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# COUPLING BETWEEN COUNTER-ROTATING BUNCHES

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## I. INTRODUCTION

Coupled bunch instabilities can occur in a storage ring filled with many bunches rotating in the same direction. Then, is it possible for coherent coupled-bunch instabilities to develop for counter-rotating bunches? In this note, we find that it is not possible for counter-rotating bunches to couple (except for the effects of bunch-bunch collision which we are not going to discuss here). The reason is that the driving force of the coupling depends on the distance between the bunches and so is the coherent coupled frequency. For bunches rotating in opposite directions, the distance between the bunches changes rapidly and the driving force therefore averages to zero. In the following, we will prove it mathematically.

## II. THE MATHEMATICS

Consider two bunches revolving in opposite directions with velocity $c$. They carry positive and negative charges respectively. Assume that their respective synchronized particles move like

$$\begin{cases} s = ct & \text{positively-charged ,} \\ s = -ct & \text{negatively-charged ,} \end{cases} \tag{1}$$

where $s$ is the distance measured along the storage ring from the point these two particles meet. Let the arrival times of the positively-charged and negatively-charged bunches be $\tau_\pm$ respectively relative to their synchronized particles. Thus, $\tau_+$ evolves as $s$ becomes more positive while $\tau_-$ evolves as $s$ becomes more negative.

At position $s$, the positively-charged bunch is affected by the wake fields of itself at the previous positions which are roughly $-cT_0$, $-2cT_0$, $-3cT_0$, etc. where $T_0$ is the revolution period of the synchronized particle. It is also affected by the wake fields of the negatively-charged bunch at roughly the positions $-s$, $-s + cT_0$, $-s + 2cT_0$, etc. The equation of longitudinal motion is

$$\frac{d^2\tau_+}{d(s/c)^2} + \omega_s^2\tau_+ = \frac{\alpha N e^2 c}{E}\Bigg\{\sum_{k=-\infty}^{\infty} W_0[kT_0 + \tau_+(s/c - kT_0) - \tau_+(s/c)]$$

$$- \sum_{k=-\infty}^{\infty} W_0[kT_0 + 2s/c + \tau_-(-s/c + kT_0) - \tau_+(s/c)]\Bigg\} \ , \tag{2}$$

where $\omega_s$ is the synchrotron frequency, $\alpha$ the momentum compaction factor, $N$ the number of particle in each bunch, $E$ the particle energy, and $W_0$ the longitudinal wake function. Here, the summation over $k$ has been extended to negative infinity because the wake function vanishes for negative time. Similarly, the equation of motion for $\tau_-$ can be written as

$$\frac{d^2\tau_-}{d(s/c)^2}+\omega_s^2\tau_- = \frac{\alpha Ne^2c}{E}\left\{\sum_{k=-\infty}^{\infty} W_0[kT_0+\tau_-(-s/c+kT_0)-\tau_-(-s/c)]\right.$$

$$\left. -\sum_{k=-\infty}^{\infty} W_0[kT_0+2s/c+\tau_+(s/c+kT_0)-\tau_-(-s/c)]\right\} , \quad (3)$$

Note that the Eq. (3) can be obtained from Eq. (2) by

$$\begin{cases} \tau_+(s/c)\to\tau_-(-s/c) , \\ \tau_-(-s/c)\to\tau_+(s/c) . \end{cases} \quad (4)$$

The wake functions in Eqs. (2) and(3) are expanded in Taylor's series keeping only the lowest-order terms (the potential distortion terms are neglected):

$$\frac{d^2\tau_+}{d(s/c)^2}+\omega_s^2\tau_+ = \frac{\alpha Ne^2c}{E}\left\{\sum_{k=-\infty}^{\infty} [\tau_+(s/c-kT_0)-\tau_+(s/c)]W_0'(kT_0)\right.$$

$$\left. -\sum_{k=-\infty}^{\infty} [\tau_-(-s/c+kT_0)-\tau_+(s/c)]W_0'(kT_0+2s/c)\right\} , \quad (5)$$

$$\frac{d^2\tau_+}{d(s/c)^2}+\omega_s^2\tau_+ = \frac{\alpha Ne^2c}{E}\left\{\sum_{k=-\infty}^{\infty} [\tau_-(-s/c+kT_0)-\tau_-(-s/c)]W_0'(kT_0)\right.$$

$$\left. -\sum_{k=-\infty}^{\infty} [\tau_+(s/c-kT_0)-\tau_-(-s/c)]W_0'(kT_0+2s/c)\right\} . \quad (6)$$

