Fermi National Accelerator Laboratory

FN-539

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April 2, 1990

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

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Abstract

In this paper we investigate the limits on beam intensity in low energy proton synchrotrons due to the instability of the beam envelope. This instability has been previously examined in the context of space charge dominated ion beams, in particular for inertial confinement fusion applications. We generalize the formalism of beam envelope evolution to include effects found in circular accelerators, *i.e.* curvature focusing and momentum dispersion. Several example lattices are analyzed to determine intensity limits imposed by onset of envelope instability; it is found that the instability can occur without depressing the phase advance below 90 degrees per cell.

Introduction

The intensity limit for operation of low energy proton synchrotrons is often determined by the space charge tune shift[1], which can drive individual particle tunes from the operating point onto a significant machine resonance. In this way a relatively small tune shift (usually a few percent) can cause the emittance of a synchrotron beam to grow significantly. Experimentally, this effect manifests itself by a linear asymptotic growth of the emittance with peak beam current. This type of behavior has been observed in experimental studies of the Fermilab Booster[2].

In contrast to scenario described above, linear transport allows much higher intensity beams to propagate without degradation of the beam emittance. Space-charge tune shifts which are nearly as large as the bare (zero-intensity) tune can be tolerated given proper conditions in linear transport channels, as the beam travels much shorter distances in general, and does not encounter the periodic resonance-driving defects found in a circular machine. The limits of stable beam propagation in a linear focusing channel have been demonstrated experimentally by Tiefenback, *et al.* [3] at LBL. It was found that as long as the bare phase advance per cell σ_0 did not exceed approximately 90 degrees, the beam emittance was stable up to very high intensities. At phase advances above 90 degrees, however, it was verified that instability of the beam envelope caused emittance growth in the transmitted beam. This result is in agreement with the theoretical predictions of Hoffman, Laslett, Smith and Haber[4]. As many synchrotrons are designed with bare phase advance per cell near 90 degrees to minimize the necessary aperture in the machine, it is reasonable to ask whether this effect is important for weaker intensity beams found in circular machines. In particular, this question was raised with regards to the Fermilab Booster, both in its present state and after the upgrade in injected energy from the linac[5].

We review below the standard derivation of rms transverse beam envelope equation, and extend it to include the effects of design orbit curvature, momentum spread and dispersion. The envelope equation obtained is then analyzed for its stability properties under perturbation. This technique is used to examine existing and proposed machines – the SSC Low Energy Booster, the Fermilab Booster, and the TRIUMF Kaon Factory Booster – using their nominal lattices, intensities, emittances and momentum spreads, to determine possible problems with envelope instability in these devices.

The Envelope Equation

The first envelope equations for describing beam propagation including the effects of the repulsive space charge self-forces were derived by Kapchinskij and Vladimirskij (K-V)[6] using a special distribution function (uniformly distributed population on a three-dimensional hyper-ellipsoidal surface in four-dimensional transverse phase space) which yields a uniform spatial density distribution within the elliptical beam boundary. This uniform density distribution yields linear space charge forces, thus allowing a straightforward extraction of equations for the evolution of the beam envelope in a focusing channel,

$$a'' + K(z)a = \frac{\epsilon_x^2}{a^3} + \frac{Q}{a+b} \tag{1}$$

for the horizontal beam size a, and

$$b'' - K(z)b = \frac{\epsilon_y^2}{b^3} + \frac{Q}{a+b} \tag{2}$$

for the vertical beam size b, where the focusing strength $K = (\partial B/\partial x)(B\rho)^{-1}$ is positive for a horizontally focusing magnetic quadrupole, $\epsilon_{x,y}$ is the total horizontal/vertical emittance, taken by convention to be four times the rms emittance, and $Q = 4I/(I_0\beta^3\gamma^3)$ with $I_0 \simeq 31$ MA for protons.

In order to avoid use of unphysical K-V distribution function, Sacherer generalized this treatment by concentrating on the evolution of the rms envelopes[7]. This approach is equivalent to following the second moments of the Vlasov equation for the four-dimensional transverse phase space of the beam. Assuming no x-y coupling, the second moments in the (x, p_x) plane, where $p_x \equiv x'$, satisfy the following equations:

$$\overline{x^2}' = 2\overline{xp_x} \tag{3}$$

$$\overline{xp_x}' = \overline{xp_x'} + \overline{p_x^2} \tag{4}$$

$$\overline{p^2}' = 2\overline{p_x p_x'}. \tag{5}$$

Analogous equations apply in the (y, p_y) plane. Following Sacherer, we formally divide the force term $p_x' \equiv F_x/\beta^2\gamma mc^2$ into a linear component due to the focusing elements and a part due to the space charge self-force $\mathcal{F}_x$. Eqs.

