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TWO-FIFTH RESONANCE ISLANDS GENERATED BY SEXTUPOLES

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ABSTRACT

We derive the Hamiltonian which describes a system under the action of the isolated nonlinear resonance $\frac{2}{5}$ generated by the sextupole term x^3. This is accomplished by successive canonical transformations. Expressions for the island width and the island tune are given in terms of the sextupole configuration. Because of the large influence of the third-integer resonance, we have this resonance included also in the analysis. The validity of this calculation is checked against single particle tracking calculation and the results are satisfactory.

I. INTRODUCTION

In the E778 experiment[1] performed at the Tevatron, five islands were observed when the horizontal tune was set near $\nu = 19.40$. It is clear that these islands are the result of the firing of the set of special sextupoles. In this paper, we are going to describe these islands using a perturbative Hamiltonian approach. To have the $\frac{2}{5}$ resonance terms, third order in sextupole strength is necessary. For the islands to exist, however, a detuning term is also required. This is the usual second-order tune-shift term. The perturbation is carried out by successive Moser transformations. This process is, in fact, equivalent to an expansion in the parameter

$$\lambda \sim x s_k \ , \tag{1.1}$$

where x is the *normalized* horizontal displacement of the particle bunch and s_k is the sextupole strength defined in Eq. (2.15) below.

In Sec. II, two successive Moser transformations are performed to arrive at the third-order sextupole terms. This appears to be a very tedious derivation. It turns out, however, that the second Moser transformation is not actually necessary. Formulas for island width and island tune are derived.

In Sec. III, the Hamiltonian derivation is compared with the tracking results of an actual E778 experimental setup at the Tevatron. Direct comparison with experiment is not possible, because the Tevatron transverse beam size was found to be larger than the island width. Because of the proximity of the islands to the third-integer separatrices, the third-integer resonance has also been included in the analysis. The inclusion of two resonances has been possible because these two resonances happen to occur in different orders of the perturbation.

II. DERIVATION OF THE ISLAND WIDTH AND ISLAND TUNE

II.1 The Hamiltonian in Action-angle Variables

The one-degree-of-freedom Hamiltonian which describes the motion of a single beam particle in the presence of a normal sextupole field is

$$H = \frac{1}{2}[P^2 + K(s)X^2] + \frac{1}{6}\frac{B_y''}{B\rho}X^3 \ , \tag{2.1}$$

where P is the canonical momentum conjugate to the horizontal displacement X, and K is proportional to the restoring force due to the ring's curvature and the field gradients of the normal quadrupoles. The term $[B''_y/6(B\rho)]X^3$ gives the normal sextupole potential with $(B\rho)$ denoting the magnetic rigidity of the particle. We perform a canonical transformation to the Floquet space using the generating function

$$G_1(x, P; s) = -\left(\frac{\beta}{\beta_0}\right)^{1/2} Px + \frac{\beta'}{4\beta_0}x^2 , \tag{2.2}$$

where β is the beta function at distance s along the ideal orbit, β_0 is a reference beta, and $\beta' = d\beta/ds$ is the derivative with respect to the ideal path length along the ring. The new Hamiltonian becomes

$$H_1 = \frac{R}{2\beta}\left(\beta_0 p^2 + \frac{x^2}{\beta_0}\right) + \frac{RB''_y}{6(B\rho)}\left(\frac{\beta}{\beta_0}\right)^{3/2} x^3 . \tag{2.3}$$

In the above, the independent variable s has been changed to the more convenient $\theta = s/R$, where R is the average radius of the storage ring. The canonical variables (X, P) have been transformed to the normalized (x, p).

This Hamiltonian is now solved exactly to zeroth order in the sextupole strength by canonical transformation to the action-angle variables (I, a). The generating function

$$G_2(a, p; \theta) = \frac{1}{2}\,\beta_0 p^2 \cot\left[\psi(\theta) - \nu\theta + a\right] \tag{2.4}$$

is used to obtain the transformation

$$x = (2I\beta_0)^{1/2} \cos\left[\psi(\theta) - \nu\theta + a\right] , \tag{2.5}$$

$$\beta_0 p = -(2I\beta_0)^{1/2} \sin\left[\psi(\theta) - \nu\theta + a\right] , \tag{2.6}$$

where $\beta_0 p = du/d\psi$ and is denoted by x' below. In the above, ν is the betatron tune and

$$\psi(s) = \int^s \frac{ds'}{\beta(s')} \tag{2.7}$$

is the Floquet phase at the location s. After the transformation, the new Hamiltonian becomes

$$H_2 = \nu I + \frac{1}{24}\frac{RB''_y}{B\rho}(2I\beta)^{3/2}\left\{\cos 3[\psi(\theta) - \nu\theta + a] + 3\cos\left[\psi(\theta) - \nu\theta + a\right]\right\} . \tag{2.8}$$

We note that the expression

$$\frac{RB_y''}{B\rho}(2I\beta)^{3/2}e^{i(\psi-\nu\theta)} \tag{2.9}$$

is periodic in θ; it can be expanded into harmonics. So we get

$$H_2 = \nu I + (2I)^{3/2}\beta_0^{1/2}\sum_m(3A_{1m}\sin q_{1m} + A_{3m}\sin q_{3m}) \ , \tag{2.10}$$

where the angles are defined by

$$q_{1m} = a - m\theta + \alpha_{1m} \ , \tag{2.11}$$

$$q_{3m} = 3a - m\theta + \alpha_{3m} \ , \tag{2.12}$$

and the Fourier coefficients are

$$A_{1m}e^{i\alpha_{1m}} = \frac{i}{24\pi}\sum_k s_k e^{i(\psi-\nu\theta+m\theta)_k} \ , \tag{2.13}$$

$$A_{3m}e^{i\alpha_{3m}} = \frac{i}{24\pi}\sum_k s_k e^{i(3\psi-3\nu\theta+m\theta)_k} \ . \tag{2.14}$$

The summation in Eq. (2.10) is over all integers or harmonics from $-\infty$ to $+\infty$. The summations in Eqs. (2.13) and (2.14) are over all sextupoles at positions θ_k around the ring. The normal sextupoles are assumed to have infinitesimal length ℓ_k with strengths

$$s_k = \left(\frac{\beta^3}{\beta_0}\right)_k^{1/2}\frac{(B_y''\ell)_k}{2(B\rho)} \ . \tag{2.15}$$

In Eqs. (2.13) and (2.14), the harmonic amplitudes A_{1m}, A_{3m} and the phases α_{1m} and α_{3m} are real numbers.

It is clear from Eq. (2.10) that the Hamiltonian has the dimension of the action I or [length], and the perturbative parameter is

$$\lambda \sim (2I\beta_0)^{1/2}A_{1m} \qquad \text{or} \qquad (2I\beta_0)^{1/2}A_{3m} \ , \tag{2.16}$$

as was mentioned in the previous section. In below, the second-order terms can therefore be referred as the I^2 terms, the third-order terms the $I^{5/2}$ terms, etc.

II.2 First Moser transformation

We now perform a Moser transformation from (I,a) to (J,b) such that the new Hamiltonian is solved exactly up to first order in sextupole strength. The generating function is

$$G_3(a,J,\theta) = aJ - (2J)^{3/2}\beta_0^{1/2}\sum_m \left(\frac{3A_{1m}}{m-\nu}\cos q_{1m} + \frac{A_{3m}}{m-3\nu}\cos q_{3m}\right) . \tag{2.17}$$

This new transformation implies

$$I = \frac{\partial G_3}{\partial a} , \tag{2.18}$$

$$b = \frac{\partial G_3}{\partial J} , \tag{2.19}$$

and

$$H_3(b,J) = H_2(b,J) + \frac{\partial G_3}{\partial \theta} . \tag{2.20}$$

Explicitly, we have

$$I = J + (2J)^{3/2}\beta_0^{1/2}\mathcal{Q}(a) , \tag{2.21}$$

where

$$\mathcal{Q}(a) = \sum_m \left[\frac{3A_{1m}}{m-\nu}\sin q_{1m}(a) + \frac{3A_{3m}}{m-3\nu}\sin q_{3m}(a)\right] , \tag{2.22}$$

and $q_{jm}(a) = ja - m\theta + \alpha_{jm}$ with $j = 1,\ 3$. From here we can calculate the term $(2I)^{3/2}$ by expanding Eq. (2.21) in powers of J

$$(2I)^{3/2} = (2J)^{3/2} + 3(2J)^2\beta^{1/2}\mathcal{Q}(a) + \frac{3}{2}(2J)^{5/2}\beta_0\mathcal{Q}^2(a) + \cdots . \tag{2.23}$$

