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A Purely Mathematical Look at the Energy-Angle Distribution of Axion-Production by Electrons

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ABSTRACT

Calculations of the energy distribution formula for axion-production by electrons are done analytically and numerically and compared with Tsai's compact energy distribution formula. Graphs of the energy-angle distribution formula, atomic form factors, and a look at how the energy distribution changes with respect to the mass of the axion are also presented.

*Work was done at Fermi National Accelerator Laboratory.

1 Introduction

The energy-angle distribution formula for axion production by electrons, $d\sigma/d\Omega_a dE_a$, is used in attempts to detect axions. By looking at the formula, one cannot easily determine its characteristics, but they are worth investigating. Therefore, the following relationships are examined: 1) How the cross section, $d\sigma/dx$, varies with x, 2) How Tsai's compact energy distribution formula compares to an analytical and numerical integration of the energy-angle distribution formula, 3) How the angular distribution $d\sigma/d\theta dx$ varies with θ, 4) How the atomic form factors vary with θ and with x, and, 5) How the cross-section varies with the mass of the axion.

2 The Energy Distribution Formula

The energy-angle distribution for axions produced by electron scattering on an atomic target is given by Tsai[1] as

$$\frac{d\sigma}{d\Omega_a dE_a} = \frac{\alpha^2\alpha_a}{\pi}\frac{E_1}{U^2}\left\{x^3 - \frac{2m_a^2x^2(1-x)}{U} + \frac{2m_a^2}{U^2}\right. \\ \left.\cdot\left[m_a^2x(1-x)^2 + m_e^2x^3(1-x)\right]\right\}\chi \tag{1}$$

where

$$\begin{aligned}
U &= m_e^2lx + m_e^2x + m_a^2(1-x)/x \\
x &= \frac{E_a}{E_1} \qquad\qquad l = \frac{E_1^2\theta_a^2}{m_e^2} \\
\chi &= \chi_{elas} + \chi_{inelas} \\
\chi_{elas} &= Z^2\left[\ln\frac{a^2m_e^2(1+l)^2}{a^2t_{min}+1} - 1\right] \\
\chi_{inelas} &= Z\left[\ln\frac{a'^2m_e^2(1+l)^2}{a'^2t_{min}+1} - 1\right] \\
t_{min} &= \left[\frac{U}{2E_1(1-x)}\right]^2 \\
a &= 184(2.718)^{-1/2}Z^{-1/3}/m_e \\
a' &= 1194(2.718)^{-1/2}Z^{-2/3}/m_e
\end{aligned}$$

$$\frac{\hbar}{\tau} = \frac{1}{2}\alpha_a m_a \left[1 - \frac{4m_e^2}{m_a^2}\right]^{\frac{1}{2}}$$

This formula, taken as given, was integrated over $d\Omega_a$. The area of interest is as θ goes from 0 to 1 milliradian, so the limits on the integral are as ϕ goes from 0 to 2 π and as θ goes from 0 to 1 milliradian. One reason for these limits is that when x is sufficiently large, 1 milliradian includes all of the integral. Another reason for the limits on θ is the approximation that $\theta << 1$ radian is needed to do the integral analytically. With the assumption that $\theta << 1$, the approximation of $\sin\theta \sim \theta$ can be made and thus $\sin\theta d\theta d\phi$ becomes $\theta d\theta d\phi$. Using this approximation, and integrating equation 1 over the limits gives the following.