3– 5 thus become

$$\overline{x^2}' = 2\overline{xp_x} \tag{6}$$

$$\overline{xp_x}' = -K(z)\overline{x^2} + \overline{x\mathcal{F}_x} + \overline{p_x^2} \tag{7}$$

$$\overline{p^2}' = -2K(z)\overline{xp_x} + 2\overline{p_x\mathcal{F}_x}. \tag{8}$$

The term $\overline{p_x\mathcal{F}_x}$ is related to emittance growth, and can be ignored if one assumes that the rms emittance

$$\epsilon_{rms} = \sqrt{\overline{x^2}\,\overline{p_x^2} - (\overline{xp_x})^2} \tag{9}$$

is either a constant or that its z dependence is a known function. If, in addition, one assumes elliptical symmetry for the beam spatial density, the term $\overline{x\mathcal{F}_x}$ depends only on the rms beam dimensions $\mathcal{X} \equiv \sqrt{\overline{x^2}}$ and $\mathcal{Y} \equiv \sqrt{\overline{y^2}}$,

$$\overline{x\mathcal{F}_x} = \frac{\mathcal{X}Q}{\mathcal{X}+\mathcal{Y}}, \tag{10}$$

where we redefine $Q \rightarrow Q/4 = I/(I_0\beta^3\gamma^3)$, In this way the hierarchy of moment equations is closed. With the additional rescaling of $\epsilon \rightarrow \epsilon/4 = \epsilon_{rms}$, the rms envelope equations, derived by differentiation of Eq. 6 and use of Eqs. 7 and 9 (along with the equivalent expressions for $\mathcal{Y}$), can be written in form identical to the K-V equation,

$$\mathcal{X}'' + K(z)\mathcal{X} = \frac{\epsilon_x^2}{\mathcal{X}^3} + \frac{Q}{\mathcal{X}+\mathcal{Y}}, \tag{11}$$

and

$$\mathcal{Y}'' - K(z)\mathcal{Y} = \frac{\epsilon_y^2}{\mathcal{Y}^3} + \frac{Q}{\mathcal{X}+\mathcal{Y}}. \tag{12}$$

All envelopes and emittances in these equations are now rms quantities; the K-V equations in rms form are contained in Eqs. 11 - 12 as a special case.

Envelope Instability

Solutions to Eqs. 11 - 12 must generally be found by numerical methods. In the case of periodic focusing structures, application of periodic boundary conditions to these equations gives the matched solution for the rms beam envelopes. Beams will always have slight mismatches to the lattice, however, and the question remains as to whether the deviations from the matched solutions are stable. To answer this concern, a perturbation analysis is performed, which entails writing the perturbed envelope equations obtained by first order Taylor expansion about the matched solutions. The eigenvalues of this system of linear equations are then examined to see if they indicate instability by moving off of the unit circle on the complex plane. These instabilities occur when two eigenvalues collide on the unit circle. If they collide at 180 degrees (on the real axis) then this is a half-integer resonance between the perturbed envelope mode and the focusing structure. This case is a termed "parametric" resonance. If both pairs of eigenvalues collide off the real axis, then this is a "confluent" resonance between the two envelope frequencies.

In the previously considered example of a symmetric FODO lattice, the resonances are mainly confluent[8]. Necessary conditions for envelope instability in this case are (i) bare phase advance per cell $\sigma_0 > 90$ degrees, and (ii) depressed phase advance per cell $\sigma < 90$ degrees. In terms of design, therefore, it is advisable to choose a FODO lattice with $\sigma_0 < 90$ degrees, or failing that to make the phase advance much larger than 90 degrees. In machines with small space charge tune depression, the only way to invite trouble with the envelope instability is to design for σ_0 just above 90 degrees. This conclusion, which was reached for symmetric FODO lattices in linear transport lines, is partially born out in our analysis of the envelope instability in circular machines presented below.

The Envelope Equation: Circular Machines

The major changes to beam behavior in circular machines which have relevance to deriving envelope equations are due to curvature focusing, which generally serves to break the horizontal-vertical symmetry of lattice, and to the presence of momentum dispersion, which causes the horizontal beam profile to expand. The curvature effects on the horizontal motion merely add to the horizontal focusing strength due to the quadrupole fields while leaving the vertical focusing strength unchanged,

$$K_x = K + \frac{1}{\rho^2}\ ,\ K_y = -K. \tag{13}$$

In addition, in circular machines the x and y phase advances are are chosen to be different to avoid coupling resonances. Thus the focusing strengths of the F and D lenses are usually different. These changes can be trivially incorporated into the envelope equations.