The new angle variable b can be calculated from Eq. (2.19)

$$b = a - 3(2J)^{1/2}\beta_0^{1/2}\mathcal{Q}_1(a) , \tag{2.24}$$

where

$$\mathcal{Q}_1(a) = \sum_m \left[\frac{3A_{1m}}{m-\nu}\cos q_{1m}(a) + \frac{A_{3m}}{m-3\nu}\cos q_{3m}(a)\right] . \tag{2.25}$$

Hence the new Hamiltonian (2.20) becomes

$$\begin{aligned} H_3(b,J) &= \nu[J + (2J)^{3/2}\beta_0^{1/2}\mathcal{Q}] \\ &+ \left\{(2J)^{3/2} + 3(2J)^2\beta_0^{1/2}\mathcal{Q} + \frac{3}{2}(2J)^{5/2}\beta_0\mathcal{Q}^2\right\}\beta_0^{1/2} \\ &\times \sum_m (3A_{1m}\sin q_{1m} + A_{3m}\sin q_{3m}) \\ &-(2J)^{3/2}\beta_0^{1/2}\sum_m m\left(\frac{3A_{1m}}{m-\nu}\sin q_{1m} + \frac{A_{3m}}{m-3\nu}\sin q_{3m}\right) . \end{aligned} \tag{2.26}$$

Or, if we collect all the terms of the same power of J together,

$$
\begin{aligned}
H_3(b,J) = \nu J + & \\
& (2J)^{3/2}\beta_0^{1/2}\sum_m \left\{3A_{1m}\sin q_{1m}\left(\frac{\nu}{m-\nu}+1-\frac{m}{m-\nu}\right)\right. \\
& \left. +A_{3m}\sin q_{3m}\left(\frac{3\nu}{m-3\nu}+1-\frac{m}{m-3\nu}\right)\right\} \\
& +3(2J)^2\beta_0\mathcal{Q}\sum_m(3A_{1m}\sin q_{1m}+A_{3m}\sin q_{3m}) \\
& +\frac{3}{2}(2J)^{5/2}\beta_0^{3/2}\mathcal{Q}^2\sum_m(3A_{1m}\sin q_{1m}+A_{3m}\sin q_{3m}) \ .
\end{aligned}
\tag{2.27}
$$

Notice that the terms proportional to $J^{3/2}$ vanish as expected. Since we would like to do perturbation up to $\mathcal{O}(\lambda^3)$ only, we can substitute a by b in the last term of Eq. (2.27). However, for the second last term we need to expand $q_{jm}(a)$ to one order in λ. From Eq. (2.24), we get

$$
a = b + 3(2J)^{1/2}\beta_0^{1/2}\mathcal{Q}_1(b) \ . \tag{2.28}
$$

Therefore, for $j = 1, 3$,

$$
\sin q_{jm}(a) = \sin(ja - m\theta + \alpha_{jm}) \approx \sin q_{jm}(b) + 3j(2J)^{1/2}\beta_0^{1/2}\mathcal{Q}_1(b)\cos q_{jm}(b) \ . \tag{2.29}
$$

Using Eq. (2.29), the second last term of Eq. (2.27) can be separated into a second-order term (J^2 term) and a third-order term ($J^{5/2}$ term). The Hamiltonian becomes

$$
\begin{aligned}
H_3(b,J) = \nu J + & \\
& 3(2J)^2\beta_0\sum_{mm'}\left(\frac{3A_{1m}}{m-\nu}\sin q_{1m}+\frac{3A_{3m}}{m-3\nu}\sin q_{3m}\right)(3A_{1m'}\sin q_{1m'}+A_{3m'}\sin q_{3m'}) \\
& + \text{“3rd-order terms”} \ ,
\end{aligned}
\tag{2.30}
$$

where the third-order terms are

$$
\begin{aligned}
& \text{“3rd-order terms”} = \\
& 27(2J)^{5/2}\beta^{3/2}\mathcal{Q}_1\sum_{mm'}\left[\left(\frac{2A_{3m}}{m-3\nu}\cos q_{3m}+\frac{A_{1m}}{m-\nu}\cos q_{1m}\right)(A_{3m'}\sin q_{3m'}+3A_{1m'}\sin q_{1m'})\right. \\
& \left. + \left(\frac{3A_{3m}}{m-3\nu}\sin q_{3m}+\frac{3A_{1m}}{m-\nu}\sin q_{1m}\right)(A_{3m'}\cos q_{3m'}+3A_{1m'}\cos q_{1m'})\right] \\
& +\frac{3}{2}(2J)^{5/2}\beta_0^{3/2}\mathcal{Q}^2\sum_{m'}(3A_{1m'}\sin q_{1m'}+A_{3m'}\sin q_{3m'}) \ .
\end{aligned}
\tag{2.31}
$$

Note that the second-order term and the first of the third-order terms come from the second last term of Eq. (2.27). In Eqs. (2.30) and (2.31), all the q_{jm}'s, $\mathcal{Q}$ and $\mathcal{Q}_1$ are now explicit functions of the angle variable b.

II.3 Second Moser Transformation

Now we want to perform another Moser transformation so that all the second-order terms in Eq. (2.30) disappear. Before doing so, let us rewrite the second-order terms using the relation

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \ . \tag{2.32}$$

The result is

$$\begin{aligned}
H_3(b,J) = \nu J + &\\
\frac{9}{2}(2J)^2\beta_0 \sum_{mm'} \Bigg\{ &\frac{3A_{1m}A_{1m'}}{m-\nu}\cos\left[-(m-m')\theta + (\alpha_{1m}-\alpha_{1m'})\right] - \frac{3A_{1m}A_{1m'}}{m-\nu}\cos Q_1 \\
&+ \frac{A_{1m}A_{3m'}}{m-\nu}\cos Q_2 - \frac{A_{1m}A_{3m'}}{m-\nu}\cos Q_3 + \frac{3A_{3m}A_{1m'}}{m-3\nu}\cos Q_4 - \frac{3A_{3m}A_{1m'}}{m-3\nu}\cos Q_5 \\
&+ \frac{A_{3m}A_{3m'}}{m-3\nu}\cos\left[-(m'-m)\theta + (\alpha_{3m'}-\alpha_{3m})\right] - \frac{A_{3m}A_{3m'}}{m-3\nu}\cos Q_6 \Bigg\} \\
&+ 9(2J)^{5/2}\beta_0^{3/2} \sum_{mm'm''}\{\cdots\} + \frac{9}{2}(2J)^{5/2}\beta_0^{3/2} \sum_{mm'm''}\{\cdots\} \ .
\end{aligned} \tag{2.33}$$

The quantities Q_1 to Q_6 are defined as follows

$$Q_1 = 2b - (m+m')\theta + (\alpha_{1m}+\alpha_{1m'}) \ , \tag{2.34}$$

$$Q_2 = 2b - (m'-m)\theta + (\alpha_{3m'}-\alpha_{1m}) \ , \tag{2.35}$$

$$Q_3 = 4b - (m+m')\theta + (\alpha_{1m}+\alpha_{3m'}) \ , \tag{2.36}$$

$$Q_4 = 2b - (m-m')\theta + (\alpha_{3m}-\alpha_{1m'}) \ , \tag{2.37}$$

$$Q_5 = 4b - (m+m')\theta + (\alpha_{3m}+\alpha_{1m'}) \ , \tag{2.38}$$

$$Q_6 = 6b - (m+m')\theta + (\alpha_{3m}+\alpha_{3m'}) \ , \tag{2.39}$$

and the triple sums in the last line of Eq. (2.33) are the third-order terms.