$$\begin{aligned}
\frac{d\sigma}{dx} &= 2\alpha^2\alpha_a \left\{ \left[\frac{Z^2}{2}\left(\ln\left\{\frac{a^2 f_2^2}{a^2 f_7^2 f_1^2 + 1}\right\} - 1\right) + \frac{Z}{2}\left(\ln\left\{\frac{a'^2 f_2^2}{a'^2 f_7^2 f_1^2 + 1}\right\} - 1\right)\right] \right. \qquad (2) \\
&\cdot \left[-\frac{x^2}{f_1} + \frac{f_5}{f_1^2} - \frac{f_6}{f_1^3}\right] + \left[\frac{Z^2}{2}\left(\ln\left\{\frac{a^2 m_e^2}{a^2 f_7^2 f_4^2 + 1}\right\} - 1\right) + \frac{Z}{2}\left(\ln\left\{\frac{a'^2 m_e^2}{a'^2 f_7^2 f_4^2 + 1}\right\} - 1\right)\right] \\
&\cdot \left[\frac{x^2}{f_4} - \frac{f_5}{f_4^2} + \frac{f_6}{f_4^3}\right] + \left[\ln\left(\frac{f_2 f_4}{f_1 m_e}\right)\right] \cdot \left[x^2 f_3 - f_3^2 f_5 + f_6 f_3^3\right](Z^2 + Z) \\
&+ \left[\frac{1}{f_1} - \frac{1}{f_4}\right]\left[(Z^2+Z)\left\{f_6 f_3^2 - f_5 f_3\right\} + f_7^2 f_6\left\{Z^2 a^2 + Z a'^2\right\}\right] + \frac{f_6 f_3}{2} \\
&\cdot \left[\frac{1}{f_1^2} - \frac{1}{f_4^2}\right](Z^2+Z) + Z^2 a f_7\left[a^2 f_7^2 f_6 - x^2\right]\left[\tan^{-1}(a f_7 f_1) - \tan^{-1}(a f_7 f_4)\right] \\
&+ Z a' f_7\left[a'^2 f_7^2 f_6 - x^2\right]\left[\tan^{-1}(a' f_7 f_1) - \tan^{-1}(a' f_7 f_4)\right] + \frac{f_7^2 f_5}{2} \\
&\cdot \left.\left[Z^2 a^2 \ln\left(\frac{f_1^2(a^2 f_7^2 f_4^2 + 1)}{f_4^2(a^2 f_7^2 f_1^2 + 1)}\right) + Z a'^2 \ln\left(\frac{f_1^2(a'^2 f_7^2 f_4^2 + 1)}{f_4^2(a'^2 f_7^2 f_1^2 + 1)}\right)\right]\right\}
\end{aligned}$$

where

$$f_1 = E_1^2 x \theta_{max}^2 + m_e^2 x + \frac{m_a^2(1-x)}{x}$$

$$f_2 = \frac{E_1^2 \theta_{max}^2}{m_e} + m_e$$

$$f_3 = \frac{x}{m_a^2(1-x)}$$

$$f_4 = m_e^2 x + \frac{m_a^2(1-x)}{x}$$

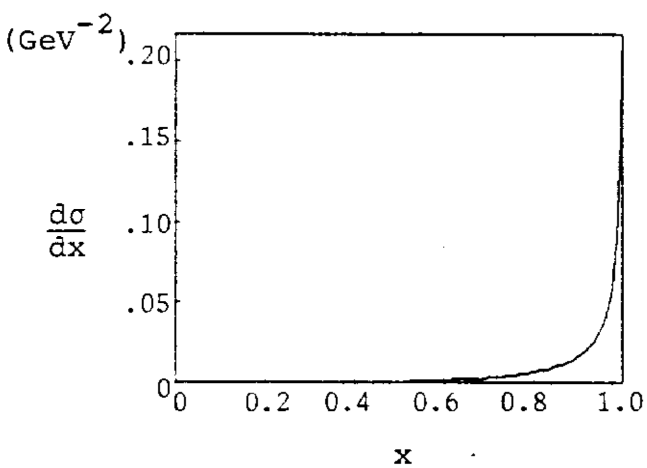
$$f_5 = m_a^2 x(1-x)$$
$$f_6 = \frac{2m_a^2}{3}\left[m_a^2(1-x)^2 + m_e^2 x^2(1-x)\right]$$
$$f_7 = \frac{1}{2E_1(1-x)}$$