The inclusion of the effects of momentum dispersion are not nearly as straightforward, and require careful treatment. The standard way of describing off-momentum orbits in accelerators is to split the motion into a betatron component x_β (the quantity we have simply termed x to this point) and a dispersion component $\eta(\Delta p/p)$. Thus we now write, for the purpose of examining rms quantities

$$x = x_\beta + \eta\frac{\Delta p}{p}, \tag{14}$$

$$p_x = x'_\beta + \eta'\frac{\Delta p}{p}. \tag{15}$$

The equation of motion for the dispersion function is taken to be

$$\eta'' + K_x(z)\eta = \frac{1}{\rho}. \tag{16}$$

This approach is only approximate in the presence of space charge, which introduces coupling that prevents this simple decomposition of the description of the motion. We have also ignored the chromatic nature of the focusing in this treatment. The rms horizontal beam width is now, using Eq. 14,

$$X = \sqrt{\mathcal{X}^2 + \eta^2\sigma_p^2}, \tag{17}$$

where $\sigma_p = \overline{(\Delta p/p)^2}$.

By repeating the above analysis on the evolution of second moments we can derive a new set of envelope equations for circular accelerators

$$X'' + [K(z) + \frac{1}{\rho^2} - \frac{\eta}{\rho}(\frac{\sigma_p}{X})^2]X = \frac{\epsilon_x^2}{X^3} + \frac{Q}{X+\mathcal{Y}}, \tag{18}$$

and

$$\mathcal{Y}'' - K(z)\mathcal{Y} = \frac{\epsilon_y^2}{\mathcal{Y}^3} + \frac{Q}{X+\mathcal{Y}}. \tag{19}$$

The left hand side of any of these expressions is equivalent to the single particle equations of motion summed in squares over all momenta. The right hand side contains the self-force term and the emittance term. The self-force term is obtained immediately from Sacherer's results. The vertical emittance is of course unchanged, but the horizontal emittance now takes the functional form

$$\epsilon_x^2 = \epsilon_x^2[1 + (\frac{\eta\sigma_p}{\mathcal{X}})^2] + \sigma_p^2(\mathcal{X}\eta' - \mathcal{X}'\eta)^2. \tag{20}$$

Use of this equation again implies that we are ignoring possible emittance growth arising from the final term in Eq. 8; in the absence of space charge ($Q = 0$) Eq. 20 is exact.

This treatment obviously has a limited range of validity; the separability of motion into betatron and dispersion components is only approximate in the presence of space charge. We thus expect that our envelope equations are valid only in the limit of small tune shifts, which fortunately serves our present purposes well.

Equations 18 and 19 are now used to provide the basis of a perturbation analysis for several example lattices of low energy booster proton synchrotrons at injection, but including the effect of the beam bunching (bunching factor) on the peak current. This is often where the space charge tune shifts are most important in a circular accelerator chain.

Envelope Instability in Circular Machines

The perturbed envelope equations that are used to test for envelope instability are, expanding Eqs. 18 and 19 about the matched solutions for $(X, \mathcal{Y})$,

$$\delta X'' + [K(z) + \frac{1}{\rho^2} + \frac{\eta}{\rho}\frac{\sigma_p^2}{X^2} + \frac{3\varepsilon_x^2}{X^4} + \frac{Q}{(X+\mathcal{Y})^2}]\delta X = 0, \tag{21}$$

and

$$\delta \mathcal{Y}'' + [-K(z) + \frac{3\epsilon_y^2}{\mathcal{Y}^4} + \frac{Q}{(X+\mathcal{Y})^2}]\delta \mathcal{Y} = 0. \tag{22}$$

Note that, in the spirit of Sacherer's treatment, we do not perturb the terms inside the emittance expression.

The method of analysis proceeds by converting Eqs. 18 and 19 to the equivalent system of first order differential equations and solving them numerically subject to the appropriate periodic boundary conditions. In our case we have employed a shooting method with a Runge-Kutta integration scheme. With the matched solutions for the given lattice, emittances, beam energy and peak current, we then solve for the eigenvalues of the system of first order equivalent to Eqs. 21 and 22. The rms phase advance per cell is calculated for each case by evaluating the phase integral over the length of the cell L,

$$\sigma_x = \epsilon_x \int_0^L \frac{dz}{\mathcal{X}^2} = \epsilon_x \int_0^L \frac{dz}{\sqrt{X^2 - (\eta\sigma_p)^2}}, \tag{23}$$

with an analogous expression holding for σ_y. In the absence of space charge, this integral gives the bare betatron phase advance of the cell. On the other hand, in the absence of momentum spread, Eq. 23 gives the *rms* betatron phase advance per cell. If both momentum spread and space charge are present, there is no obvious interpretation of Eq. 23, as it contains some information about the off-momentum orbits, coupled to the betatron orbits through space charge forces.

We now move on to the analysis of several low energy synchrotron lattices, which will illustrate some interesting aspects of the envelope instability in these machines. The first example we take is that of the SSC Low Energy Booster (LEB) FODO lattice[9], which has an injection energy of 600 MeV, 360 mA peak current, and normalized rms emittance 0.75 mm-mrad. The maximum β-function is 11.9 m and the maximum dispersion is $\eta = 84$ cm.