We are looking at a coherent growth like

$$\begin{cases} \tau_+(s/c)=\tilde{\tau}_+e^{-i\Omega s/c} , \\ \tau_-(-s/c)=\tilde{\tau}_-e^{-i\Omega s/c} , \end{cases} \quad (7)$$

so that when $\mathcal{I}m\,\Omega > 0$, both $\tau_+(s/c)$ and $\tau_-(s/c)$ are growing. The longitudinal impedance $Z_{||}$ is related to the wake function by

$$\frac{Z_{||}(\omega)}{C} = \int_{-\infty}^{\infty} ds W_0(s/c)e^{i\omega s/c} , \quad (8)$$

where $C$ is the circumference of the storage ring. Substituting Eqs. (7) and (8) in Eqs. (5) and (6), we get

$$(-\Omega^2+\omega_s^2)\tilde{\tau}_+ = -i\xi(a\tilde{\tau}_+ - b\tilde{\tau}_-) , \quad (9)$$

$$(-\Omega^2 + \omega_s^2)\tilde{\tau}_- = -i\xi(a\tilde{\tau}_- - b\tilde{\tau}_+) \ , \tag{10}$$

where $\omega_0$ is the revolution frequency,

$$\xi = \frac{\alpha N e^2}{E T_0^2} \ ,$$

$$a = \sum_{p=-\infty}^{\infty} \left[\nu_p Z_{||}(\nu_p\omega_0) - p Z_{||}(p\omega_0) e^{-i2p\omega_0 s/c}\right] \ ,$$

and

$$b = \sum_{p=-\infty}^{\infty} \left[\nu_p Z_{||}(\nu_p\omega_0) e^{-i2\nu_p\omega_0 s/c}\right] \ . \tag{11}$$

Eqs. (9) and (10) can now be written in the matrix form,

$$\begin{pmatrix} a-\lambda & -b \\ -b & a-\lambda \end{pmatrix} \begin{pmatrix} \tilde{\tau}_+ \\ \tilde{\tau}_- \end{pmatrix} = 0 \ , \tag{12}$$

where the eigenvalue is

$$\lambda = \frac{-\Omega^2 + \omega_s^2}{-i\xi} \ . \tag{13}$$

The solution is simple:

$$\lambda_\pm = a \pm b \qquad \begin{pmatrix} \tilde{\tau}_+ \\ \tilde{\tau}_- \end{pmatrix} = \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} \ . \tag{14}$$

Thus, it appears that the two bunches are coupled. However, when the eigenvalues are written out explicitly,

$$\lambda_\pm = \sum_p \left\{\nu_p Z_{||}(\nu_p\omega_0)\left[1 \pm e^{-i2\nu_p\omega_0 s/c}\right] - p Z_{||}(p\omega_0)\left[1 + e^{-i2p\omega_0 s/c}\right]\right\} \ , \tag{15}$$

we see that they contain terms with fast oscillating phases. Take, for example, the term $\exp(-i2\nu_p\omega_0 s/c)$; it oscillates $2\nu_p$ times in one revolution around the storage ring. Coupled-bunch effects are usually driven by sharp resonances with frequency at least of the order of the rf frequency. Thus, $\nu_p$ is a big number and the exponential will go to zero after averaging over one turn. As a result, the eigenvalues become

$$\lambda_\pm = \sum_p [\nu_p Z_{||}(\nu_p\omega_0) - p Z_{||}(p\omega_0)] \ , \tag{16}$$

and are degenerate. This implies that the two bunches are not coupled at all. Similar argument and derivation can be made for the transverse motion and also for situation with many bunches.

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