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Radiative Electron Polarization: Theoretical Predictions and Explanation of the SPEAR Data

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RADIATIVE ELECTRON POLARIZATION: THEORETICAL PREDICTIONS AND EXPLANATION OF THE SPEAR DATA

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I make various theoretical predictions, which should be testable experimentally, for the behavior of the polarization of an electron or positron beam circulating in a high energy storage ring. In particular, I treat the effects of synchrotron sidebands, including tune modulation and chromaticity. I also offer a theoretical explanation of experimental data taken at SPEAR, and am able to resolve various puzzling features of the data, which have remained unexplained up to now.

1 Introduction

It is known that beams of electrons and positrons circulating in high energy storage rings become spontaneously polarized by the emission of synchrotron radiation; this is now known as the Sokolov-Ternov effect [1]. A theoretical formula for the equilibrium degree of polarization, applicable to realistic accelerator models, and for arbitrary spin resonances, was given by Derbenev and Kondratenko [2], but they did not give an algorithm to actually *evaluate* the formal functions in their formula. I developed such an algorithm, within the framework of perturbation theory, treating only linear orbital dynamics, in Ref. [3], and wrote a computer program called SMILE to implement it. Experimental measurements of the polarization as a function of various accelerator parameters, in particular the beam energy, had previously been made at SPEAR [4], and they found several depolarizing resonances, including higher order ones. Theoretical formalisms to calculate the polarization at that time, however, could only fit one of the first order resonances they saw [4], and not the other resonances, and so the experimenters only published a guide to the eye through their data.

In this report, I make various theoretical predictions for the behavior of the polarization, which should be directly testable experimentally. In addition, I offer an explanation of the SPEAR polarization data [4]. Although my formalism can calculate higher order resonances, I did not publish a fit to the SPEAR data in Ref. [3]. Partly this was because the program was not really ready, it was not able to handle the full range of magnets required, and I did not have a copy of the SPEAR lattice. Program development has continued, however, and recently I obtained a copy of the SPEAR lattice, and was able to fit the data using SMILE. I present my results below. Because of the calculations, I am able to resolve various puzzling features in the data, which have remained unexplained up to now, and to show that the data contain evidence for more resonances than the guide to the eye in Ref. [4] indicates. In this context, one must realize that the experimenters only identified resonances for which there was unambiguous evidence in the data. They did not claim that these were the *only* resonances.

The fit to the data was written up in a letter to the experimenters [5], and presented at a recent conference [6]. I have decided that the simplest way to write this report is just to reproduce the contents of the letter in an Appendix, with appropriate citations and footnotes. First, however, I describe the theoretical predictions for the behavior of the polarization.

Theoretical Predictions

In this part of the report several theoretical predictions are listed, which should be directly testable experimentally. The use of the Compton backscattering laser method in Ref. [4] demonstrated clearly the ability of this technique to measure the polarization to the required degree of accuracy, in addition to showing that the resonances were isolated and clearly discernible in a machine like SPEAR.

2 Technical Details

2.1 First order resonances

The $\hat{n}$ axis of the Derbenev-Kondratenko formula [2] is a solution of the Thomas-BMT equation for spin motion $d\hat{n}/d\theta = \vec{\Omega} \times \hat{n}$, where θ is the ring azimuth and $\vec{\Omega}$ is the spin precession vector in dimensionless units. I decompose $\vec{\Omega} = \vec{\Omega}_0 + \vec{\omega}$, where $\vec{\Omega}_0$ is due to motion on the closed orbit and $\vec{\omega}$ is due to betatron and synchrotron oscillations. The full technical details can be found in many references, e.g. [7] and [8]. It is not necessary to repeat them here. To calculate $\hat{n}$, I use a basis of three orthonormal vectors $\{\hat{l}_0, \hat{m}_0, \hat{n}_0\}$, which are the solutions of the Thomas-BMT equation $d\vec{s} = \vec{\Omega}_0 \times \vec{s}$ on the closed orbit. The vector $\hat{n}_0$, usually vertical, is the value of $\hat{n}$ on the closed orbit. Next I define $\vec{k}_0 = \hat{l}_0 + i\hat{m}_0$ and write

$$\hat{n} = \sqrt{1 - |\zeta|^2}\,\hat{n}_0 + \mathrm{Re}(\zeta \vec{k}_0^*)\ . \tag{1}$$

The fundamental point is that if one can solve for ζ then one knows $\hat{n}$. Note that ζ is complex. I find it useful to employ a complex exponential notation. I shall refer only to ζ below.

The equation of motion for ζ is [7,8]

$$\frac{d\zeta}{d\theta} = -i\vec{\omega}.\vec{k}_0\sqrt{1 - |\zeta|^2} + i\vec{\omega}.\hat{n}_0\zeta\,. \tag{2}$$

I decompose $\vec{\omega}$ into a linear combination of normal modes

$$\vec{\omega} = \sum_{\lambda=1}^{3} \vec{\omega}_\lambda + \mathrm{c.c.} \tag{3}$$

where $\lambda = x, y, z$, say, in an uncoupled machine ($\lambda \equiv$ horizontal betatron, vertical betatron, and synchrotron modes). Then $|\vec{\omega}_\lambda| \propto I_\lambda^{1/2}$, where I_λ is the appropriate orbital action. The first-order solution for ζ (in powers of the orbital actions) is

$$\zeta(\theta) \;=\; \frac{-i}{e^{i2\pi(\nu-\nu_{x,y})} - 1} \int_\theta^{\theta+2\pi} \vec{\omega}^*_{x,y}.\vec{k}_0\, d\theta' \qquad (+\text{ other oscs})$$

$$\propto \int_{\theta}^{\theta+2\pi} k_{x,y}(\theta')\sqrt{\beta_{x,y}}\, e^{-i\psi_{x,y}} e^{i\nu\theta'} f_{x,y}\, d\theta' + \dots \tag{4}$$

I shall focus on the betatron modes for now. Here k is the quadrupole focusing strength, and $f_{x,y}$ is a geometrical factor whose value is proportional to the dot product $\vec{\omega}^*_{x,y}.\vec{k}_0$. The spin integrals for ζ are typically dominated by the quadrupoles, hence only they have been retained above. The above integral is quite similar in form to those from orbital dynamics theory for calculating the action-angle variables for a ring with sextupoles etc., but with a spin phase factor in place of the sextupole strength. Note that ζ diverges whenever the "resonance denominator" $e^{i2\pi(\nu-\nu_\lambda)} - 1$ vanishes, i.e.

$$\nu - \nu_\lambda = 0, \pm 1, \pm 2, \pm 3, \dots \tag{5}$$

Resonances also arise when the complex conjugates diverge, i.e.

$$\nu + \nu_\lambda = 0, \pm 1, \pm 2, \pm 3, \dots \tag{6}$$

2.2 Test: harmonics

The following prediction does not originate with me, but has long been known by workers in the field. It can be found, for example, in the "spin-matching" results of Chao and Yokoya [9]. In an ideal planar ring, $\vec{k}_0$ is horizontal, whereas $\vec{\omega}$, in a quadrupole, is vertical(horizontal) for a horizontal(vertical) betatron oscillation. Hence, for horizontal betatron oscillations, $\vec{\omega}^*_x.\vec{k}_0 = 0$ and so $f_x = 0$ in an ideal ring, but for vertical betatron resonances, $\vec{\omega}^*_y.\vec{k}_0 \neq 0$ and so $f_y \neq 0$ in an ideal ring. Hence, for vertical betatron resonances, the integrand is determined by the ideal lattice, and not by the imperfections. Hence, if the integrand $k\sqrt{\beta}$ has a superperiodicity, say 2 as in SPEAR, then all odd harmonics of the vertical betatron resonances will vanish, and we should see only

$$\nu \pm \nu_y = 0, \pm 2, \pm 4, \pm 6, \dots \tag{7}$$

This will not happen exactly, because imperfections and the nonlinear tunespreads will still yield "forbidden resonances," but they will be much narrower than the "allowed resonances." This explains why the odd harmonic $\nu = 3 + \nu_y$ at 3.605 GeV in SPEAR is so narrow. One can test this claim by measuring the resonances $\nu = \nu_y \pm 1, 2, 3, 4, \dots$, not necessarily in SPEAR, but in any planar ring with a superperiodicity greater than one. Note that the above argument does not apply to horizontal betatron resonances because $f_x = 0$ in a planar ring, and so the integrand is zero in an ideal ring, and is thus proportional to the imperfections (which have no superperiodicity) in a real ring. Therefore one does not get "forbidden" horizontal betatron resonances, e.g. the resonance $\nu = 3 + \nu_x$ at 3.65 GeV in SPEAR is not suppressed by virtue of being an odd harmonic.

2.3 Tune modulation

One of the big difficulties in polarization calculations is that in an ideal planar ring all the resonances vanish. The resonance widths are therefore proportional to the imperfections in the ring. (In the case of vertical betatron resonances the spin integral is nonzero in a perfect planar ring but the vertical emittance is zero. The resonance width is proportional to the spin integral multiplied by the emittance, and is therefore proportional to the imperfections.) This makes predictions of absolute resonance widths difficult. However, the *ratios* of resonance widths are in many cases *independent* of the imperfections, and can be predicted absolutely. The simplest is to consider synchrotron sideband resonances of the first order betatron resonances $\nu \pm \nu_{x,y} = 0, \pm 1, \ldots$

Defining $a = (g-2)/2$, note that synchrotron oscillations lead to a modulation of the spin tune because for a planar ring $\nu = a\gamma$, hence

$$\nu = a\gamma = a\gamma_0 \left(1 + \frac{\delta\gamma}{\gamma}\right) = \nu_0(1 + \sqrt{2I_z}\cos\psi_z) \tag{8}$$

Therefore

$$\zeta(\theta) \to \int k\sqrt{\beta}\, e^{-i\psi} e^{i\left[\nu_0\theta' + \nu_0\sqrt{2I_z}\nu_s^{-1}\sin\psi_z\right]}\, d\theta' + \ldots \tag{9}$$

Using the identity

$$e^{ir\sin\psi} = \sum_m e^{im\psi} J_m(r)\ , \tag{10}$$

this becomes a sum of resonances

$$\zeta(\theta) \propto \sum_m \int k\sqrt{\beta}\, e^{i[\nu_0\theta' - \psi_{x,y} + m\psi_z]} J_m\left(\frac{\nu_0\sqrt{2I_z}}{\nu_s}\right)\, d\theta' + \ldots \tag{11}$$

and this has resonances whenever $\nu_0 = \text{integer} + \nu_{x,y} + m\nu_s$, which are the synchrotron sidebands. The detailed expressions for ζ and $|\gamma(\partial\hat{n}/\partial\gamma)|^2$ are given in Ref. [7]. If we write

$$\left\langle \frac{1}{|\rho|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle = F\left\langle \frac{1}{|\rho_0|^3}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right\rangle_{1^{st}\ order}, \tag{12}$$

we can express the contributions of the synchrotron oscillations by an "enhancement factor" F,

$$F = \sum_{m=-\infty}^{\infty} \frac{W_m^2}{(\delta + m\nu_s)^2} = \sum_{m=-\infty}^{\infty} e^{-\alpha}\left[\left(1 + \frac{J_\epsilon}{J_x}|m|\right)I_{|m|}(\alpha) + \frac{J_\epsilon}{J_x}\alpha I_{|m|+1}(\alpha)\right]\frac{\delta^2}{(\delta + m\nu_s)^2}, \tag{13}$$

where $\delta = \nu - \nu_{first\ order\ resonance}$, the J's are damping partition numbers, the I's are modified Bessel functions, and

$$\alpha = \frac{\nu_0^2\sigma_\epsilon^2}{\nu_s^2}. \tag{14}$$

For *isolated* resonances, the resonance width is proportional to $W_m^{1/2}$.