The bare phase advances per cell are $\sigma_{x0} = 93.6$ degrees and $\sigma_{y0} = 90.0$ degrees, and we can anticipate that this lattice may be susceptible to beam envelope instability. The instability growth rate is plotted along with the phase advances as a function of peak beam current in Fig. 1(a) for the case of no momentum spread. We see that the instability can occur, somewhat surprisingly, without both σ_x and σ_y being depressed to less than 90 degrees. The onset of (confluent) envelope instability ($\sigma_x = 90.6$ degrees) occurs at 1120 mA, or approximately three times the design, which is a comfortable margin.

Inclusion of the design energy spread of $\sigma_p = 9.3\times = 10^{-4}$, however, causes the picture to change somewhat for the worse. This case is shown in Fig. 1(b). The onset of the instability occurs at about 400 mA, which is quite close to the design value. This result can be explained by examining the space charge force term in the envelope equations. If $\sigma_0 < 90°$ and there is no momentum spread, the quantity $\mathcal{X} + \mathcal{Y}$ is nearly constant, and the space charge force has little modulation. The increase in severity of the instability with momentum spread is due to the larger modulation of the space charge force brought about by the periodicity of the dispersion. The dispersion contribution to the beam width adds approximately in phase with the horizontal betatron contribution, introducing a strong periodic driving term to the envelope equations.

In the Fermilab Booster,[10] one can expect that the large bare tunes ($\sigma_{y0} = 102°$ and $\sigma_{x0} = 100.5°$) would keep the machine far from conditions which would result in envelope instability. This is indeed true for the case with no momentum spread, shown in Fig. 2(a). The normalized rms emittance is taken to be 1.66π mm-mrad (10π 'Fermilab' emittance) in both dimensions, and the peak operating current is 560 mA, assuming $N = 2.5\times10^{12}$ protons/batch and the bunching factor si $B = 0.25$ at 200 MeV injection energy. Instability onset occurs at approximately twice this current, assuming constant emittance. The calculation for momentum spread $\sigma_p = 1.5 \times 10^{-3}$, displayed in Fig. 2(b), is again more pessimistic. The space charge force modulation lowers the instability threshold current to less than 600 mA, which is right on the assumed maximum operating current. This parametric resonance has a large growth rate ($\sim$ 15 percent/cell), which peaks at 1.2 A and then is joined by the other unstable mode above 1.45 A, generating a new large confluent resonant band at higher currents.

It should be noted that the envelope equations can all be recast in a

dimensionless form which explicitly shows that the calculations presented here are dependent only on the ratio Q/ϵ. Thus if the injection energy is raised while keeping the number of protons per batch and normalized emittance constant the threshold current rises as $\beta\gamma^2$. For the linac upgrade this factor is 1.75, which raises the threshold to over 1 A.

The case of the TRIUMF Kaon Factory low energy booster looks interesting on the surface, as peak currents are very high and the initial vertical phase advance per cell is 93 degrees. On the other hand, the horizontal phase advance is far smaller, at $\sigma_{x0} = 78.4°$. This large tune splitting makes the machine relatively insensitive to the envelope instability. Also, the horizontal betatron width is larger than the vertical width due tune and emittance differences (normalized rms $\epsilon_{x0} = 23$ mm-mrad and $\epsilon_{x0} = 10.3$ mm-mrad), and is much larger than the width due to dispersion. The cases without and with momentum spread in this machine are shown in Figs. 3(a) and 3(b), respectively. The momentum spread actually has a small stabilizing effect in the TRIUMF machine. The envelope instability in both cases has a threshold current at 14 A, which is well above the design peak current of about 2 A.

Conclusions

This treatment of the envelope instability in circular machines has pointed to the possibility of problems near the nominal operating point of the Fermilab and SSC low energy synchrotrons at injection. There is good news for each case: at Fermilab, the linac upgrade takes the threshold current up to an acceptably high level, and at the SSC the LEB is currently under redesign, with phase advance per cell near 113 degrees[11]. This design should provide for a larger threshold current, although calculations should be performed to check this conclusion.

In order to understand the potential harm that envelope instability can pose to circular machines better, it would be instructive to run some multi-particle tracking codes which include space charge self-consistently. These codes would be able to better include the effects of chromaticity (which may provide damping) and the effect of space charge on the dispersion function. Multi-particle tracking has been done by Machida[12], for a variety of ma-

chines. These simulations display some unexplained emittance growth in the case of the SSC LEB. This effect could not have been due to an accurate calculation of envelope instability, however, as that would entail evaluating the space charge kick many times per focusing cell. This scheme could be easily implemented to observe the instability computationally, as the growth rate is high enough that one would only need track through a few tens of cells to see an effect.