We shall transform from (J,b) to (I,a). To avoid proliferation of the notation, we have chosen to call the new variables (I,a) again. We need a generating function

$$G_4(b,I;\theta) = bI + \bar{G}_4(b,I,\theta) \ , \tag{2.40}$$

with

$$J = I + \frac{\partial \bar{G}_4}{\partial b} \ , \tag{2.41}$$

such that in the new Hamiltonian we have

$$\nu \frac{\partial \bar{G}_4}{\partial b} + \frac{\partial \bar{G}_4}{\partial \theta} + \text{ 2nd-order term of Eq. (2.30)} = 0 \tag{2.42}$$

at least up to second order. This will not be possible, however, because when $m = m'$ there are two terms in Eq. (2.33),

$$\frac{9}{2}(2I)^2\beta_0 \sum_m \left(\frac{3A_{1m}^2}{m-\nu} + \frac{A_{3m}^2}{m-3\nu} \right) \ , \tag{2.43}$$

which are independent of θ and b. Note that $\bar{G}_4$ must be of second order; therefore $(2J)$ has been replaced by $(2I)$ in above. Aside from these terms, $\bar{G}_4$ can be found readily,

$$\begin{aligned} \bar{G}_4(b, I; \theta) \ = \ & \frac{9}{2}(2I)^2\beta_0 \sum_{mm'} \left\{ -\frac{3A_{1m}A_{1m'}}{(m-\nu)[(m+m')-2\nu]} \sin Q_1 \right. \\ & + \frac{A_{1m}A_{3m'}}{(m-\nu)[(m'-m)-2\nu]} \sin Q_2 - \frac{A_{1m}A_{3m'}}{(m-\nu)[(m'+m)-4\nu]} \sin Q_3 \\ & + \frac{3A_{3m}A_{1m'}}{(m-3\nu)[(m-m')-2\nu]} \sin Q_4 - \frac{3A_{3m}A_{1m'}}{(m-3\nu)[(m+m')-4\nu]} \sin Q_5 \\ & \left. - \frac{A_{3m}A_{3m'}}{(m-3\nu)[(m+m')-6\nu]} \sin Q_6 \right\} \\ & - \frac{9}{2}(2I)^2\beta_0 {\sum_{mm'}}' \left\{ -\frac{3A_{1m}A_{1m'}}{(m-\nu)(m-m')} \sin\left[-(m-m')\theta + (\alpha_{1m} - \alpha_{1m'})\right] \right. \\ & \left. - \frac{A_{3m}A_{3m'}}{(m-3\nu)(m'-m)} \sin[-(m'-m)\theta + (\alpha_{3m'} - \alpha_{3m})] \right\} \ . \end{aligned} \tag{2.44}$$

The prime in the last summation signifies that the situation of $m = m'$ has been excluded.

The new Hamiltonian becomes

$$H_4(a, I) = \nu I + \frac{9}{2}(2I)^2\beta_0 \sum_m \left[\frac{3A_{1m}^2}{m-\nu} + \frac{A_{3m}^2}{m-3\nu} \right] + \text{ “3rd-order terms”} \ . \tag{2.45}$$

Note that the third-order terms are intact as in Eq. (2.30) except for the substitution of $(2J)$ by $(2I)$ and b by a. In fact, this is a general result. That is, in the second Moser transformation, $\bar{G}_4$ is always of second order. Therefore, the substitution of (J, b) for this transformation into the second-order terms of Eq. (2.33) will produce only second-order terms and fourth-order or still higher-order terms, and no third-order terms. For this reason, the third-order terms of the result of the second Moser transformation can be read off directly from Eq. (2.30) with the old action-angle variables replaced by the new ones and without actually performing the transformation. The second Moser transformation generates only the detuning term [Eq. (2.43)], which can also be read off easily or obtained[2] by averaging the phase of Eq. (2.30). In other words, the second Moser transformation can actually be avoided.

II.4 Evaluation of Third-order Terms

The two third-order terms of Eq. (2.31) can be evaluated easily, by expanding the trigonometric functions with the use of the Eq. (2.32) and

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] \ . \tag{2.46}$$

Since we are interested in the $\frac{2}{5}$ resonance, we keep only those terms which will produce an argument of $5a$ in the final sine or cosine. This can be easily tracked, because each q_{jm} contains ia ($i = 1$ or 3).

Let us take up the second of the third-order terms. First we evaluate $\mathcal{Q}^2(a)$ defined by Eq. (2.22),

$$\begin{aligned}\mathcal{Q}^2 \ &\doteq\ \frac{9}{2}\sum_{mm''}\left\{-\frac{A_{3m}A_{3m''}}{(m-3\nu)(m''-3\nu)}\cos(q_{3m}+q_{3m''}) - \frac{A_{1m}A_{1m''}}{(m-\nu)(m''-\nu)}\cos(q_{1m}+q_{1m''})\right.\\ &\left. +\frac{2A_{1m}A_{3m''}}{(m-\nu)(m''-3\nu)}\left[\cos(q_{1m}-q_{3m''}) - \cos(q_{1m}+q_{3m''})\right]\right\} \ .\end{aligned} \tag{2.47}$$

In the above, terms involving $\cos(q_{3m}-q_{3m''})$ and $\cos(q_{1m}-q_{1m''})$ have been dropped, because they will not lead to an argument of $5a$ eventually. The second of the third-order terms then gives

$$\begin{aligned}&\frac{3}{2}(2I)^{5/2}\beta_0^{3/2}\mathcal{Q}^2\sum_{m'}(A_{3m'}\sin q_{3m'} + 3A_{1m'}\sin q_{1m'}) =\\ &\quad \frac{27}{8}(2I)^{5/2}\beta_0^{3/2}\sum_{mm'm''}\left\{\frac{3A_{3m}A_{1m'}A_{3m''}}{(m-3\nu)(m''-3\nu)}\sin(q_{3m}-q_{1m'}+q_{3m''})\right.\\ &\quad -\frac{A_{1m}A_{3m'}A_{1m''}}{(m-\nu)(m''-\nu)}\sin(q_{1m}+q_{3m'}+q_{1m''})\end{aligned}$$

$$+\frac{2A_{1m}A_{3m'}A_{3m''}}{(m-\nu)(m''-3\nu)}\sin(q_{3m'}+q_{3m''}-q_{1m})$$
$$-\frac{6A_{1m}A_{1m'}A_{3m''}}{(m-\nu)(m''-3\nu)}\sin(q_{1m}+q_{1m'}+q_{3m''})\Bigg\} \ , \tag{2.48}$$

where only the relevant terms have been included.

Similarly, the first third-order term in Eq. (2.31),

$$27(2I)^{5/2}\beta_0^{3/2}\mathcal{Q}_1\sum_{mm'}\Bigg\{\frac{3A_{3m}A_{3m'}}{m-3\nu}\sin(q_{3m}+q_{3m'})+\frac{3A_{1m}A_{1m'}}{m-\nu}\sin(q_{1m}+q_{1m'})$$
$$+\frac{A_{3m}A_{1m'}}{m-3\nu}\left[-3\sin(q_{3m}-q_{1m'})+6\sin(q_{3m}+q_{1m'})\right]$$
$$+\frac{A_{1m}A_{3m'}}{m-\nu}\left[-\sin(q_{3m'}-q_{1m})+2\sin(q_{1m}+q_{3m'})\right]\Bigg\} \ , \tag{2.49}$$

becomes, after the substitution of $\mathcal{Q}_1$ from Eq. (2.25),

$$\frac{27}{2}(2I)^{5/2}\beta_0^{3/2}\sum_{mm'm''}\Bigg\{-\frac{3A_{3m}A_{1m'}A_{3m''}}{(m-3\nu)(m''-3\nu)}\sin(q_{3m}-q_{1m'}+q_{3m''})$$
$$+\frac{6A_{1m}A_{3m'}A_{1m''}}{(m-\nu)(m''-\nu)}\sin(q_{1m}+q_{3m'}+q_{1m''})+\frac{8A_{1m}A_{3m'}A_{3m''}}{(m-\nu)(m''-3\nu)}\sin(q_{3m'}+q_{3m''}-q_{1m})$$
$$+\frac{21A_{1m}A_{1m'}A_{3m''}}{(m-\nu)(m''-3\nu)}\sin(q_{1m}+q_{1m'}+q_{3m''})\Bigg\} \ . \tag{2.50}$$