2.4 Test: ratios of widths

Note that the *ratio* of the resonance widths is deterministic, because the imperfections are the same for all the resonances. There was not enough data in the SPEAR measurements to verify this quantitatively, but several tests are possible:

1. The ratios should be independent of the imperfections. This can be tested by exciting closed orbit bumps. As with all statements from perturbation theory, there will be corrections, but the principal contribution will be independent of imperfections.
2. The $+m\nu_s$ and $-m\nu_s$ resonances should have equal width. Again there will be corrections because the energy spreads are not equal since the energies of the resonances are not equal, but the asymmetry should be small.
3. The absolute value of the ratio of the resonance widths, to that of the parent first order resonance, is calculable, so that in test 1 above the ratios should not only be independent of the imperfections, but the absolute value can also be checked.

2.5 Test: chromaticity

The synchrotron oscillations also modulate the betatron tune via the chromaticity, viz.

$$\nu_{x,y} \to \nu_{x0,y0} + \xi_{x,y} \frac{\delta\gamma}{\gamma} = \nu_{x0,y0} + \xi_{x,y} \sqrt{2I_z} \cos\psi_z \tag{15}$$

and so the synchrotron sidebands in ζ above are actually given by

$$\zeta(\theta) \to \int k\sqrt{\beta}\, e^{-i\psi} \epsilon^{i\nu_0\theta'} \epsilon^{i(\nu_0-\xi)\sqrt{2I_z}\nu_s^{-1}\sin\psi_z}\, d\theta' + \ldots \tag{16}$$

Note therefore that if $\nu_0 = \xi$, the tune modulation will vanish and there will be NO SIDEBANDS, regardless of the imperfections or the value of the energy spread.

This is a very powerful test. One can change the energy spread using resonances at different energies, and the imperfections by closed orbit bumps, and this should not excite the sidebands. Of course the parent first order resonance will not vanish. As always, there will be corrections, because of other terms in the perturbation expansion for $\hat{n}$, but they should be much narrower, so "no sidebands" actually means "very much narrower sidebands."

In SPEAR the spin tune is about $7 - 9$, and in CESR it is about $12 - 14$. In rings like HERA or LEP the spin tune will be about $70 - 100$, and this test may thus be impractical. Note that this test will only work for resonances of the form $\nu - \nu_{x,y} = \ldots$, because for the others $\nu + \nu_{x,y} = \ldots$ the condition for no sidebands is $\nu_0 + \xi = 0$, which requires a negative chromaticity, which would cause an unstable beam. As the chromaticity increases the sidebands of such resonances should grow *wider*. One can test this too. For *clearly discernible resonances*, the resonance widths are still given by the enhancement factor Eq. (13), but with the modifications

$$\begin{aligned}
\text{Width} \quad &\propto \quad W_m\left(\alpha \to \frac{(\nu_0 - \xi)^2\sigma_\epsilon^2}{\nu_s^2}\right) \qquad (\nu - \nu_{x,y} = \pm m\nu_s + \text{integer}) \\
&\propto \quad W_m\left(\alpha \to \frac{(\nu_0 + \xi)^2\sigma_z^2}{\nu_s^2}\right) \qquad (\nu + \nu_{x,y} = \pm m\nu_s + \text{integer}). \qquad (17)
\end{aligned}$$

So for a given resonance one can step the chromaticity and measure the dependence of sideband width on chromaticity, even for resonances of the form $\nu + \nu_{x,y} = \ldots$.

N.B. In Ref. [7], the ratios of the widths of the resonances $\nu = 3 + \nu_x$ and $\nu = 3 + \nu_x - \nu_s$, $\nu = 3 + \nu_x - 2\nu_s$ were fitted without taking the horizontal chromaticity into account, but a satisfactory fit was nevertheless obtained. This is because $\xi_x \simeq 0.3$ and $\nu_0 \simeq 8.28$ [10] so that the chromaticity made a negligible contribution.

2.6 Cancellation of first order resonances

In orbital dynamics, one eliminates resonances by cancelling out the driving term of that resonance. This involves making certain integrals around the ring circumference vanish. Something similar exists in polarization theory, and the procedure is called "spin matching" [9]. The simplest example is to adjust the machine superperiodicity so that various harmonics vanish. I mentioned this above, but I also have in mind something slightly different. Note that the first order solution for ζ involved a sum over the various oscillations. Therefore one adds the integrals *before* averaging over the distribution of amplitudes and phases of the oscillations. One expects that the average will kill any correlation between integrals pertaining to different modes, e.g. horizontal and vertical oscillations.

Now it has been shown, e.g. in Ref [11], that for first order resonances, the relevant integrals *do not depend on the actions and angles*, and so if they cancel each other, they do so *throughout the whole beam*. This sounds counter-intuitive, and indeed it is. However, the resonances are actually determined by the derivative $\gamma(\partial\hat{n}/\partial\gamma)$, which I have ignored up to now, and in $\gamma(\partial\hat{n}/\partial\gamma)$ the first order integrals are independent of the actions and angles. It is not necessary to worry about the subtleties here. Clearly, however, the integrals cannot cancel if the tunes are different — they would diverge at different tunes. However, if one could make the tunes nearly equal, then one could adjust the machine *so as to make the horizontal and vertical betatron resonances nearly cancel each other.*

Since the resonances are determined by imperfections, this will require some empirical closed orbit adjustment. For example one could adjust the machine to make each resonance $\nu = \nu_x + \ldots$ and $\nu = \nu_y + \ldots$ vanish — this has been done successfully at PETRA — then deliberately introduce a controlled closed orbit bump to excite each resonance by a known magnitude and phase. This would prove that the resonance integrals really do add before averaging, even if they belong to different orbital modes — one naively thinks that a horizontal and vertical betatron integral should be "incoherent."

3 Summary

Let us summarize the proposed tests of the theoretical predictions:

1. A scan of the polarization in two energy ranges, so that one can verify the suppression of forbidden harmonics, in a ring with superperiodicity > 1.
2. Ratios of resonance widths, especially synchrotron sidebands (probably the simplest higher order resonances to measure), and independence of these widths on imperfections.
3. Elimination of synchrotron sidebands by adjustment of chromaticity. Also variation of widths with respect to chromaticity.
4. Elimination of horizontal and vertical betatron resonances by playing them off against each other. Requires empirical adjustment of imperfections.

Acknowledgements

I thank the members of the SLAC-Wisconsin Collaboration for their helpful responses to my letter, and D.P. Barber and K. Yokoya for their comments on the theoretical predictions. This work was supported by the Universities Research Association Inc., under Contract DE-AC02-76CH03000 from the Department of Energy.

Appendix: Theoretical interpretation of the SPEAR polarization data

This appendix contains a copy of my overall theoretical fit to the SPEAR data [4], followed by a transcript of a letter written to the SLAC-Wisconsin Collaboration which took the data published in Ref. [4]. The phrase "your paper" below always means Ref. [4]. The letter contains a a copy of the experimental data, then my theoretical fits, and then an explanation of the above.

4 Specific points

I would like to draw your attention to specific features in the experimental and theoretical data, which I think are significant. I invite your opinions on these and other features. I also enclose a summary of the theory and computer program at the end.

4.1 Notation

The notation employed follows that in your paper. The symbols ν_x, ν_y and ν_s denote the tunes of the particle orbits, for the horizontal betatron, vertical betatron and longitudinal (synchrotron) oscillations, respectively. The tune is the orbital oscillation frequency divided by the revolution (Larmor) frequency. The quantity ν is the "spin tune," i.e. the tune of the spin precession in the storage ring. Resonances occur because of coupling between the spin and orbital precessions/oscillations, and generally lead to strong depolarization. A resonance occurs whenever the condition