If one becomes convinced that the envelope instability may be a problem in a machine, then observation of the instability becomes an important issue. This could be accomplished using the turn-by-turn beam profile monitor currently under development for use in the Fermilab Booster [13]. A beam which is envelope unstable, where the growth rate is smaller than a turn period, should display a growing oscillation in beam size when observed turn-by-turn. These oscillations should damp as the emittance grows due to phase space filamentation induced by the beam's nonlinear self-fields. As the emittance grows the beam will become larger and the space charge tune shift will diminish, taking the system below the instability threshold.

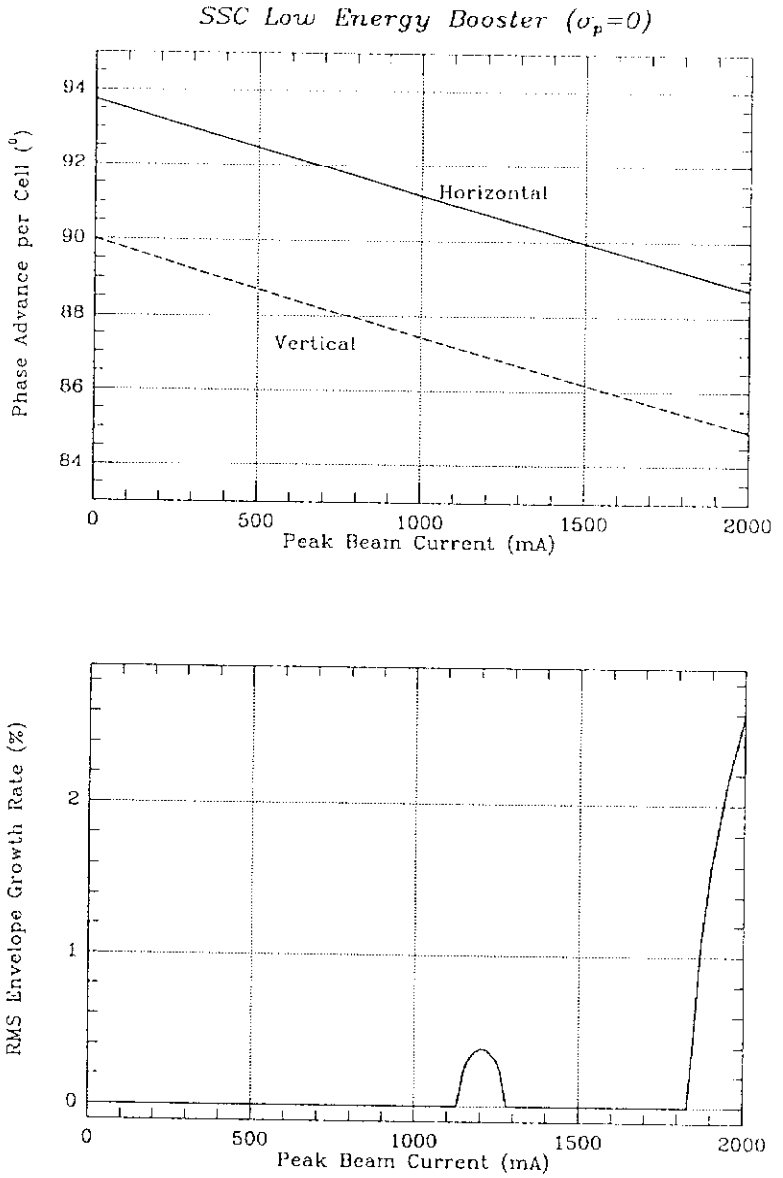

Figure 1(a). Phase advance and growth rate per cell plotted against peak current at injection energy, SSC LEB, no momentum spread.