Summing up Eqs. (2.48) and (2.50), the new Hamiltonian becomes

$$H_4(a,I)=\nu I+\frac{9}{2}(2I)^2\beta_0\sum_m\left[\frac{3A_{1m}^2}{m-\nu}+\frac{A_{3m}^2}{m-3\nu}\right]$$
$$+\frac{9}{8}(2I)^{5/2}\beta_0^{3/2}\left\{234S_1+69S_2+102S_3-27S_4\right\} \ , \tag{2.51}$$

where

$$S_1=\sum_{mm'm''}\frac{A_{1m}A_{1m'}A_{3m''}}{(m-\nu)(m''-3\nu)}\sin\left[5a-(m+m'+m'')\theta+(\alpha_{1m}+\alpha_{1m'}+\alpha_{3m''})\right] \ , \tag{2.52}$$

$$S_2=\sum_{mm'm''}\frac{A_{1m}A_{3m'}A_{1m''}}{(m-\nu)(m''-\nu)}\sin\left[5a-(m+m'+m'')\theta+(\alpha_{1m}+\alpha_{3m'}+\alpha_{1m''})\right] \ , \tag{2.53}$$

$$S_3=\sum_{mm'm''}\frac{A_{1m}A_{3m'}A_{3m''}}{(m-\nu)(m''-3\nu)}\sin\left[5a-(m-m'+m'')\theta+(\alpha_{3m'}+\alpha_{3m''}-\alpha_{1m})\right] \ , \tag{2.54}$$

$$S_4=\sum_{mm'm''}\frac{A_{3m}A_{1m'}A_{3m''}}{(m-3\nu)(m''-3\nu)}\sin\left[5a-(m-m'+m'')\theta+(\alpha_{3m}+\alpha_{3m''}-\alpha_{1m'})\right] \ . \tag{2.55}$$

II.5 The Detuning Term

The expression in Eq. (2.43) is called the detuning term because it consists of a constant multiplied by I^2. This constant is one half of the detuning c, which can be expressed as

$$c = 36\beta_0 \sum_m \left[\frac{3A_{1m}^2}{m-\nu} + \frac{A_{3m}^2}{m-3\nu}\right] \ . \tag{2.56}$$

The harmonics can be readily summed[2, 3] using the formulas

$$\sum_{m=-\infty}^{\infty} \frac{e^{i(m\theta+b)}}{m-\nu} = \begin{cases} -\dfrac{\pi}{\sin \pi\nu} e^{i[b+\nu(\theta-\pi)]} & 0 < \theta < 2\pi \\ \\ -\pi \cot \pi\nu \, e^{ib} & \theta = 0 \end{cases} \tag{2.57}$$

to give

$$c = -\frac{\beta_0}{2\pi} \sum_k (3B_1 s + B_3 s)_k \tag{2.58}$$

where the distortion functions are defined as

$$\begin{aligned} B_1(\psi) &= \frac{1}{2\sin \pi\nu} \sum_k \frac{s_k}{4} \cos\left(|\psi_k - \psi| - \pi\nu\right) \qquad 0 \le |\psi_k - \psi| \le 2\pi\nu \ , \\ B_3(3\psi) &= \frac{1}{2\sin 3\pi\nu} \sum_k \frac{s_k}{4} \cos 3(|\psi_k - \psi| - \pi\nu) \qquad 0 \le |\psi_k - \psi| \le 2\pi\nu \ . \end{aligned} \tag{2.59}$$

II.6 The Triple Sums of the Resonance Terms

Next we are going to calculate the triple sums of S_1, S_2, S_3 and S_4, and express the S_i's in a closed form. We shall demonstrate the way to calculate S_1 and give the results for the other three sums. First we rewrite it as

$$S_1 = \mathcal{I}m \sum_{mm'm''} \frac{(A_{1m}e^{i\alpha_{1m}})(A_{1m'}e^{i\alpha_{1m'}})(A_{3m''}e^{i\alpha_{3m''}})}{(m-\nu)(m''-3\nu)} e^{i[5a-(m+m'+m'')\theta]} \ , \tag{2.60}$$

where the expressions

$$A_{jm}e^{i\alpha_{jm}} \qquad j = 1, 3 \tag{2.61}$$

are given by Eqs. (2.13) and (2.14). From the above sum, we will keep only the slowly varying terms, that is, terms of the form $e^{i(5a-97\theta)}$, since the tune of the machine is close to 19.40.

Hence the above triple sum — over m, m' and m'' — is actually constrained by the condition

$$m + m' + m'' = 97 \ . \tag{2.62}$$

Eliminating m', we obtain

$$S_1 = \mathcal{I}m\, \vec{S}_1 e^{i(5a-97\theta)} \ , \tag{2.63}$$

where the complex number $\vec{S}_1$ is defined as

$$\vec{S}_1 = \sum_{mm''} \frac{(A_{1m}e^{i\alpha_{1m}})(A_{1(97-m-m'')}e^{i\alpha_{1(97-m-m'')}})(A_{3m''}e^{i\alpha_{3m''}})}{(m-\nu)(m''-3\nu)} \ . \tag{2.64}$$

Substituting Eqs. (2.13) and (2.14) into Eq. (2.63), we get

$$\vec{S}_1 = \left(\frac{i}{24\pi}\right)^3 \sum_{k_1 k_2 k_3} s_{k_1} s_{k_2} s_{k_3} e^{i(\psi-\nu\theta+97\theta)_{k_2}} \times$$
$$\times \sum_{m} \frac{e^{i[\psi_{k_1}-\nu\theta_{k_1}+m(\theta_{k_1}-\theta_{k_2})]}}{m-\nu} \sum_{m''} \frac{e^{i[3\psi_{k_3}-3\nu\theta_{k_3}+m''(\theta_{k_3}-\theta_{k_2})]}}{m''-3\nu} \ . \tag{2.65}$$

The two sums over m and m' can now be performed exactly using the formulas given by Eq. (2.57). We obtain

$$\vec{S}_1 = \frac{-i}{(24)^3\pi \sin\pi\nu \sin 3\pi\nu} \Bigg[\cos\pi\nu \cos 3\pi\nu \sum_{k_2} s_{k_2}^3 e^{i(5\psi_{k_2}-5\delta\theta_{k_2})}$$
$$+ \cos\pi\nu \sum_{\substack{k_2 k_3 \\ (k_3 \neq k_2)}} s_{k_2}^2 s_{k_3} e^{i(2\psi_{k_2}+3\psi'_{k_3}-5\delta\theta_{k_2}-3\pi\nu)} + \cos 3\pi\nu \sum_{\substack{k_1 k_2 \\ (k_1 \neq k_2)}} s_{k_1} s_{k_2}^2 e^{i(\psi'_{k_1}+4\psi_{k_2}-5\delta\theta_{k_2}-\pi\nu)}$$
$$+ \sum_{\substack{k_1 k_2 k_3 \\ (k_1, k_3 \neq k_2)}} s_{k_1} s_{k_2} s_{k_3} e^{i(\psi'_{k_1}+\psi_{k_2}+3\psi'_{k_3}-5\delta\theta_{k_2}-4\pi\nu)} \Bigg] \ , \tag{2.66}$$

where use has been made of the following notations:

$$\delta = \nu - \frac{97}{5} \ , \tag{2.67}$$

and

$$\psi'_k = \begin{cases} \psi_k & \theta_k > \theta_{k_2} \\ \psi_k + 2\pi\nu & \theta_k < \theta_{k_2} \ . \end{cases} \tag{2.68}$$

The expression for $\vec{S}_1$ appears to be rather lengthy and complicated, the reason being that we need to separate the various situations when some or all of k_1, k_2, and k_3 are the same.