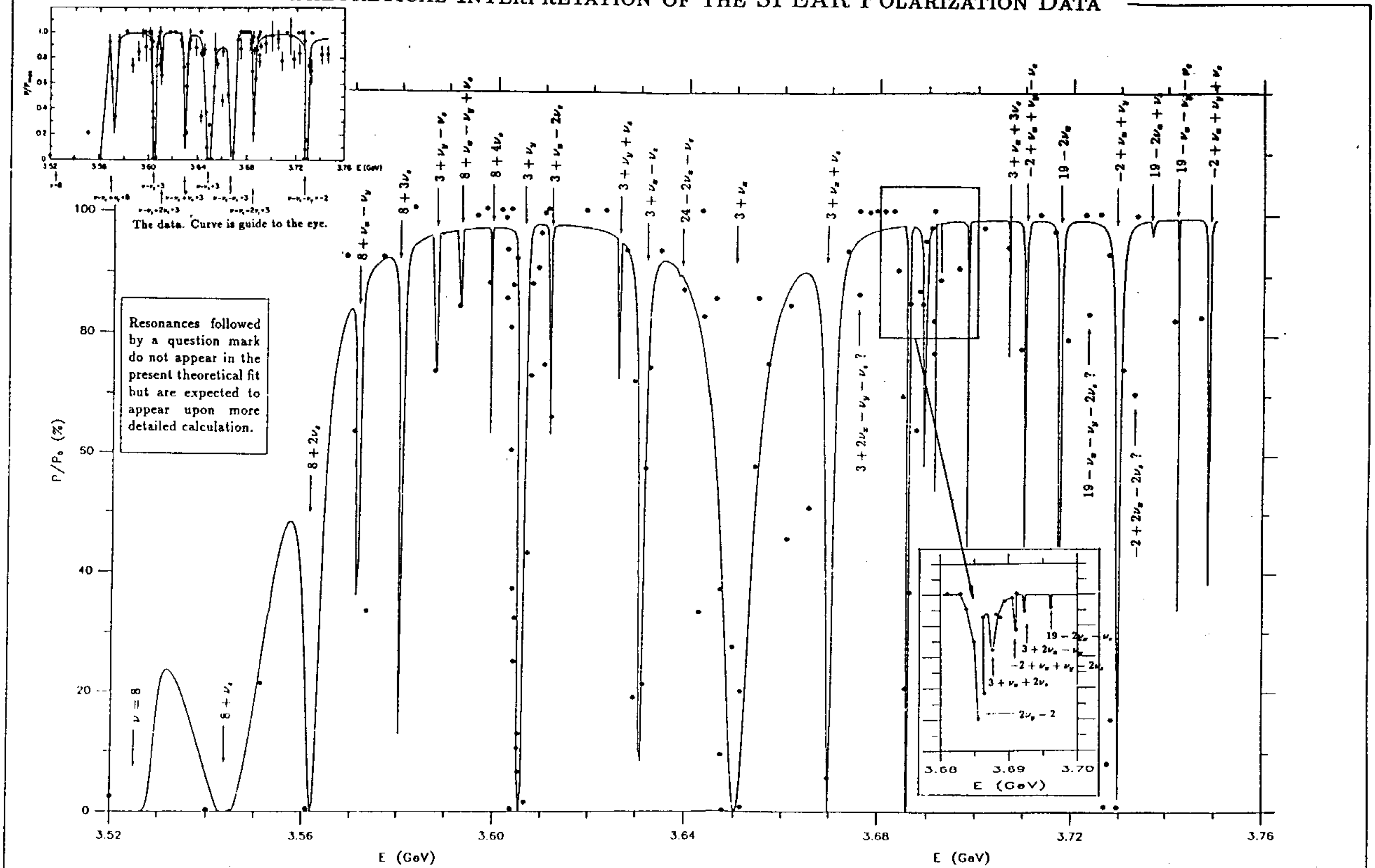
$$\nu = m_0 + m_1\nu_x + m_2\nu_y + m_3\nu_s \tag{18}$$

is satisfied. Here m_0, m_1, etc. are integers, including zero. A low order resonance has small absolute values for m_1, m_2 and m_3, e.g. $\nu = 3 + \nu_x$, while higher order resonances have larger values, e.g. $\nu = 3 + \nu_x \pm \nu_s$ or $\nu = \nu_x + \nu_y - 2$. You identified various low order resonances in your published data, up to the third order resonances $\nu = 3 + \nu_x \pm 2\nu_s$ at approximately 3.61 and 3.69 GeV, respectively.

4.2 Orders of calculation

Calculating the polarization to high orders using SMILE is a fairly time-consuming business (and for machines like HERA and LEP a heavily disk-space-consuming business too), because the number of integrals to be calculated increases rapidly with the order of perturbation theory. The polarization was calculated up to third order resonances in any combination of orbital modes (horizontal and/or vertical betatron, etc.), in steps of $\Delta(a\gamma) = 0.001$, i.e. $\Delta E = 0.440652$ MeV, over the whole energy range. It was calculated to higher orders in selected modes in various smaller energy intervals, when investigating specific higher order resonances, and these "subgraphs" were spliced into the "main" graph. This necessarily implies a choice of the resonances one believes to be important: the theory curves are not a general scan for all possible resonances up to some ridiculous order. I shall return to this point below.

Theoretical Interpretation of the SPEAR Polarization Data



S.R. Mane, Fermilab, September 1988.

Fig. 1 Overall Theoretical Fit to SPEAR data

COPY

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September 7, 1988

The SLAC-Wisconsin Collaboration,
Drs. J.R. Johnson, R. Prepost, D.E. Wiser,
J.J. Murray, R.F. Schwitters, C.K. Sinclair,
Dear Sirs,

Some years ago you measured the positron beam polarization in SPEAR using a laser Compton backscattering technique, and published, amongst other results, a graph of the polarization vs. accelerator energy. You found several depolarizing resonances, and published a fit, by Dr. A. Chao, to the first order ones. Due to lack of a suitable theory, no fit was published to the higher order resonances, and you drew a guide to the eye through your data.

In 1986 I developed a formalism to calculate these higher order resonances, and coded it into a computer program called SMILE. I did not publish a fit to the data at that time due to the lack of a copy of the SPEAR magnet lattice. I have recently obtained a copy, and have applled SMILE to it. I believe, on the basis of my calculations, that you actually found many more resonances than you claimed in your paper. Some are very narrow, shows that your measurements were very precise. I enclose a copy of my results, and invite your comments. The enclosed material contains (i) a copy of your graph, (ii) my theoretical fits, and (iii) a commentary on the above.

I hope to present some of these results at the 8^{th} International Symposium on High-Energy Spin Physics, to take place soon in Minneapolis. Perhaps I shall meet some of you there. I regret that I could not contact you sooner, but I only recently obtained a copy of the SPEAR lattice, and then had to perform various checks before I could circulate these results.

I would be grateful if you would contact me directly with your questions and comments, as I could then answer them more fully, and it would avoid any misunderstandings as to what exactly I claim about your work.

Thanking you in advance.