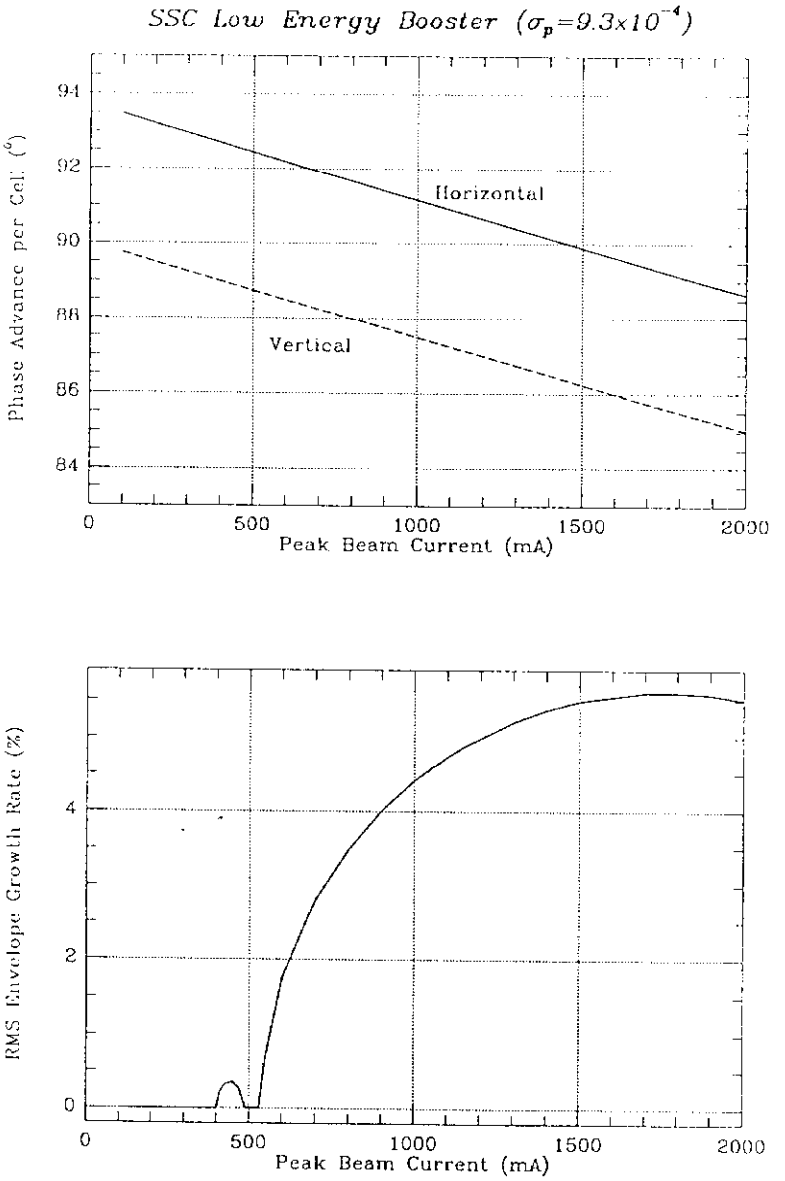

Figure 1(b). Phase advance and growth rate per cell plotted against peak current at injection energy, SSC LEB, $\sigma_p = 9.3 \times 10^{-4}$.

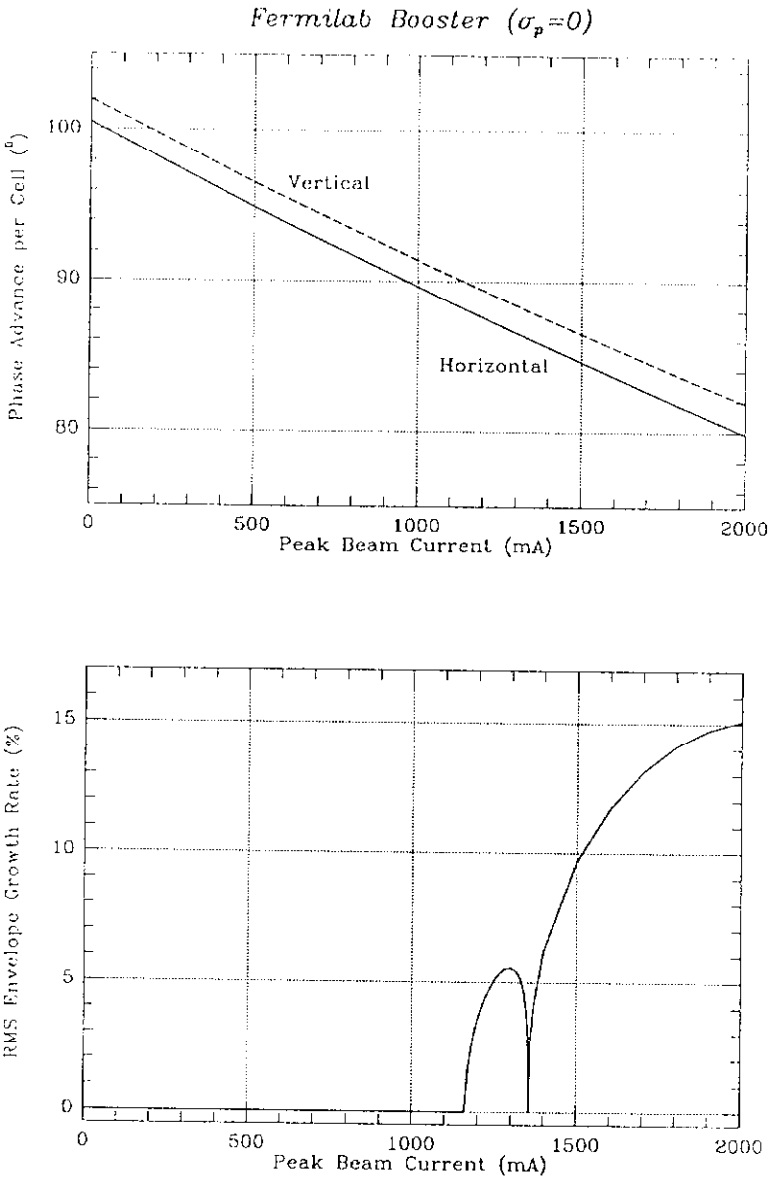

Figure 2(a). Phase advance and growth rate per cell plotted against peak current at injection energy, Fermilab Booster, no momentum spread.