Similarly, the triple summations in S_2, S_3, and S_4 can be performed. We define further, for $j = 2,\ 3,\ 4$,

$$S_j = \mathcal{I}m\, \vec{S}_j e^{i(5a-97\theta)} , \tag{2.69}$$

and the other three complex numbers

$$\vec{S}_2 = \frac{-i}{(24)^3\pi \sin^2 \pi\nu} \Bigg[\cos^2 \pi\nu \sum_{k_2} s_{k_2}^3 e^{i(5\psi_{k_2} - 5\delta\theta_{k_2})}$$
$$+2\cos\pi\nu \sum_{\substack{k_2 k_3 \\ (k_3 \neq k_2)}} s_{k_2}^2 s_{k_3} e^{i(4\psi_{k_2} + \psi'_{k_3} - 5\delta\theta_{k_2} - \pi\nu)} + \sum_{\substack{k_1 k_2 k_3 \\ (k_1, k_3 \neq k_2)}} s_{k_1} s_{k_2} s_{k_3} e^{i(\psi'_{k_1} + 3\psi_{k_2} + \psi'_{k_3} - 5\delta\theta_{k_2} - 2\pi\nu)} \Bigg] , \tag{2.70}$$

$$\vec{S}_3 = \frac{-i}{(24)^3\pi \sin\pi\nu \sin 3\pi\nu} \Bigg[\cos\pi\nu \cos 3\pi\nu \sum_{k_2} s_{k_2}^3 e^{i(5\psi_{k_2} - 5\delta\theta_{k_2})}$$
$$+ \cos\pi\nu \sum_{\substack{k_2 k_3 \\ (k_3 \neq k_2)}} s_{k_2}^2 s_{k_3} e^{i(2\psi_{k_2} + 3\psi'_{k_3} - 5\delta\theta_{k_2} - 3\pi\nu)} + \cos 3\pi\nu \sum_{\substack{k_1 k_2 \\ (k_1 \neq k_2)}} s_{k_1} s_{k_2}^2 e^{i(-\psi'_{k_1} + 6\psi_{k_2} - 5\delta\theta_{k_2} + \pi\nu)}$$
$$+ \sum_{\substack{k_1 k_2 k_3 \\ (k_1, k_3 \neq k_2)}} s_{k_1} s_{k_2} s_{k_3} e^{i(-\psi'_{k_1} + 3\psi_{k_2} + 3\psi'_{k_3} - 5\delta\theta_{k_2} - 2\pi\nu)} \Bigg] , \tag{2.71}$$

$$\vec{S}_4 = \frac{-i}{(24)^3\pi \sin^2 3\pi\nu} \Bigg[\cos^2 3\pi\nu \sum_{k_2} s_{k_2}^3 e^{i(5\psi_{k_2} - 5\delta\theta_{k_2})}$$
$$+2\cos 3\pi\nu \sum_{\substack{k_2 k_3 \\ (k_3 \neq k_2)}} s_{k_2}^2 s_{k_3} e^{i(2\psi_{k_2} + 3\psi'_{k_3} - 5\delta\theta_{k_2} - 3\pi\nu)} + \sum_{\substack{k_1 k_2 k_3 \\ (k_1, k_3 \neq k_2)}} s_{k_1} s_{k_2} s_{k_3} e^{i(3\psi'_{k_1} - \psi_{k_2} + 3\psi'_{k_3} - 5\delta\theta_{k_2} - 6\pi\nu)} \Bigg] . \tag{2.72}$$

So the Hamiltonian now has the following form

$$H_4(I, a) = \nu I + \frac{1}{2} c I^2 + \epsilon I^{5/2} \sin(5a - 97\theta + \varphi) \ , \tag{2.73}$$

where the detuning c is given by Eq. (2.58), and the real numbers ϵ and φ are defined by

$$\epsilon e^{i\varphi} = \frac{9}{8} 2^{5/2} \beta_0^{3/2} [234\vec{S}_1 + 69\vec{S}_2 + 102\vec{S}_3 - 27\vec{S}_4] \ . \tag{2.74}$$

Here, we are not going to give the explicit expression for ϵ because it is extremely complicated. In the actual computation for an array of sextupoles using a computer, however, the evaluation of ϵ is in fact pretty simple.

The expressions for $\vec{S}_j$ are obviously dependent on the absolute locations of the sextupole configuration. However, ϵ needs to be independent of the absolute sextupole locations. If there is only one sextupole, there is only one term in each of $\vec{S}_j$. The phase of each of these terms is the same and therefore does not contribute to ϵ at all. The absolute sextupole location goes into the phase φ. The situation of two sextupoles can also be worked out easily, and we find that ϵ depends on $|\psi_1 - \psi_2|$ only. It is then not hard to convince oneself that ϵ indeed does not depend on the absolute sextupole locations.

II.7 The Fifth-order Resonance

Starting from the sextupole Hamiltonian, we have arrived at a form which clearly describes a system under the action of the $\frac{2}{5}$ resonance. Following the traditional technique, we make a canonical transformation to a rotating system in phase space with the generating function

$$F_2 = \left(a - \frac{97}{5}\theta + \frac{\varphi}{5} \right) I_1 \ . \tag{2.75}$$

Then

$$\Psi = a - \frac{97}{5}\theta + \frac{\varphi}{5} \ , \tag{2.76}$$

$$I = I_1 \ , \tag{2.77}$$

$$H_5(\Psi, I_1) = \delta I_1 + \frac{1}{2} c I_1^2 + \epsilon I_1^{5/2} \cos(5\Psi) \ , \tag{2.78}$$

with $\delta = \nu - \frac{97}{5}$.

In order to have a resonance, the tune must reach $\frac{97}{5}$ at a certain amplitude. Since we start with a base tune of $\nu = 19.415$ which is above $\frac{97}{5}$, we expect the detuning c to be negative. One usually defines a resonance action I_r by

$$|\delta| - |c| I_r = 0 \ . \tag{2.79}$$

The fixed points are given by

$$\frac{\partial H_5}{\partial I_1} = \frac{\partial H_5}{\partial \Psi} = 0 \ , \tag{2.80}$$

or

$$\begin{cases} \sin 5\Psi = 0 \ , \\ \delta + c I_1 + \frac{5}{2} I_1^{3/2} \cos 5\Psi = 0 \ . \end{cases} \tag{2.81}$$

To check whether a fixed point is stable or not, we can expand the Hamiltonian around the fixed point by substituting

$$\begin{cases} I_1 = I_0 + \Delta I \ , \\ \Psi = 2\pi n + \Delta\Psi \qquad n \text{ an integer} \ , \end{cases} \tag{2.82}$$

with I_0 satisfying Eq. (2.81), and get, after dropping the constant terms,

$$H_5 = \frac{1}{2}\left(c \pm \frac{15}{2} I_0^{1/2}\right)(\Delta I)^2 \mp \frac{25}{2}\epsilon I_0^{5/2}(\Delta\Psi)^2 \qquad \text{for} \qquad \cos 5\Psi = \pm 1 \ . \tag{2.83}$$

We can then conclude that the action I_s (I_u) at the stable (unstable) fixed point satisfies

$$|\delta| - |c| I_{s,u} \pm \frac{5}{2}|\epsilon| I_{s,u}^{3/2} = 0 \ . \tag{2.84}$$

Therefore,

$$I_{s,u} \approx I_r \pm \frac{5}{2}\frac{|\epsilon|}{|c|} I_r^{3/2} \ , \tag{2.85}$$

so that $I_u < I_r < I_s$. As is defined in Eq. (2.74), ϵ should be positive. Here, however, we stick to the more general situation that ϵ can be negative.

The boundaries of the stable islands are formed by trajectories joining the unstable fixed points. They are called separatrices and their equation can be easily found by the fact that the Hamiltonian is a constant on the curve. Setting the constant value of the Hamiltonian equal to its value at the unstable fixed point, we get therefore for the separatrices,

$$\delta I_1 + \frac{1}{2}\, c I_1^2 + \epsilon I_1^{5/2} \cos 5\Psi = \delta I_u + \frac{1}{2}\, c I_u^2 \mp |\epsilon| I_u^{5/2} \qquad \delta \gtrless 0 \; . \tag{2.86}$$

This is illustrated in Fig. 1.