Yours sincerely,
SRMane
S.R. MANE

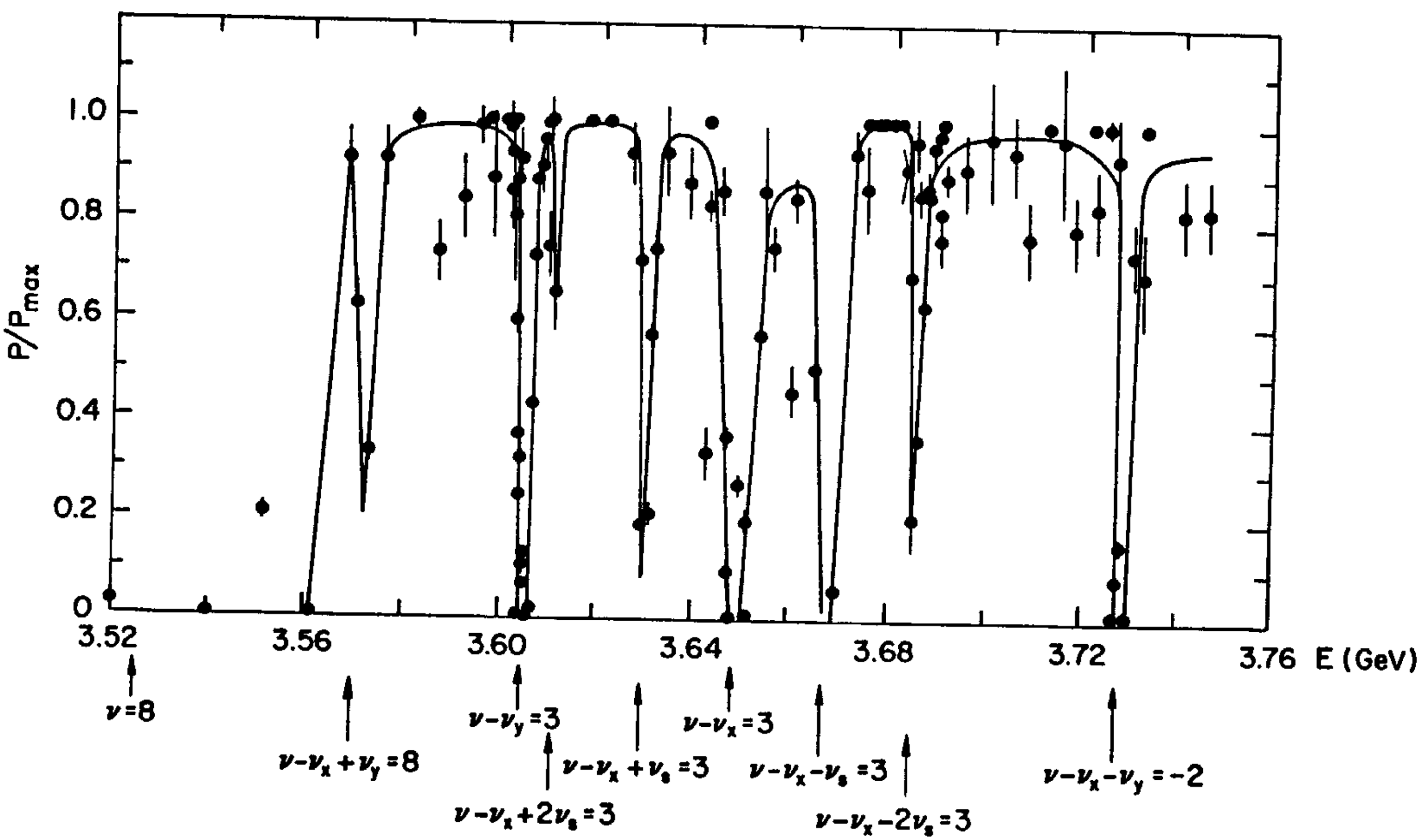


Fig 2 Copy of data published in Ref 4

RANDOM SEED 1

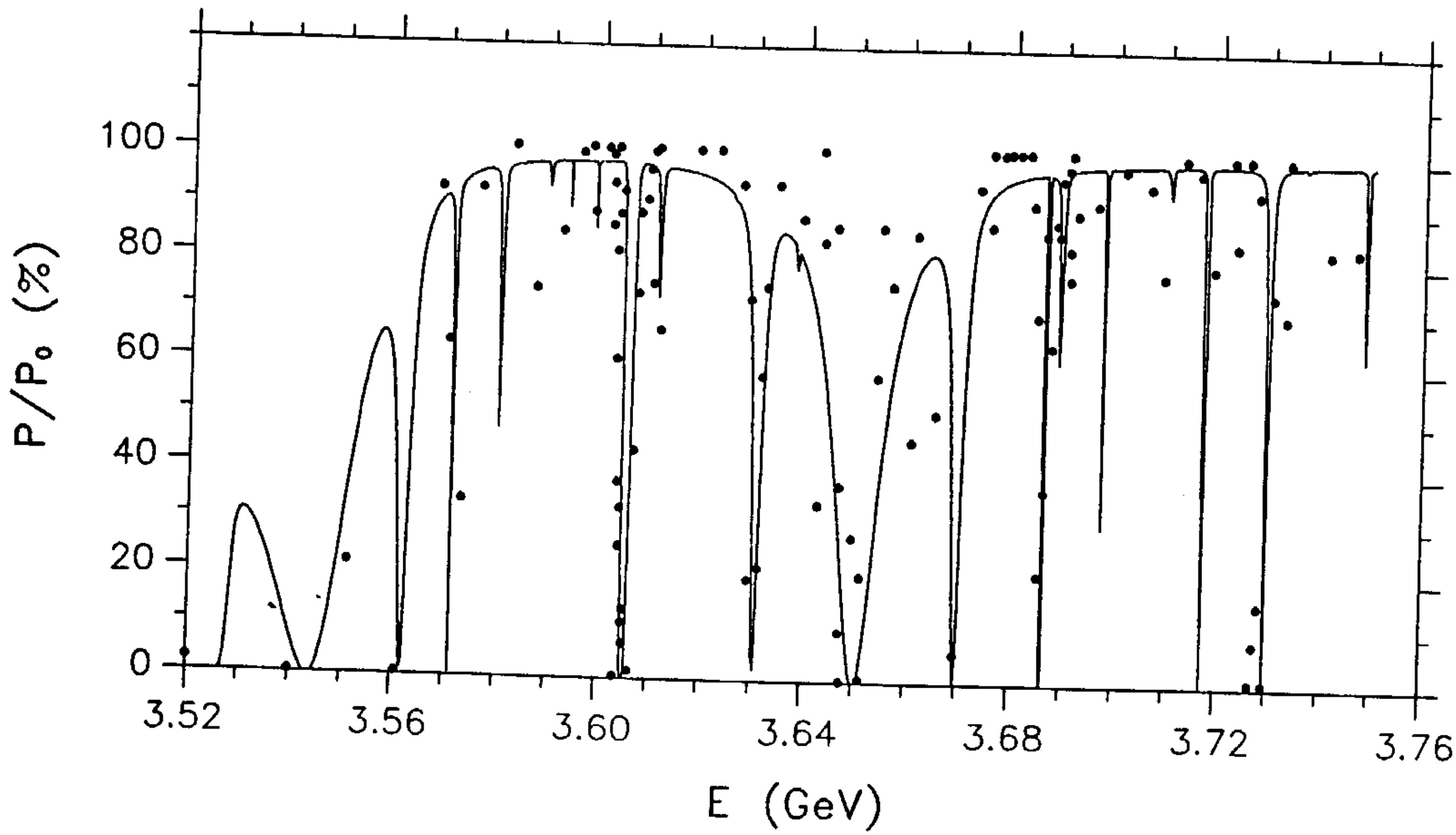


Fig. 3. Fit to data. First random seed.

RANDOM SEED 2

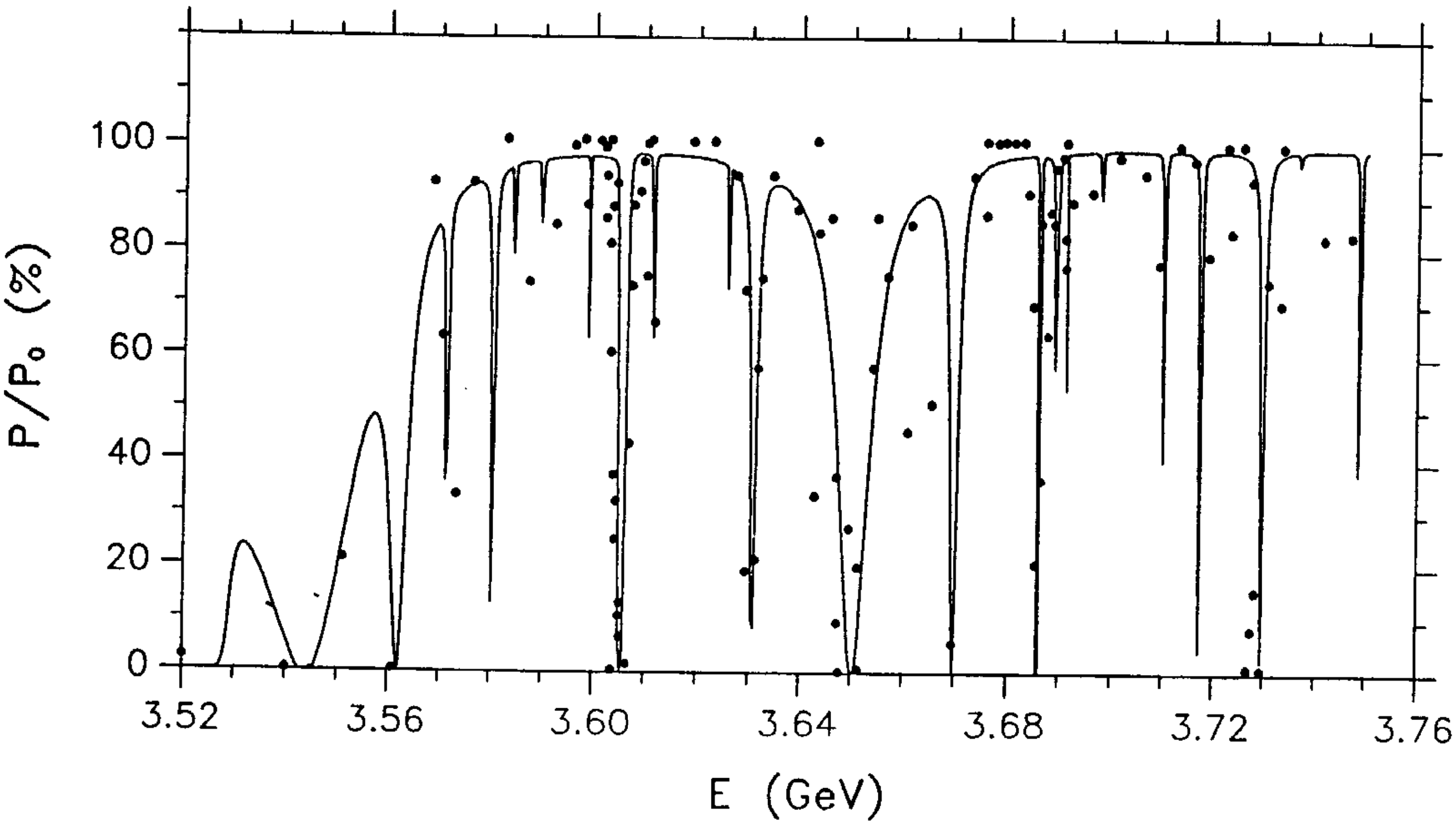


Fig. 4. Fit to data. Second random seed.

RANDOM SEED 3

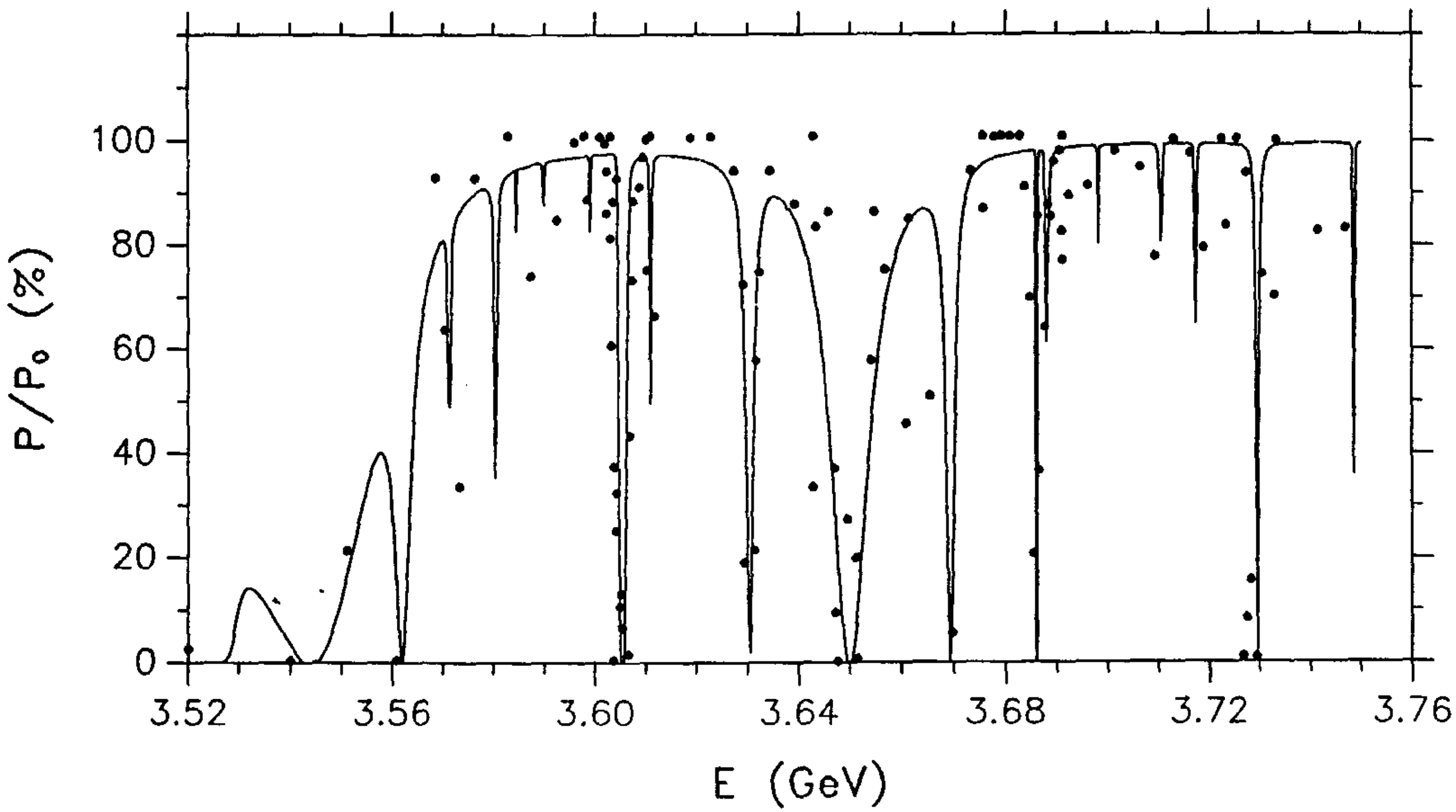


Fig. 5. Fit to data. Third random seed.

Magnified view of energy range 3.59 - 3.61 GeV.

Scan more finely to determine true width of $\nu = 8 + 4\nu_s$ resonance
Finer scan ⇒ very narrow width. Also find another
very narrow resonance $\nu = 24 - 3\nu_x$?