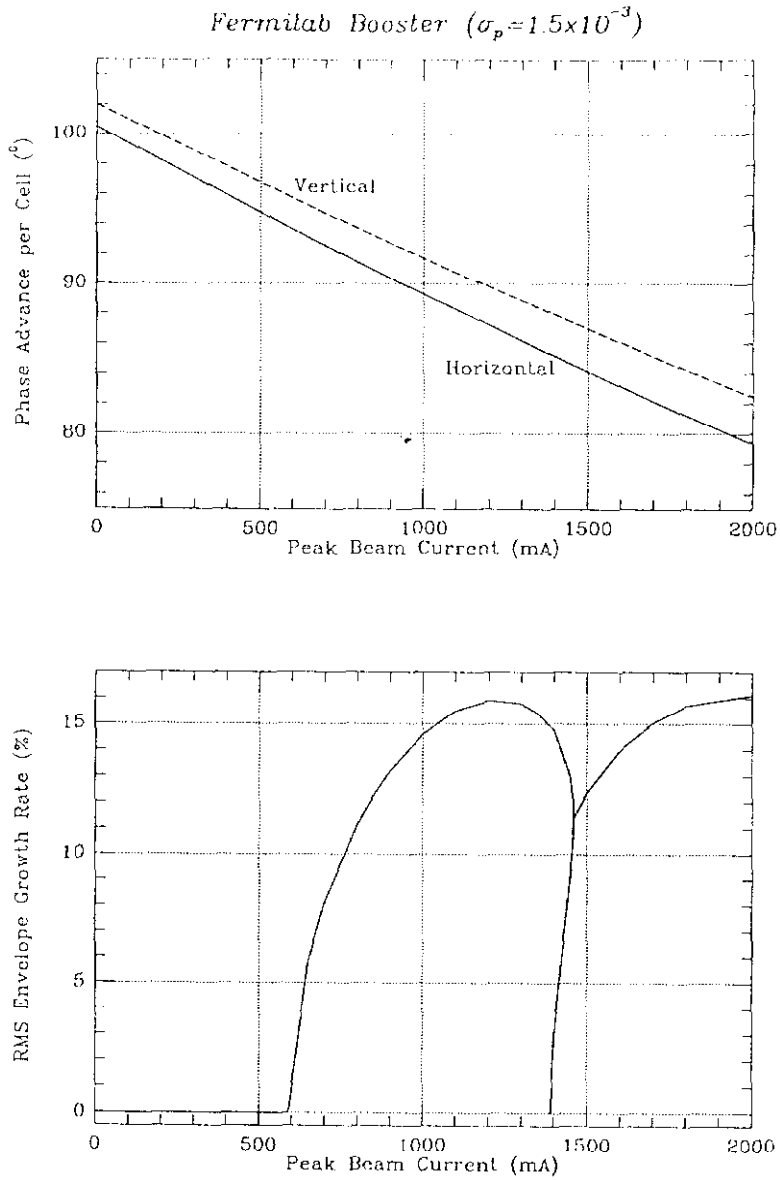

Figure 2(b). Phase advance and growth rate per cell plotted against peak current at injection energy, Fermilab Booster, $\sigma_p = 1.5 \times 10^{-3}$.