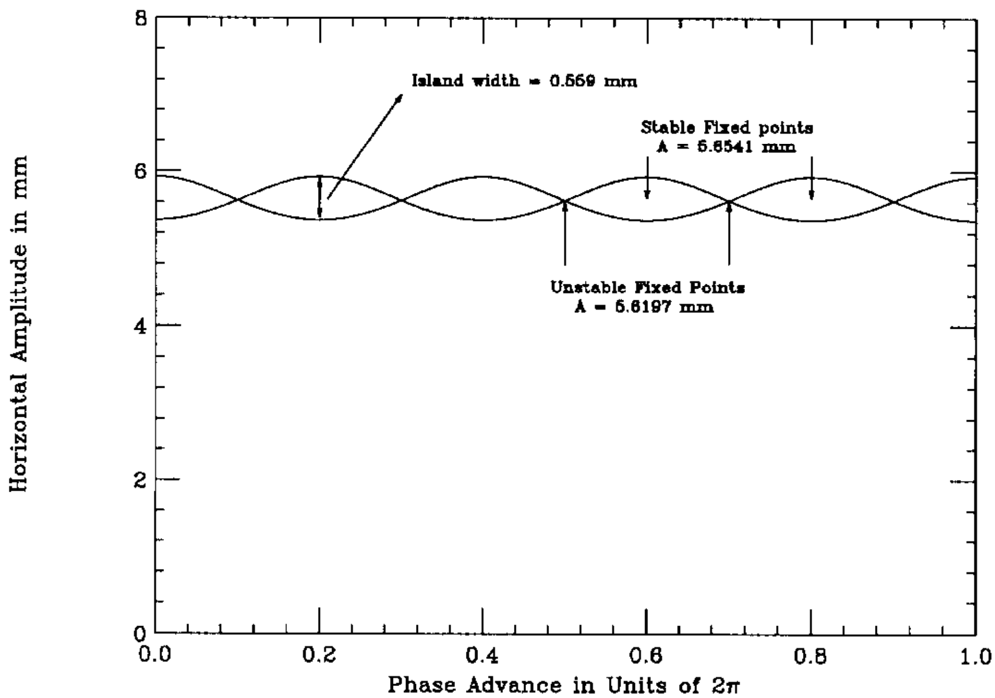

Fig. 1. Contours of the Hamiltonian describing motion under the action of a fifth-order resonance. The sextupole excitation is 25 amperes and the base tune is 19.415.

The island width can be found by evaluating the action of the separatrices when the phase Ψ is at the stable fixed point, or $\epsilon \cos 5\Psi = +|\epsilon|$. With the help of Eq. (2.79), we obtain from Eq. (2.86),

$$|c| I_1 I_r - \frac{1}{2}\, |c| I_1^2 - |\epsilon| I_1^{5/2} = |c| I_u I_r - \frac{1}{2}\, |c| I_u^2 + |\epsilon| I_u^{5/2} \; . \tag{2.87}$$

Noting that I_r is very close to I_u, we have

$$(I_1 - I_u)^2 \approx \frac{4|\epsilon| I_u^{5/2}}{|c|} \ . \tag{2.88}$$

Or, the island's *total* width is[4, 5]

$$\Delta I_w = 4\sqrt{\frac{|\epsilon| I_u^{5/2}}{|c|}} \ . \tag{2.89}$$

A very fundamental concept of the resonance island structure is the frequency of the oscillation of a particle around the center of the island. This quantity can also be expressed in terms of the coefficients ϵ, c and the resonance action I_r which is defined by Eq. (2.79). The island tune Q_I can be read out[5] easily from Eq. (2.83):

$$Q_I^2 = 5^2 |c\epsilon| I_r^{5/2}. \tag{2.90}$$

III. APPLICATION TO E778 - COMPARISON WITH TRACKING

In this section, we will actually compute the island width and island tune for one of the experimental setups[6] used in E778. Unfortunately, the island width was found to be bigger than the actual transverse size of the Tevatron beam. As a result, both the island width and island tune could not be measured directly from the experiment. Instead, we shall compare computed results with the prediction of single-particle tracking calculations.

III.1 Resonance parameters

We first calculate the coefficient c of the detuning term $\frac{1}{2}\, cI^2$, with the use of Eq. (2.58). For the E778 sextupole configuration[6] with sextupole excitation of 25 amperes and a tune of 19.415, the detuning is calculated to be

$$c = -94.42 \text{ mm}^{-1} \ , \tag{3.1}$$

where the reference β_0 has been taken as 100 m.

The calculation of ϵ is relatively straightforward although the explicit expressions for $\vec{S}_j$ ($j = 1, \ \ldots, \ 4$) appear to be lengthy. For the above experimental conditions, ϵ turns out to be

$$\epsilon = 18.30 \text{ mm}^{-3/2} \ . \tag{3.2}$$

Finally we can calculate I_u, the action at an unstable fixed point which satisfies Eq. (2.84), or

$$|\delta| - |c|I_u - \frac{5}{2}|\epsilon|I_u^{3/2} = 0 \ . \tag{3.3}$$

This is a cubic equation in $I_u^{1/2}$ so it can be easily solved. Using the above values of c and ϵ_0 we find that the physically acceptable solution is

$$I_u = 1.579 \times 10^{-4} \text{ mm} \ . \tag{3.4}$$

The amplitude of the particle is related to the action approximately by

$$\mathcal{A} \approx \sqrt{2I_1\beta_0} \ . \tag{3.5}$$

Therefore, at the unstable fixed point it becomes

$$\mathcal{A} = 5.62 \text{ mm} \ . \tag{3.6}$$

III.2 Island Width

Using the above resonance parameters and Eq. (2.89), the island width in terms of action is,

$$\Delta I_w = 3.12 \times 10^{-5} \text{ mm} \ ; \tag{3.7}$$

and in terms of amplitude,

$$\Delta\mathcal{A}_w = 0.555 \text{ mm} \ . \tag{3.8}$$

Figure 2 displays in action-angle representation, the single particle tracking results obtained for the same conditions as these of the above calculation.

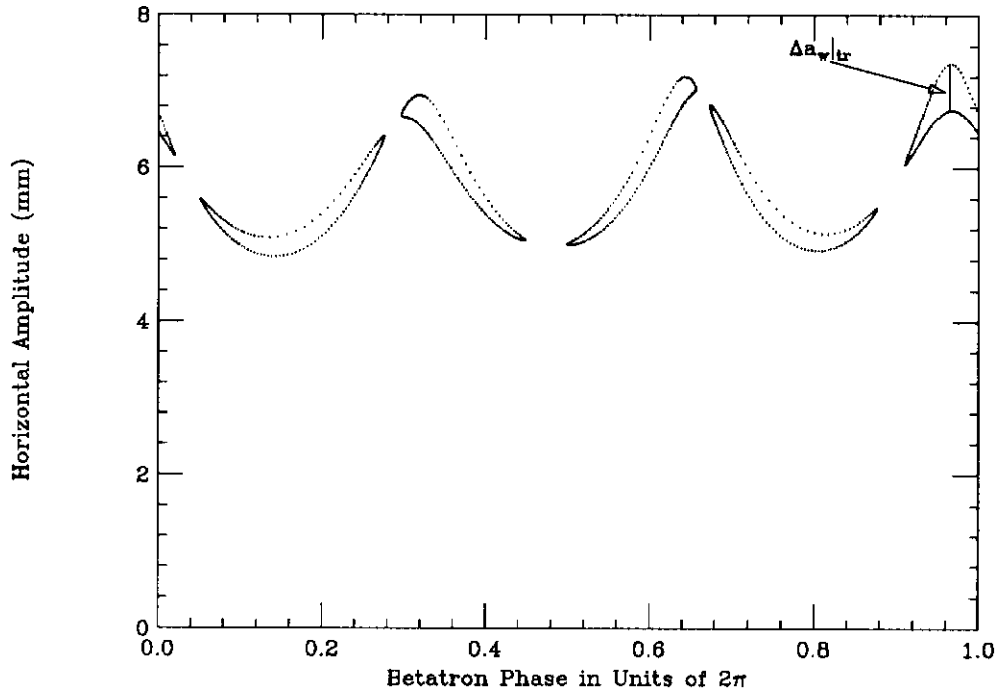

Fig. 2. Action-angle representation of single particle tracking using the code EVOL. The sextupole excitation is 25 amperes, the kick amplitude is 5.25 mm and the initial tune 19.415. $\Delta a_w|_{tr}$ denotes a "possible" definition of the island width in this particular case.

We see that the shape of the five islands is very different from that shown in Fig. 1. This is not very surprising. The Hamiltonian H_5 that we used to derive the islands contains only one resonance explicitly. In fact, the islands are situated very near to the separatrices of the third-integer resonance, as is shown in Appendix A and indicated by the tracking result in Fig. 3. The third-integer resonance driving term has distorted the phase space to a triangular shape on which the five islands are superimposed.