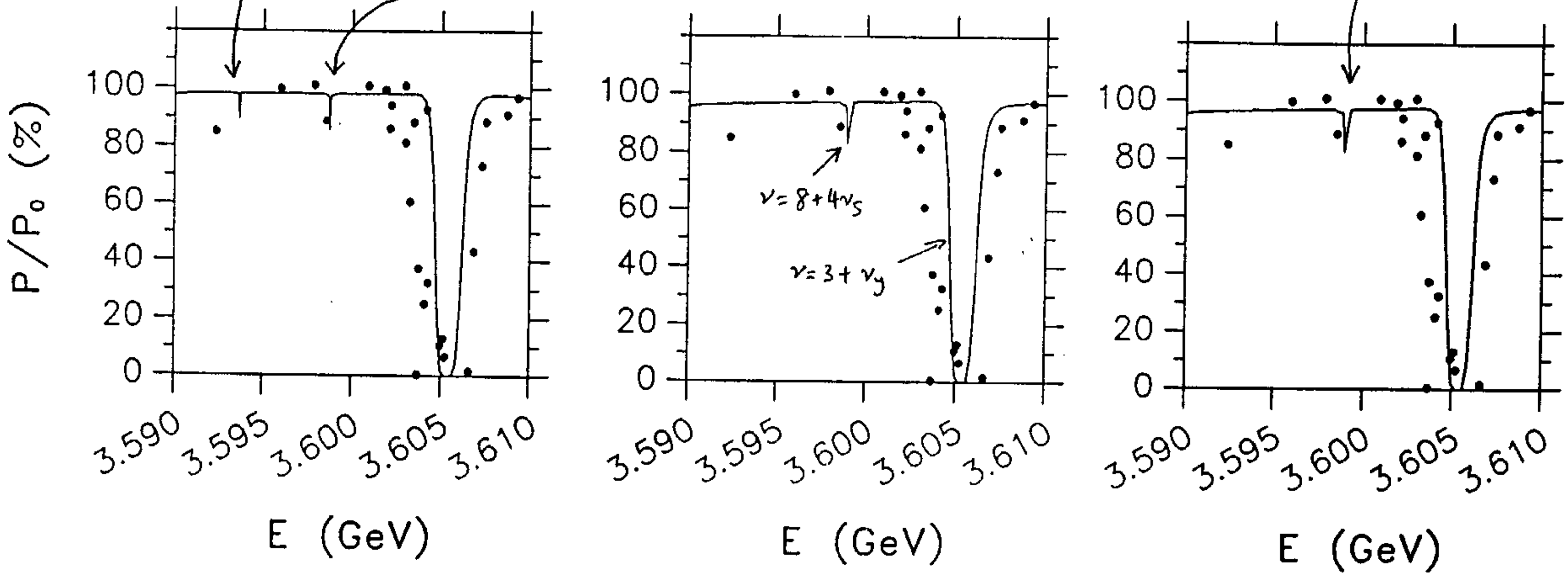
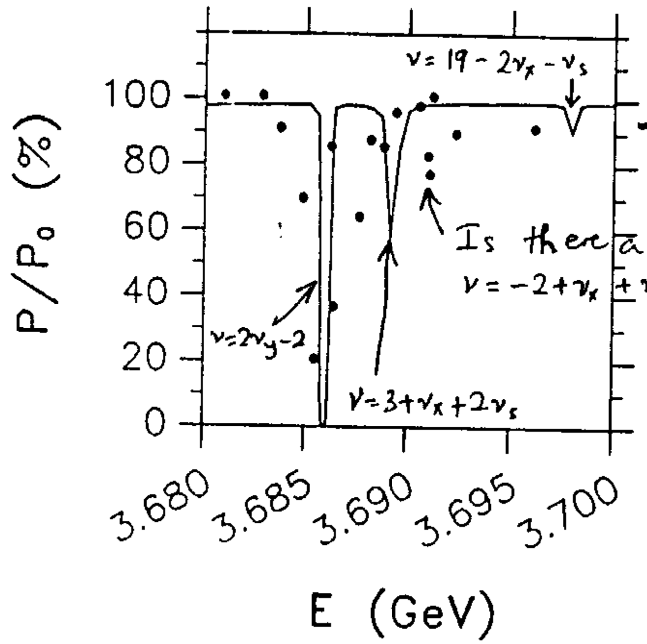


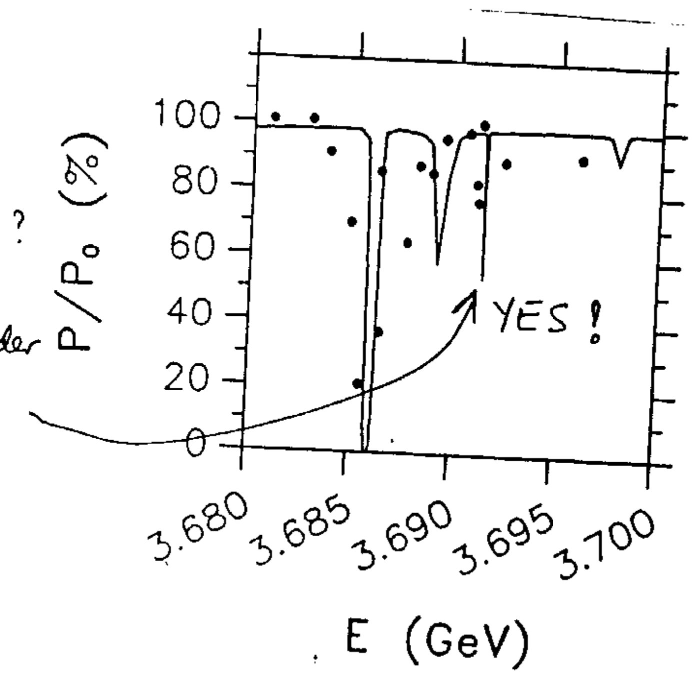
Fig. 6 Magnified view of theoretical fit

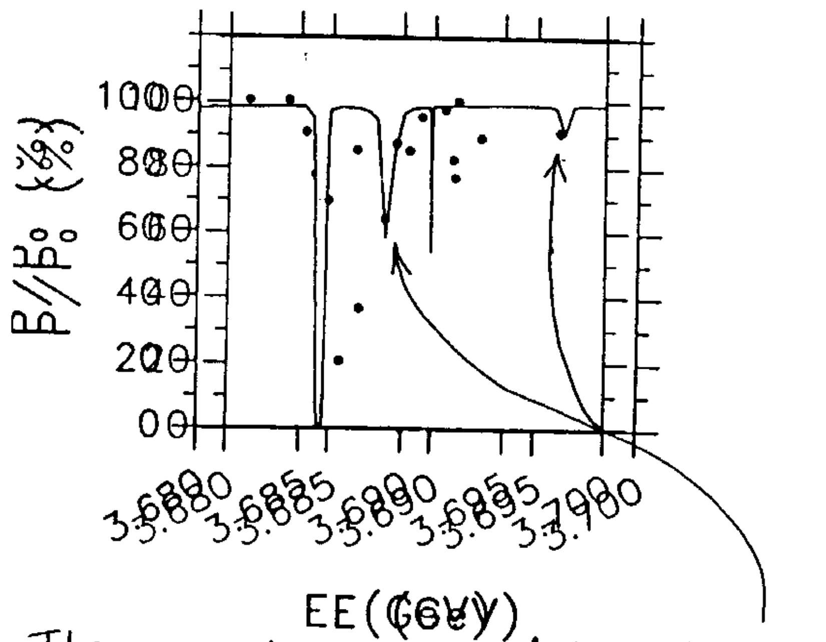
Fig. 7. Magnified view of energy region 3.68–3.70 GeV



Is there a resonance here? $\nu = -2 + \nu_x + \nu_y - 2\nu_s$?

Scan 4th order resonances

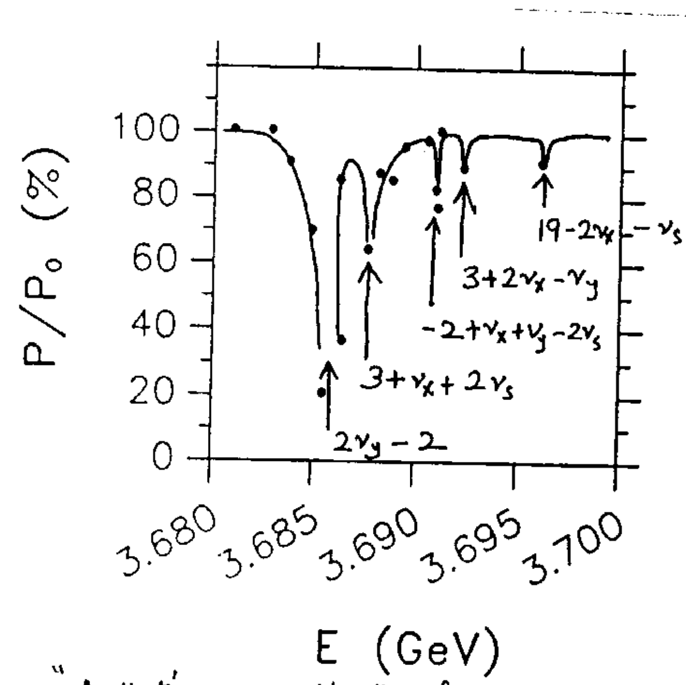




Theory graph shifted relative to expt

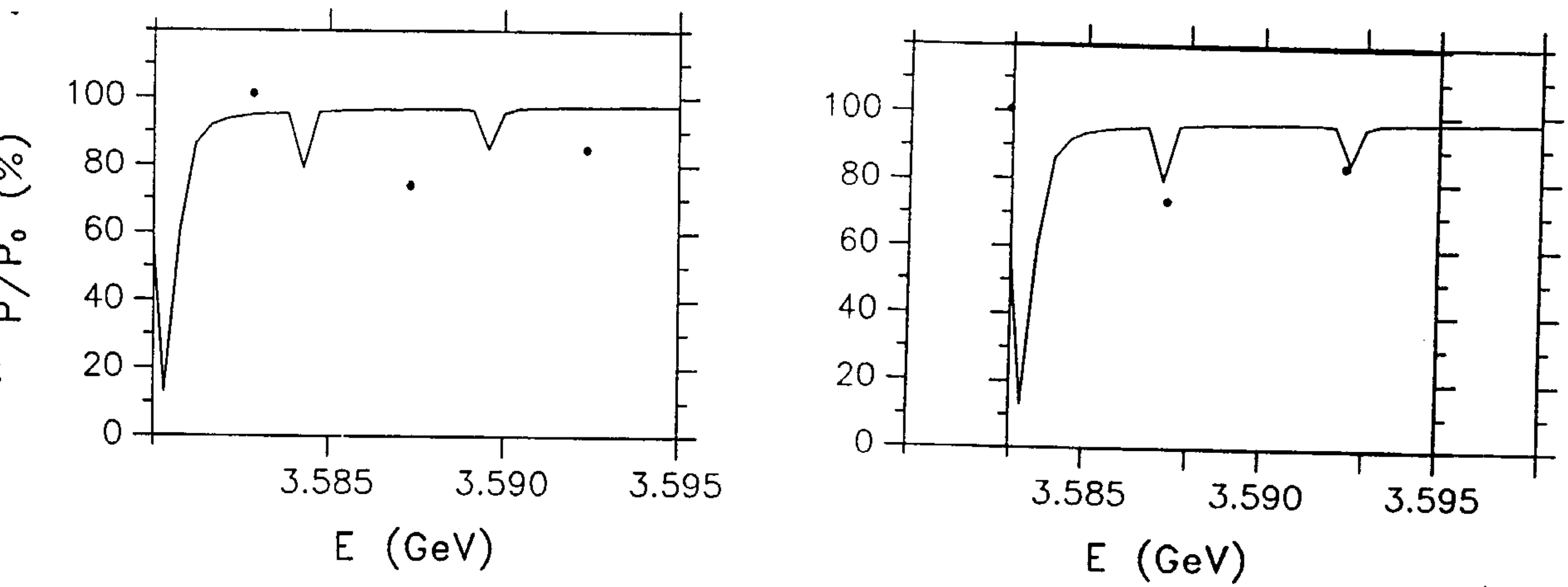
→ resonances involving ν_x fit data much better

resonances involving ν_y no longer fit.