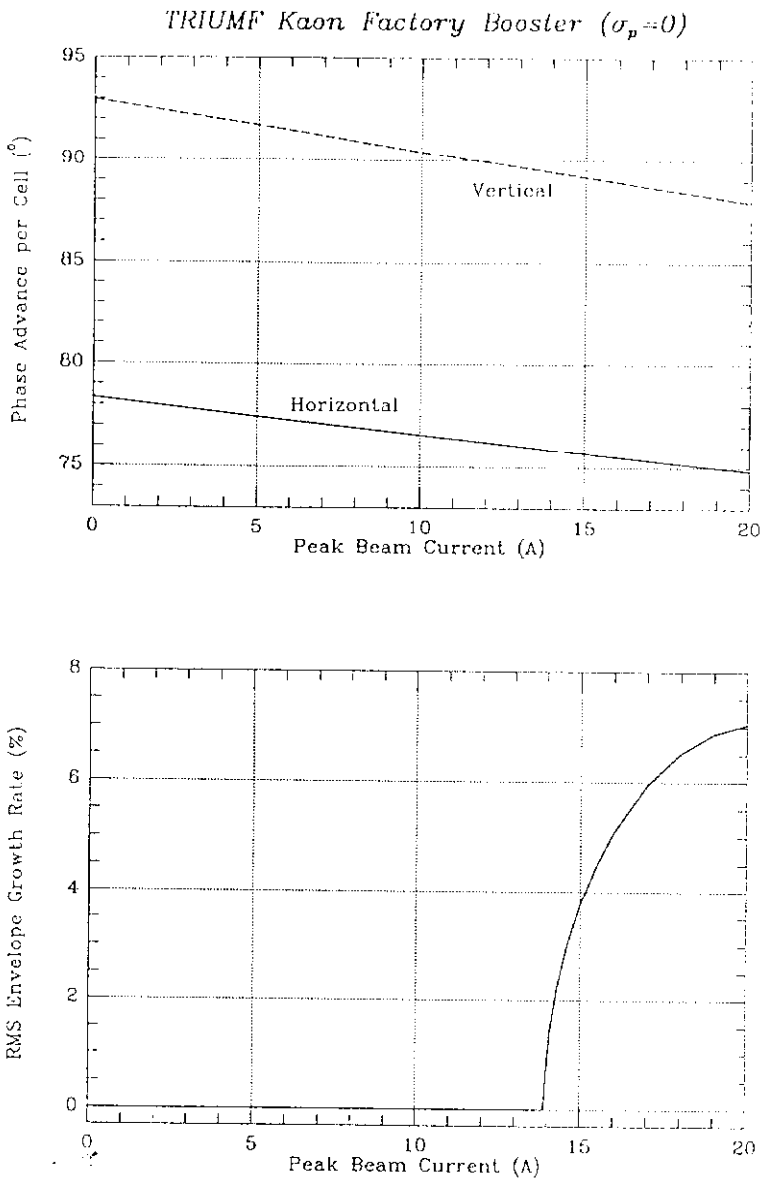

Figure 3(a). Phase advance and growth rate per cell plotted against peak current at injection energy, TRIUMF Kaon Factory Booster, no momentum spread.

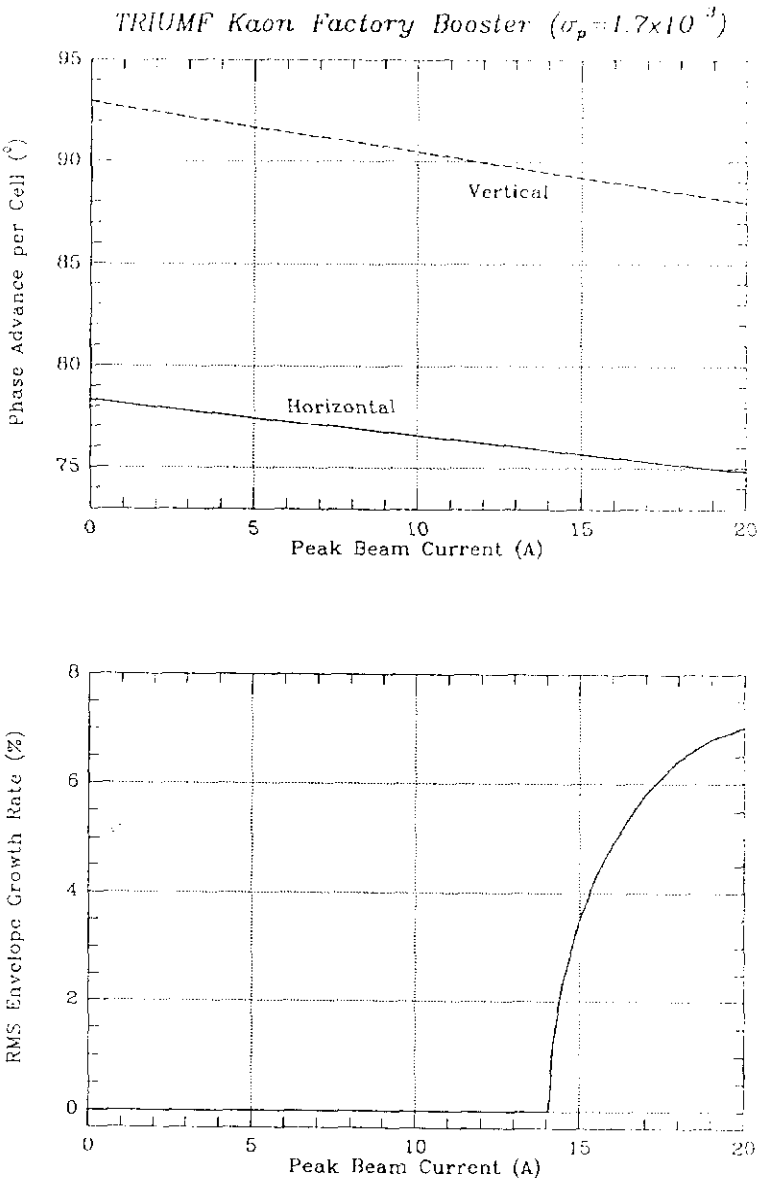

Figure 3(b). Phase advance and growth rate per cell plotted against peak current at injection energy, TRIUMF Kaon Factory Booster, $\sigma_p = 1.7 \times 10^{-3}$.

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