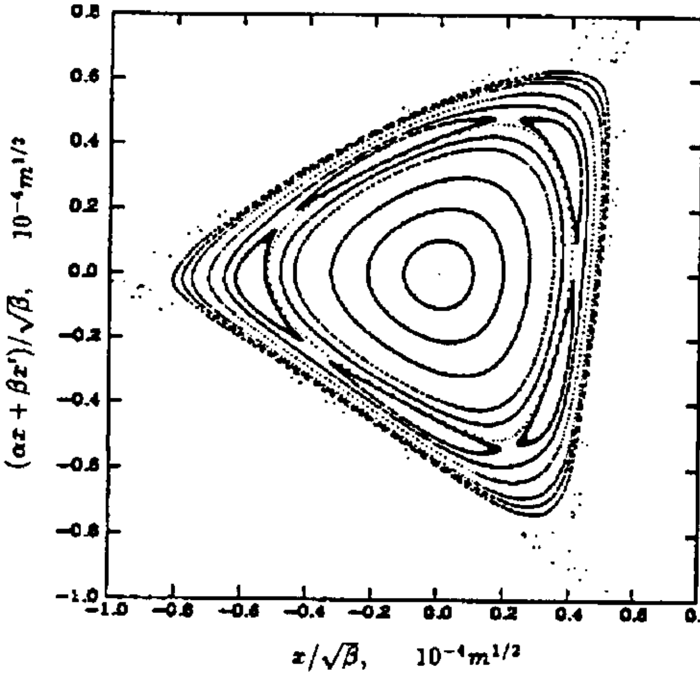

Fig. 3. Poincaré plot generated by numerical tracking particles of various amplitudes.

There has been always a difficulty to include more than one resonance in an analysis. However, such a difficulty can be avoided here. The $\frac{2}{5}$ resonance starts occurring in the third-order perturbation only, while the third-integer resonance occurs in the first-, second-, third-, and higher-order perturbations. As will be shown in the Appendix B the $\frac{2}{5}$ resonance is far away from all other resonances in the third-order perturbation, implying that the third-integer in the third-order perturbation does not influence the $\frac{2}{5}$ resonance at all. This justifies the dropping of all the third-order terms in H_5 when we discuss the $\frac{2}{5}$ resonance. On the other hand, the third-integer resonance in the first-order and second-order perturbations has already been taken care of exactly during the canonical transformations G_3 and G_4 of Eqs. (2.17) and (2.40). What we need to do is to work out the third-integer resonance content in the final action I_1.

The relation between amplitude and action given by Eq. (3.5) is only approximate. The correct relation is

$$\mathcal{A} \approx \sqrt{2I_{\rm old}\beta_0}\ , \tag{3.9}$$

where $I_{\rm old}$ is the original action introduced in Eqs. (2.5); the subscript "old" has been added to distinguish it from the one defined in Eq. (2.40). This original action is related to the final action I_1 through Eqs. (2.21), (2.41), and (2.77). If we keep only the third-integer resonance in its lowest-order sextupole strength, we need only Eq. (2.21) and arrive at

$$\mathcal{A} \approx \sqrt{2\beta_0 I_1}\left[1+\sqrt{2\beta_0 I_1}\sum_m \frac{3A_{3m}}{m-3\nu}\sin(3a-m\theta+\alpha_{3m})\right] , \tag{3.10}$$

which clearly depicts a third-integer wave, into which the $\frac{2}{5}$ resonance islands will be imbedded. In the summation, the most important term is obviously $m=58$. However, we can also perform the summation exactly to give

$$\sum_m \frac{3A_{3m}}{m-3\nu}\sin(3a-m\theta+\alpha_{3m}) = -\frac{\hat{s}}{8\sin 3\pi\nu}\cos\left[3\Psi-3\left(\nu-\frac{97}{5}\right)\theta-3\pi\nu-\frac{3}{5}\varphi+\hat{\varphi}\right] , \tag{3.11}$$

where Eq. (2.76) has been used, and the real numbers $\hat{s}$ and $\hat{\varphi}$ are defined by

$$\sum_k s_k e^{i\psi_k} = \hat{s}e^{i\hat{\varphi}} . \tag{3.12}$$

In the above, θ is the location of turn-by-turn observation along the ring, which is an input of the tracking calculation. Thus, there is no free parameter at all. Unfortunately, some inputs to the tracking calculation are not available. Here, we assume the argument of cosine in Eq. (3.10) to be $3\Psi-\pi/2$ and take only the $m=58$ term with $A_{3m}=0.0022\ \mathrm{mm}^{-1}$. For each value of Ψ, the action I_1 at the separatrices is solved from Eq. (2.86). It is then substituted into Eq. (3.10) to give a plot of amplitude $\mathcal{A}$ versus phase Ψ. The result, shown in Fig. 4, is now very similar to Fig. 2. The width of each island appears to be larger than the corresponding one in the tracking result. The maximum total island width in Fig. 4 is 0.66 mm.

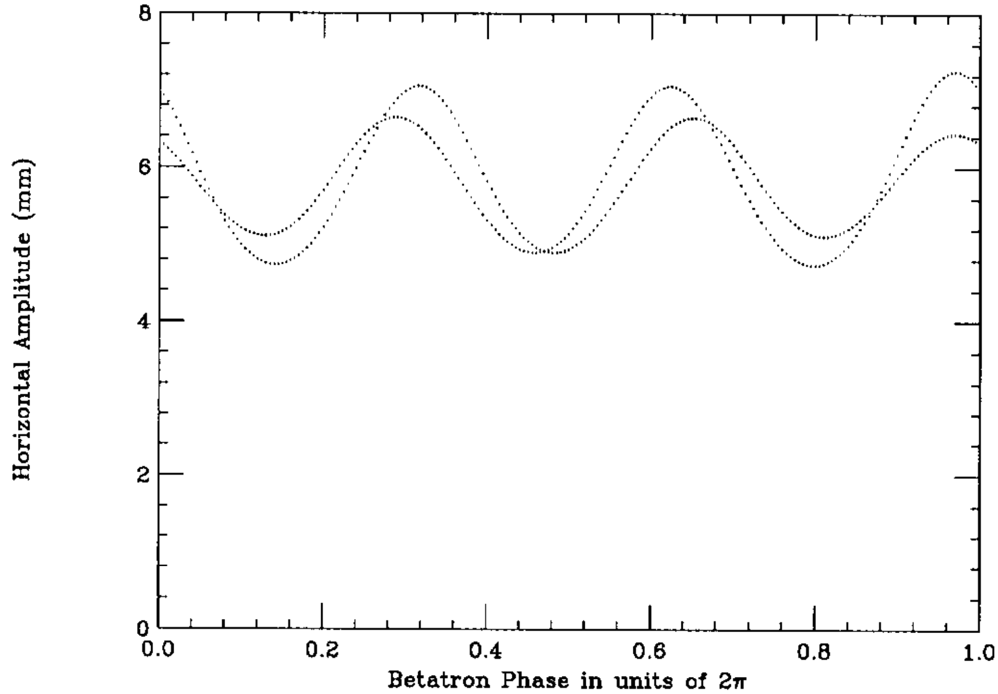

Fig. 4. Contours of the Hamiltonian describing motion under the action of a fifth-order resonance. The distortion due to the presence of a third-integer resonance has been included. The sextupole excitation is 25 amperes, the kick amplitude is 5.25 mm and the initial tune 19.415.

We need to point out that Eq. (3.11) does not describe the third-integer resonance completely. For example, it does not reproduce the third-integer separatrices. This is because so far we have kept only the first-order sextupole term in Eq. (3.11). For a complete description, terms corresponding to all-order-sextupole strength are required. If we only keep the A_{3m} term with $m = 58$, the second-order sextupole term can be included easily. This amounts to the addition of

$$+(2\beta_0 I_1)\frac{9A_{3m}^2}{4(m-3\nu)^2}\left\{5+\cos 2(3a-m\theta+\alpha_{3m})\right\}$$

inside the squared brackets of Eq. (3.11). However, this term does not alter the plot in Fig. 4 appreciably.

III.2 Island Tune

Let us now calculate the small amplitude tune Q_I around the center of the island. Recall that Q_I is given in Eq. (2.90) by

$$Q_I^2 = 5^2|c\epsilon|I_r^{5/2} , \tag{3.13}$$

where I_r is given by Eq. (2.79). A glance at Fig. 2 showing the five islands located at different amplitudes may lead one to think naively that each island would have a different tune which varies with the amplitude to the $\frac{5}{4}$ power. This idea is incorrect. For a $\frac{2}{5}$ resonance, a particle in one island will pop over to the second next island after one turn of the Poincaré map. The fact that a particle will be popping over all the islands makes it meaningless to talk about the island tune of one island. Although the five islands center at different amplitudes, there is only one island tune. In fact, it is better to talk in the language of the final action I_1. Although the amplitudes for the five islands are all different, there is only one resonance action I_r given by Eq. (2.79). The relation between the amplitudes of the five unstable fixed points and I_r can be obtained from Eq. (3.10). As was given by Eq. (3.13), there is only one island tune for a given base tune and sextupole configuration.