"Artists' conception" of resonance spectrum

Fig. 8 Magnified view of the energy range 3.580 – 3.595 GeV



Theory resonances appear to be displaced from data by a constant amount.

Theory graph shifted relative to data. Both resonances match.

4.3 Values of the tunes

It is stated in your paper that the machine tunes varied slightly between the various runs of data taking, and that this was compensated for, when publishing the data, by using the relationship between the spin tune and the beam energy ($\nu = \gamma(g-2)/2$) to position the resonances appropriately. However, if, say, the horizontal betatron tune was *above* average and the vertical betatron tune was *below* average in a particular scan, there is no way to compensate for both tunes simultaneously.

For example, in the magnified view above of the polarization in the region $E = 3.68 - 3.70$ GeV, there is no way to fit the ν_x and ν_y resonances simultaneously to the data. It can be verified that there is no single set of three numbers (tunes) that will simultaneously fit the published positions of all the resonances exactly — some of the resonances are 1 or 2 mm off on the scale of the graph in your SLAC publication [12]. Since some of the resonances are about 4 or 5 mm wide, this variation is small but noticeable if one attempts to fit the positions of the resonances theoretically, using the same values for ν_x, ν_y and ν_s throughout the whole energy range of the experiment. Therefore, the following values were used to fit the positions of the resonances:

$$\begin{aligned} \nu_x &= 5.281\,, \qquad \nu_y = 5.176\,, \qquad \nu_s = 0.0417\,, \qquad 3.52 < E < 3.60 \text{ GeV} \\ \nu_x &= 5.282\,, \qquad \nu_y = 5.182\,, \qquad \nu_s = 0.0446\,, \qquad 3.60 < E < 3.76 \text{ GeV}\,. \end{aligned} \tag{19}$$

This does not, of course, depend on my theory, or anyone else's. The numbers are chosen by examining the experimental graph. Notice that the theory curves join smoothly; the above variation of the tunes does not visibly affect the polarization. The relevant quantities are the resonance *widths*, not positions, and they are not significantly affected by the above changes in the tunes.

4.4 Imperfections

In a perfect machine all the resonances vanish. Hence imperfections were generated by applying vertical kicks to the orbit around the ring, using random numbers with a Gaussian distribution with zero mean. A simple orbit correction technique was applied, which was to Fourier analyze the closed orbit,

$$y_{c.o.}(\theta) = \sum_k \tilde{y}_k\, e^{ik\theta} \tag{20}$$

and to suppress the two harmonics closest to the vertical betatron tune. Since $\nu_y \simeq 5.18$, the real and imaginary parts of $\tilde{y}_5$ and $\tilde{y}_6$ were suppressed. Such correction is necessary in a simulation because the spectrum of the harmonics $\tilde{y}_k$ changes greatly after correction — the harmonics before correction tend to be peaked near the betatron tune, whereas after correction the spectrum is white. This was observed to be the case in the above fits.

4.5 Resonance $\nu = 3 + \nu_x$ at E=3.65 GeV

As mentioned above, the theory curves depend on random numbers (to determine the absolute resonance widths), so some criterion must be used to determine the standard deviation of the random numbers which generate the imperfections. This was done by fitting the theory curve to the resonance $\nu = 3 + \nu_x$ at 3.65 GeV, so this cannot be considered as a theoretical prediction.

4.6 Resonances $\nu = 3 + \nu_x \pm \nu_s$ at E=3.63/3.67 GeV

Although the absolute resonance widths depend on the imperfections, it can be shown that the *ratios* of the widths of certain resonances do not, and can thus be calculated deterministically (see the figures below). The resonance width can be defined as the interval in which $P/P_0 < 50\%$. In particular, the ratios of the widths of the synchrotron sideband resonances $\nu = m_0 + \nu_{x,y} \pm m_3\nu_s$, $m_3 = 1, 2, 3, \ldots$, to that of the parent resonance $\nu = m_0 + \nu_{x,y}$ are independent of the imperfections. It can be seen that although the three theory curves are generated using different random number seeds, the ratio of the width of the resonance $\nu = 3 + \nu_x - \nu_s$ at 3.63 GeV to that of $\nu = 3 + \nu_x$ is the same (about 0.2), and in all cases the second order resonance fits the experimental data. *This was achieved without further adjustment of the parameters once the orbital tunes and the standard deviation of the random numbers had been fixed* [13]. For the resonance $\nu = 3 + \nu_x + \nu_s$ at 3.67 GeV, there is not enough data to make any quantitative statement. My calculations predict it should have the same width as its "twin" $\nu = 3 + \nu_x - \nu_s$ at 3.63 GeV, to leading order in perturbation theory, with small less singular corrections.

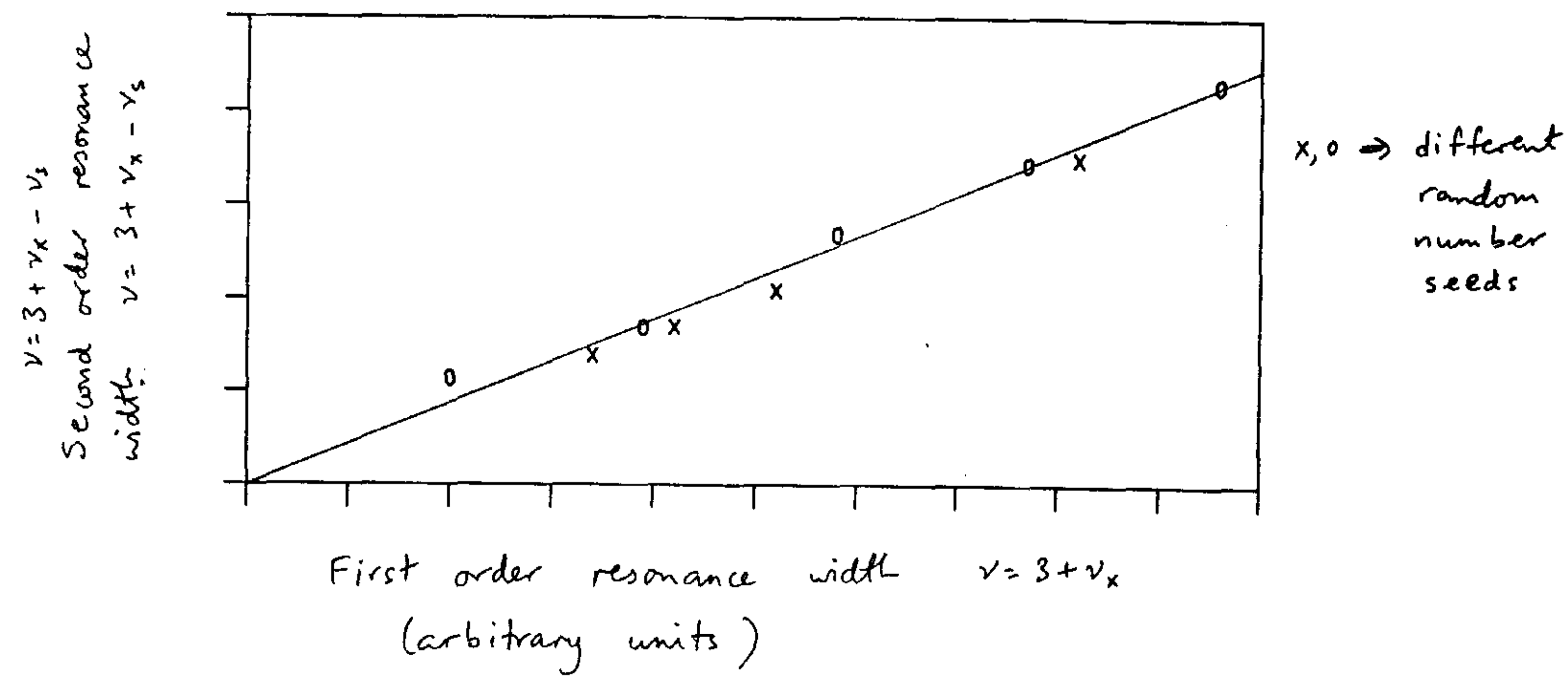
4.7 Resonance $\nu = 3 + \nu_y$ at E=3.605 GeV

4.7.1 Nonlinear tune spread

The width of this resonance was claimed in your paper to be determined by the tune spread of the vertical betatron oscillations, caused by nonlinear dynamics in the machine [14]. The fit by Chao published in your paper is narrower than the observed width. My own calculations confirm this conclusion, but I was able to obtain a reasonable fit in the following way.

My computer algorithm was formulated using only linear dynamics: one of the points I proved [3] is that linear dynamics can generate higher order resonances by itself. However, the algorithm can be modified to include nonlinear dynamics using the following approximation: we assume that the principal effect of the nonlinearities

Fig. 9 Ratio of resonance widths in SPEAR.
Width ($\nu = 3 + \nu_x - \nu_s$) vs. Width ($\nu = 3 + \nu_x$)



Although the <u>absolute</u> resonance widths depend on imperfections, the <u>ratio</u> of resonance widths is independent of the imperfections, to leading order in perturbation theory. The theoretical ratio agrees with the expt data. The scatter in the points can be attributed to nonleading terms in the perturbation expansion, whose values <u>do</u> depend on the imperfections.

Fig. 10. Ratio of resonance widths

NB: This is not the SPEAR lattice.

