial condition $\phi = \phi_0$ and $\delta_p = \delta_{p0}$ at $t = 0$. We introduce the new variable

$$\bar{\phi} = \frac{eV}{2\pi E \nu_s}\phi = \frac{\nu_s}{\eta h}\phi , \tag{4.9}$$

so that the elliptic trajectory becomes circular. Using Laplace transform, we obtain the solution in complex notation

$$(\bar{\phi}, \delta_p) = \vec{a}_0 e^{-i2\pi\nu_s N} + \frac{i\nu_\Omega^2}{\nu_s}\frac{\sin\pi(\nu_m - \nu_s)N}{\nu_m - \nu_s} e^{-i2\pi\nu_s N} - i\frac{\nu_\Omega^2}{\nu_s}\frac{\cos\pi(\nu_m - \nu_s)N}{2\nu_s}\sin(2\pi\nu_s N)\ , \tag{4.10}$$

where

$$\nu_\Omega^2 = \frac{2\pi\nu_s^2 D b_m}{\eta C_0 f_q}\ , \tag{4.11}$$

and the approximation that the perturbing tune ν_m is close to the synchrotron tune ν_s has been used. In the above, the first term is the evolution of the initial position $\vec{a}_0 = (\bar{\phi}_0, \delta_{p0})$ of a particle in the bunch. The third term shifts the center of rotation upward and downward, which is equivalent to the shift of the synchronous center in the case of a constant kick discussed in the last section. The second term represents the synchronous oscillation of the particle in the bunch with an amplitude due to the periodic kick. This term can become very big because of the small denominator. If the rf bucket is big enough, there is still no increase in bunch area. It is the dependence of synchrotron tune on amplitude that smears out the bunch leading to increase in bunch area.

IV.4 Estimation and Discussions

If we take $b_m = 0.1\ \mu$ for the horizontal kick amplitude due to ground motion (quarry blast) at a frequency $f_m = 3$ Hz, with the frequency-flip factor $\eta = 1.1\times10^{-4}$, rf harmonic $h = 1.05\times10^5$, and synchrotron frequency $f_s = 4$ Hz, a point bunch in the longitudinal phase space can smear to a size with

$$\Delta\delta_p \approx \frac{2\pi b_m}{f_q}\frac{f_s}{|f_m - f_s|} = 3.12\times10^{-7}\ ,$$

$$\Delta\phi = \frac{\eta h}{\nu_s}\Delta\delta_p = 3.08\times10^{-3}\ \text{rf rad}\ . \tag{4.12}$$

For 1000 quadrupoles, just multiply the result by $\sqrt{1000}$.

We can draw the following conclusions:

(1) Unless the perturbing frequency is exactly the same as the synchronous frequency and is *locked on* to it exactly, our estimation shows that there is not a continuous

growth.

(2) If the perturbation is a continuum $\mu(\omega_m)$, which centers at the synchrotron frequency ω_s and is narrow, the solution of the equations of motion (4.8) gives

$$(\tilde{\phi}, \delta_p) \approx \frac{i\pi\nu_\Omega^2\omega_0^2}{2\omega_s}\mu(\omega_s)e^{-i2\pi\nu_s N} \ . \tag{4.13}$$

So there is no resonance with the synchrotron motion.

(3) If the perturbation is random, the above treatment does not apply and the kick can be in resonance with synchrotron oscillation.

(4) With the synchrotron frequency dependent on amplitude, the story may be very different.

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