The $\frac{5}{2}$ power dependency on the action can, however, be tested. Both c and ϵ cannot be changed because we do not want to change the sextupole configuration. The base tune ν or $\delta = \nu - \frac{97}{5}$ can be varied by varying the quadrupole current. This will give rise to a different I_r. Such a single-particle simulation was performed using EVOL[7]. The results are plotted in Fig. 5, where each point corresponds to one base tune and the amplitude plotted is the "average" amplitude of the particle as it pops over from one island to the next. The tracking calculations suggest that

$$Q_I = 3.8 \times 10^{-5} A_r^{5/2}, \tag{3.14}$$

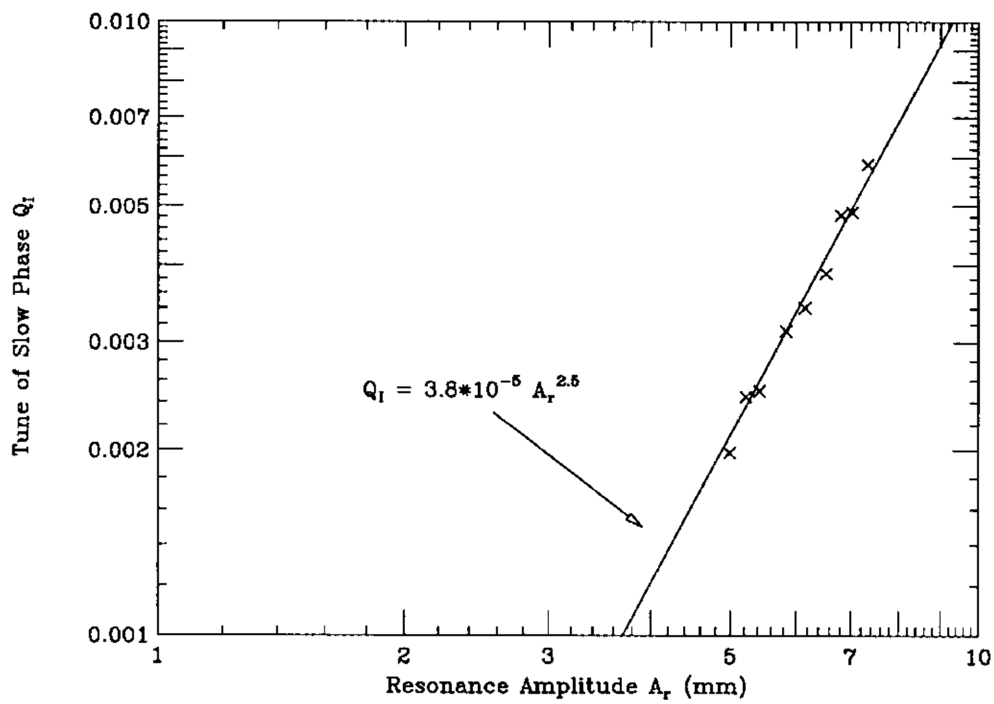

Fig. 5. Single particle tracking results of the island tune Q_I, versus the resonance amplitude A_r in mm, using the code EVOL. The sextupole excitation is 25 amperes and the base tune is 19.415.

To compare tracking with the Hamiltonian theory, we substitute the values c and ϵ into Eq. (3.13) and use Eq. (3.5) to convert action into amplitude. The result is

$$Q_I = 4.9 \times 10^{-5} A_r^{5/2} , \tag{3.15}$$

which is close to the tracking result.

Here, we want to make some remarks about the result comparison. As was noted in the previous subsection, Eq. (3.5) is only approximate in converting action into amplitude. Equation (3.7) should have been used. However, we would also need to know the observation location along the ring and the phase φ and $\hat{\varphi}$ to work out the amplitudes at the island centers or the "average" amplitude. In the tracking, the "average" amplitude is also not easy to determine accurately because the particle is popping from one island to another all the time. A better way to compare result may be to resort to the action I_r instead of amplitude. However, the action is not a physically measurable quantity and it cannot be determined directly from the tracking result. For this reason, it is not surprising to see some discrepancy between theory and tracking.

IV. CONCLUSION

We have successfully derived the $\frac{2}{5}$ resonance through third-order perturbation in a Hamiltonian theory. We derive expressions for the detuning c and the resonance-driving coefficient ϵ, in terms of which the island width and island tune are obtained.

Since the $\frac{2}{5}$ islands are very near to the separatrices of the third-integer resonance, we have the latter resonance also included in the analysis. The inclusion of two resonances has been successful, because these two resonances occur in different orders of the perturbation: the $\frac{2}{5}$ resonance in the third-order while the third-integer in the first order.

The computed island width and island tune are compared with results of single-particle tracking and the agreement has been satisfactory.

APPENDIX

A. THIRD-INTEGER SEPARATRICES

Let us start from Eq. (2.10) and delete all sextupole terms except for A_{3m} with $m = 58$. We then go to a stationary frame and derive the unstable fixed points and separatrices. We obtain for the unstable fixed points:

$$(2\beta_0 I_u)^{1/2} = \frac{|\delta|}{3|A_{3m}|} , \tag{A.1}$$

where $\delta = \nu - m/3$. The separatrices are:

$$(2\beta_0 I_u)^{1/2} = \frac{|\delta|}{6|A_{3m}|\sin(\psi \pm \pi/3)} , \tag{A.2}$$

where ψ is some convenient phase. Therefore, the closest amplitude of the separatrices is

$$\mathcal{A}_{\min} = \frac{|\delta|}{6|A_{3m}|} = 6.31 \text{ mm} , \tag{A.3}$$

where $A_{3m} = 0.0022$ mm and $\nu = 19.415$ have been used. In comparison, the $\frac{2}{5}$ resonance unstable fixed points have an amplitude of 5.62 mm.

B. ISLAND SEPARATION

There are numerous bands of islands, each of which corresponds to a resonance. In Sec. II, we retained only the $\frac{2}{5}$ resonance but dropped all the others that occur in the third-order perturbation. This is justified only when the neighboring island bands are far away. In this section we want to check the validity of this assumption.

The $\frac{n}{m}$ resonance action I_r is defined in Eq. (2.79) as

$$\left|\nu - \frac{n}{m}\right| - |c| I_r = 0 . \tag{B.1}$$

The next neighboring island series with resonance action I_r' is given by

$$\left|\nu - \frac{n'}{m'}\right| - |c| I_r' = 0 , \tag{B.2}$$

where

$$\begin{cases} n' = n \\ m' = m \pm 1 \end{cases} \qquad \text{or} \qquad \begin{cases} n' = n \pm 1 \\ m' = m \end{cases} \quad . \tag{B.3}$$

We therefore have, for island separation,

$$\Delta I_r \equiv |I_r' - I_r| = \frac{1}{|c|}\Delta\left(\frac{n}{m}\right) \ , \tag{B.4}$$

where

$$\Delta\left(\frac{n}{m}\right) \equiv \left|\frac{n'}{m'} - \frac{n}{m}\right| \ . \tag{B.5}$$

With the aid of Eq. (B.3), we obtain

$$\Delta\left(\frac{n}{m}\right) = \frac{1}{m} \quad \text{or} \quad \frac{n}{m(m \pm 1)} \ . \tag{B.6}$$

Using Eqs. (2.89), (B.4), and (B.6), the ratio of total island width to island separation is therefore

$$\frac{\Delta I_w}{\Delta I_r} \approx \frac{4}{\Delta\left(\frac{n}{m}\right)}\sqrt{|c\epsilon| I_r^{5/2}} \ . \tag{B.7}$$

Putting in $n = 97$, $m = 5$, as well as the other computed values for the resonance, we arrive at

$$\frac{\Delta I_w}{\Delta I_r} \approx 0.015 \quad \text{or} \quad 0.00063 \ , \tag{B.8}$$

which is indeed small.

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