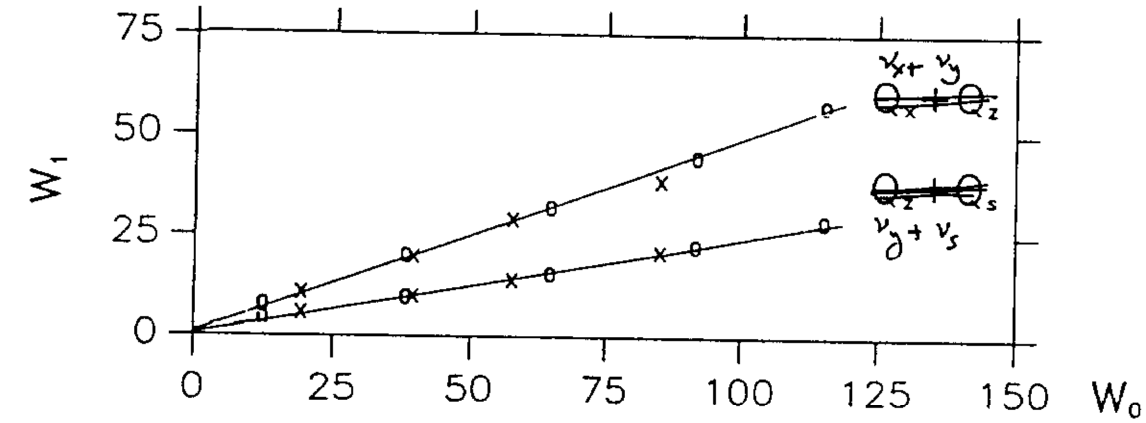


Fig. 2 Higher order resonance widths (denoted W_1) vs. first order resonance width (W_0) in arbitrary units, for two sets of imperfections (crosses & circles).

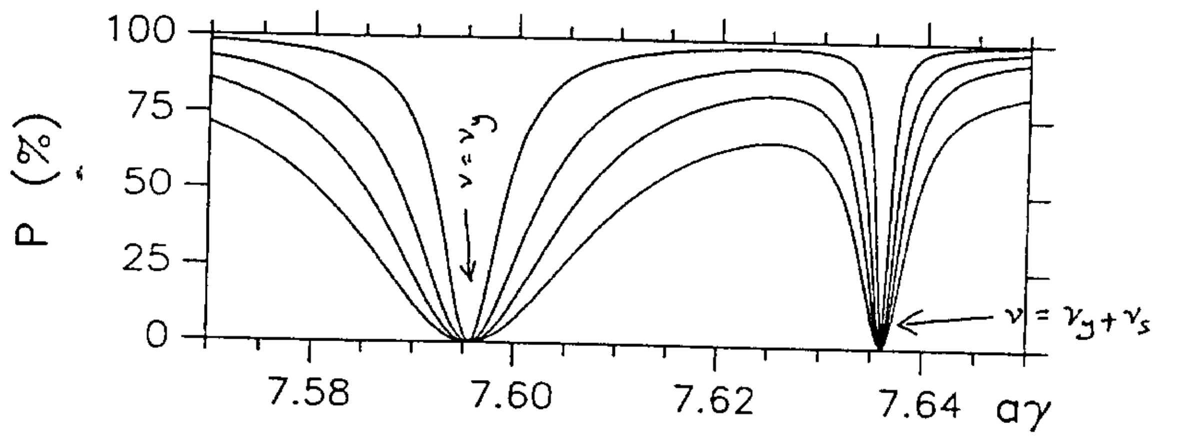


Fig. 3 Polarization vs. $a\gamma$ curves for one set of imperfections in Fig. 2. The linearity in Fig. 2 works even for nearly overlapping resonances.

Ref: S.R. MANE, to be published in a CERN Report on Workshop on Polarization in LEP. ~~(index number not yet available)~~. 88-06 (1988) (Ref. 15).

is to create a tune spread (of the vertical betatron oscillations), but without significantly affecting the amplitudes of the betatron oscillations. Therefore we can express a "nonlinear betatron oscillation" in the form

$$\begin{aligned} y(\theta) &= a_y \cos(\nu_y \theta + \phi_0) && \text{(linear)} \\ \nu_y &\to \nu_y + \nu'_y a_y^2 && \\ y &\to a_y \cos((\nu_y + \nu'_y a_y^2)\theta + \phi_0) && \text{(nonlinear)}\,. \end{aligned} \tag{21}$$

Here a_y is the amplitude, and for sextupole-induced tune spreads it is known that $\Delta\nu_y \propto a_y^2$. Since the solutions for $\hat{n}$ and $\gamma(\partial\hat{n}/\partial\gamma)$ are expressed as power series in a_y, the above nonlinear oscillation is expanded in powers of a_y, and added to the other terms in the series. I repeat that this is an approximate procedure.

In one of your graphs, you published the value of the r.m.s. vertical betatron tune spread, and I used this to fit ν'_y above. Then I ran the program with the approximate nonlinear betatron tune and obtained the above theory curves. Note that I do not claim that the resonance width is equal to the vertical betatron tune spread: it is merely determined by the existence of the tune spread. Without tune spread (linear dynamics only) the theoretical width is much narrower than the observed value, consistent with Chao's finding.

A magnified view of this resonance was given above. I shall discuss it in more detail below.

By the way, I retained the nonlinear tune spread throughout the whole energy scan. It also helps to determine the widths of the resonances $\nu = \nu_y - \nu_s$ (3.585 GeV), $\nu = \nu_y + \nu_s$ (3.625 GeV), and $\nu = 2\nu_y - 2$ (3.686 GeV). Note that they are all narrower than the vertical betatron tune spread. There is no significant effect for other resonances.

4.7.2 Nonlinear resonances

In my nomenclature, the resonance $\nu = 3 + \nu_y$ at 3.605 GeV is an example of a true "nonlinear resonance." Resonances such as $\nu = 3 + \nu_x \pm \nu_s$ are "higher order," *but linear*, resonances, because they can be explained using only linear dynamics. I use the term nonlinear resonance only for cases where the resonance width is determined by nonlinear dynamics. Many other authors refer to the first order resonances as linear resonances, and to *all* higher order resonances as nonlinear resonances. From above, you can see that I refer to the resonance $\nu = 3 + \nu_y$ as nonlinear even though it is of first order.

4.7.3 Magnified view

A magnified view of the energy range 3.59 – 3.61 GeV, which contains the resonance $\nu = 3 + \nu_y$, is enclosed. Notice that the observed width appears to be larger than the theory fit. It suggests that the nonlinearities have a bigger effect than is accounted for by using the approximation I made above. We shall see another example below, where the same thing happens. A curious feature of nonlinear resonances is that they seem to have a different characteristic shape from linear resonances: they are U shaped, whereas the linear ones are V shaped, i.e. the former have steeper walls and flatter valleys. I shall discuss the other very narrow resonances in the graph later.

4.8 Resonances $\nu = 3 + \nu_x \pm 2\nu_s$ at E=3.61/3.688 GeV

4.8.1 Asymmetry of widths

These are two narrow, higher order resonances. Their widths are also determined without further adjustment of the theory after fitting the first order resonance $\nu = 3 + \nu_x$. According to theory, their widths should be equal (to leading order in perturbation theory — I expect any corrections to this to be small [7]). The experimental curve indicates a large difference in the resonance widths — the resonance at 3.688 GeV is much wider than the one at 3.61 GeV. My explanation is as follows: according to my calculations, the wider resonance at 3.688 GeV is actually made up of *two* nearly overlapping resonances, viz. $\nu = 3 + \nu_x + 2\nu_s$ as indicated in the experiment, and $\nu = 2\nu_y - 2$ at a slightly lower energy. The energies are approximately 3.69 and 3.686 GeV, respectively. I think there is some experimental evidence to support this conclusion because there are points with high polarization ($P/P_0 > 80\%$) in the middle of the "resonance," suggesting that there are two narrow, nearly overlapping, resonances there.

4.8.2 Magnified view, evidence for additional resonances

A magnified view of the resonance at 3.688 GeV is enclosed. On this scale, one can see that some of the theory resonances are displaced relative to the data. If the theory graph is shifted relative to the data, a much better fit is obtained for those resonances involving only ν_x. This is in keeping with the spirit of adjusting the energy positions without changing the resonance widths. Looking at the data on this scale, I think there really are two resonances there.

Note also that, on this magnified scale, we can discern evidence for another narrow resonance, which seems to fit the assignment $\nu = -2 + \nu_x + \nu_y - 2\nu_s$. Being a fourth order resonance, it is beyond the maximum order of the general theory scan, which only calculates up to third order resonances. Hence I ran a fine scan to search for fourth order resonances, and there it was, as shown in the top right hand subgraph!

Note, as I pointed out above, that there is no way to fit the experimental locations of all the resonances using fixed values for the tunes. However, a theory scan necessarily requires fixed tunes. Consequently, based on the theoretical calculations, I constructed a hand-drawn "artist's conception" of the experimental resonance spectrum. There is enough data to justify at least three of the five resonances in this tiny energy interval — the experiment really was able to see narrow resonances.

4.8.3 Nonlinear resonance

The $\nu = 2\nu_y - 2$ resonance is nonlinear. Linear dynamics alone predicts a much narrower width. Recall that I said above that I retained the approximate nonlinear dynamics throughout the calculations, and I also said that I would show another example of a nonlinear resonance. Once again, the observed width seems to exceed the theoretical fit, which suggests that the nonlinearities have a bigger effect than my approximate method accounts for. The resonances $\nu = 3 + \nu_x + 2\nu_s$ and $\nu = -2 + \nu_x + \nu_y - 2\nu_s$ on the other hand, are linear, and seem to be fine.

4.9 Narrow higher order resonances

The experimental guide to the eye only identified the low order resonances visible in the data. However, there are several isolated points in the data at which P/P_0 is distinctly lower than the background data, but a smooth curve is drawn through such regions. We have seen some examples already. Such points are marked with an arrow in the figure below. I think that these points of reduced polarization (usually $60 - 70\%$) are evidences of real physics, and should not be ignored. I think that the experiment found several narrow resonances, but because they are so narrow, the centers of the resonances were not hit, and so the polarization did not drop to zero. In particular, I interpret the point at 3.598 GeV, where $P/P_0 = 88\%$ while the other points are approximately 100%, as evidence that you found the very narrow resonance $\nu = 8 + 4\nu_s$ — a fourth order resonance. It can be seen in the magnified view of the region $E = 3.59 - 3.61$ GeV above. Curiously, the first three members of this family, viz. $\nu = 8 + \{\nu_s, 2\nu_s, 3\nu_s\}$, at 3.543 GeV, 3.562 GeV and 3.58 GeV respectively, cannot be positively identified in the experiment, even though they are all wider, due to the scarcity of experimental data below 3.59 GeV. The experimental guide to the eye does not try to identify them either, for the same reason. Too bad.

I think the other arrowed points can also be explained this way. Recall that the experimental values of ν_x, ν_y and ν_s were not exactly constant throughout the energy scan — it is a compilation of many scans. However, the tunes were constant in the theory fits, except as indicated above, and I think this explains why the theoretical locations of these narrow resonances do not exactly coincide with some of the points. There are also some points at which the theoretical fits do not indicate a narrow resonance, although the data do. These are marked with a question mark. This may be because the perturbation expansion was not carried far enough — the resonances only appear at a higher order than that calculated. Recall that I stated above that the theory curves are not a brute force scan of all possible resonances to some very high order.

Note also that, according to theory, the polarization drops to zero at the exact center of a resonance (actually, perturbation theory is then not valid), so the theory curves should, in principle, drop to zero every time a resonance is encountered, but for a narrow resonance this requires a very fine scan, and so the theory curves generally do not drop to zero. It is an indication of narrowness, in its own way.

4.10 The region below 3.59 GeV

The data are sparse in this region, and no attempt is made to draw a guide to the eye below 3.56 GeV. Only one resonance is identified in this region, viz. $\nu = \nu_x - \nu_y + 8$ at 3.57 GeV. The theoretical width of this resonance is narrower than that indicated experimentally, but I think there is no contradiction, because the data are clearly consistent with a narrower resonance. The resonance $\nu = 8 + 3\nu_s$ at 3.58 GeV seems to slot in between the data perfectly, and is therefore not observed, even though it is as wide as many other resonances which *are* observed. Again I think there is no contradiction with experiment.

4.11 The region from 3.64 – 3.67 GeV

None of the theoretical curves really fit the data in this region, even though I stated above that the resonance $\nu = 3 + \nu_x$ was used to calibrate the magnitudes of the random numbers. The "calibration" was an approximate one, to get the general order of magnitude correct — basically, to get the heights of the shoulders on either side to approximately fit the data. I have no real explanation of the data here. Unlike the other clearly discernible resonances, where the data form a distinct U or V shape, the data here do not. The experimental guide to the eye also ignores any "fine structure" in the data in this energy range. It is curious that, even close to the center of the resonance $\nu = 3 + \nu_x$ at 3.65 GeV, there are data with $P/P_0 \simeq 30\%$ at the same energies where $P/P_0 \simeq 0$. The theory just predicts a smooth curve.

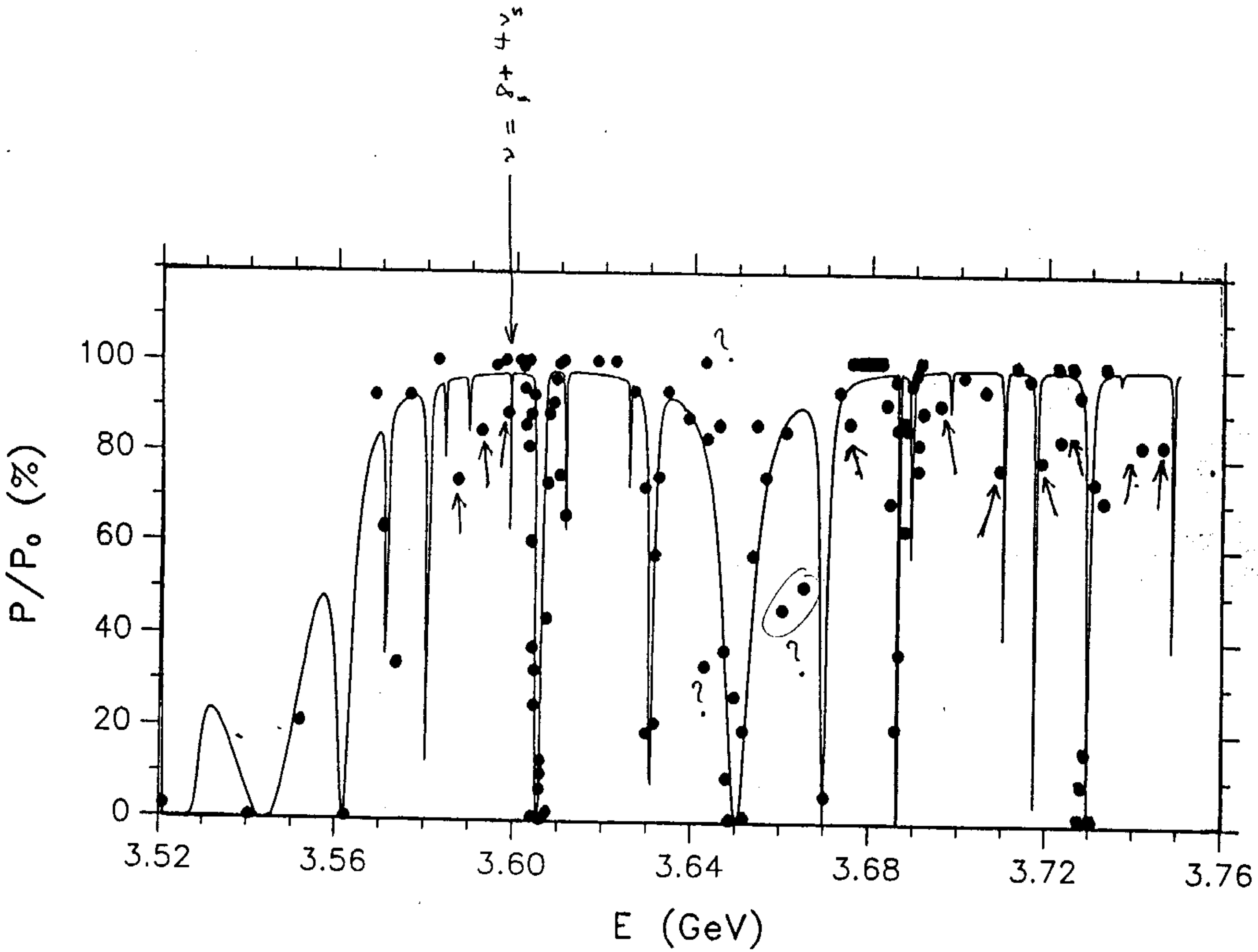


Fig. 11 Additional resonances, not treated in this report.

5 Conclusions

Your experiment provides a clean determination of many spin resonances, including some very narrow ones, to an amazingly high order. There is a rich spectrum of resonances visible in your data. I would not be surprised if you could find fifth order resonances, with more detailed scans. Speaking as a theorist, the data indicate that SPEAR is really an excellent machine to test theoretical predictions because the resonances are well-separated, thus enabling a precise determination of the resonance widths, and resonances of high order are visible, as well as resonances whose widths are clearly determined by nonlinear dynamics. It is not high-energy physics *per se*, but it is an excellent display of Hamiltonian dynamics coupled with statistical mechanics.

A summary of the theory and computer program follows. Thank you.

6 Summary of theory and computer program

The polarization is calculated using the Derbenev-Kondratenko formula [2]. You may recall that the formula is

$$P = \frac{8}{5\sqrt{3}} \frac{\left\langle |\rho|^{-3}\, \hat{b}.\left[\hat{n} - \gamma\frac{\partial\hat{n}}{\partial\gamma}\right]\right\rangle}{\left\langle |\rho|^{-3}\left[1 - \frac{2}{9}(\hat{n}.\hat{v})^2 + \frac{11}{18}\left|\gamma\frac{\partial\hat{n}}{\partial\gamma}\right|^2\right]\right\rangle} \,. \tag{22}$$

Here $\vec{v}$ is the particle velocity, γ is the particle energy in units of rest mass energy, $\hat{b} \equiv \vec{v}\times\dot{\vec{v}}/|\vec{v}\times\dot{\vec{v}}|$, $\hat{n}$ is the spin quantization axis and ρ is the radius of curvature of the particle trajectory. The angular brackets denote an average over the distribution of particle orbits and the ring azimuth. I evaluate this formula using a suitably defined perturbation expansion to calculate the spin-orbit coupling, hence the vectors $\hat{n}$ and $\gamma(\partial\hat{n}/\partial\gamma)$. The details are given in Ref. [3]. It is a fairly long paper (14 pages), and I do not think it is essential for you to digest its contents. A description of the algorithm is given below, and I hope it is adequate.

The solutions for $\hat{n}$ and $\gamma(\partial\hat{n}/\partial\gamma)$, in terms of the accelerator parameters (such as the beta functions, beam energy, quadrupole focusing strengths, etc.) are expressed as series expansions in powers of the amplitudes of the orbital oscillations, i.e., the N^{th} term contains N powers of betatron oscillations, and yields an expression for the N^{th} order resonances (plus corrections to the lower order resonances)...

[*Note added in present report:* The reader should be aware that higher order spin integrals can modify the widths of lower order resonances. This has been observed in results from SMILE. Perhaps the simplest way to see this is to calculate $\hat{n}$ *exactly* for a simple, but nontrivial, model. This has been done in Ref. [15], for a planar ring with only one resonance. The reader can then compare the exact solution with, say, a first order perturbation theory approximation, to see the corrections induced by higher order terms in the perturbaton series.]

...Thus the first order term is linear in the betatron oscillations, and coincides with Chao's results — at the first order of the perturbation expansion, it reproduces Chao's results for the first order resonances [11] (your paper contains a graph where those results were used to fit some of the resonances you found). However, the perturbation expansion also goes to higher orders, and yields expressions for higher order spin resonances. The technical details of how I calculate these terms (they are integrals around the storage ring circumference) are boring, and not essential, I think. Explicit expressions for these higher order integrals are complicated, unless one makes drastic simplifying approximations, and are also not essential.

The computer program is called SMILE, and can, in principle, calculate to arbitrary orders, given enough time and disk space. It is also possible to instruct the program to calculate to a high order in only the longitudinal (synchrotron) oscillations, say, leaving the transverse betatron oscillations alone, if it is felt that the longitudinal oscillations are more important: the user specifies the maximum order of calculation for each mode of oscillation. Provision is made to include the effects of machine imperfections. I have obtained a copy of the SPEAR magnet lattice, and run SMILE with it, with different random number seeds to generate different sets of imperfections. It is obviously not exactly the same magnet lattice as the one you worked with, but not too far different, I hope.

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[13] In practice, the chromaticity also affects the ratio. The effects of chromaticity are not discussed in the appendix. They are treated in the section on theoretical predictions. They have a negligible effect for the SPEAR data, as proved earlier in the section on theoretical predictions.

[14] It was *conjectured* by the authors of Ref. [4] that the width of the resonance $\nu = 3 + \nu_y$ was due to power supply ripple. I attribute it to real dynamics (nonlinear tunespread) and, by using an approximate model described in the above section, I can fit the width of this and another nonlinear resonance (viz. $\nu = 2\nu_y - 2$ at 3.686 GeV).

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