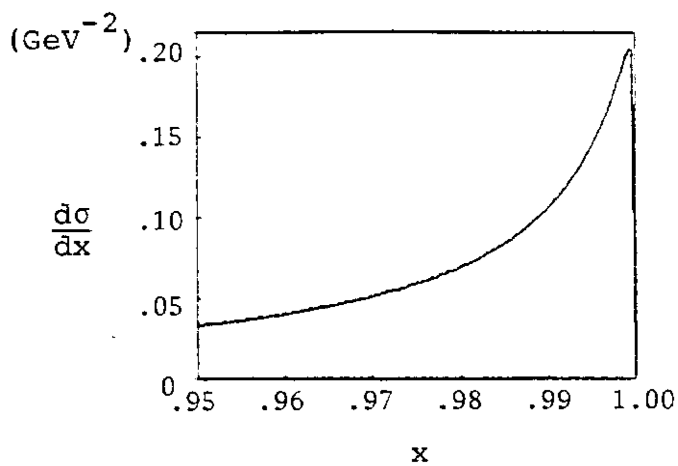
where θ_{max} is the upper limit of the integration.

Using $E_1 = 100$ GeV, $m_a = 5.0\text{x}10^{-3}$ GeV, $Z = 74$ (tungsten), $\tau = 10^{-14}$ sec and $\theta_{max} = 1$ milliradian, this formula gives the following values.

Table 1

x	$d\sigma/dx\ GeV^{-2}$	x	$d\sigma/dx\ GeV^{-2}$
0.00041	2.824E-17	0.750	4.693E-03
0.050	3.320E-07	0.800	6.759E-03
0.100	3.998E-06	0.850	1.014E-02
0.150	1.559E-05	0.900	1.651E-02
0.200	3.970E-05	0.950	3.305E-02
0.250	8.146E-05	0.955	3.632E-02
0.300	1.471E-04	0.960	4.024E-02
0.350	2.445E-04	0.965	4.502E-02
0.400	3.833E-04	0.970	5.102E-02
0.450	5.768E-04	0.975	5.875E-02
0.500	8.423E-04	0.980	6.913E-02
0.550	1.204E-03	0.985	8.383E-02
0.600	1.698E-03	0.990	0.106
0.650	2.377E-03	0.995	0.145
0.700	3.326E-03	.99999578	5.773E-05





Tsai's energy-angle distribution formula, equation 1, was evaluated numerically over the same limits by using the sum of the areas of small rectangles (Binning) and by Simpson's method. The results corresponded very closely with those obtained from the analytical integration, equation 2.

Table 2

x	Binning/Analytic	Simpsons Method/Analytic
.00041	100.02 %	100.00 %
.05	94.33	100.00
.10	99.81	100.00
.15	99.82	99.99
.20	99.98	99.97
.25	100.22	99.95
.30	100.40	99.98
.35	100.48	100.03
.40	100.49	100.04
.45	100.53	100.02
.50	100.62	99.99
.55	100.75	99.99
.60	100.88	100.00
.65	100.98	100.03
.70	101.01	100.05
.75	100.98	100.07
.80	100.89	100.08
.85	100.74	100.08
.90	100.55	100.07
.95	100.33	100.04
.955	100.30	100.04
.96	100.25	100.04
.965	100.25	100.04
.97	100.23	100.03
.975	100.20	100.03
.98	100.17	100.02
.985	100.14	100.02
.990	100.11	100.02
.995	100.08	100.01
.99999578	100.06	100.01

3 Comparison with Tsai's Formula

Tsai[1] has calculated the following formula for $d\sigma/dx$ using the Weizsacker-Williams method and the assumption that $a^2 t_{min} << 1$.

$$\frac{d\sigma}{dx} = 2r_o^2\alpha_a x \frac{(1+2/3f)}{(1+f)^2}\left[Z^2\ln(184Z^{-1/3}) + Z\ln(1194Z^{-2/3})\right] \qquad (3)$$
$$+2r_o^2\alpha_a x\left[\frac{1}{3f^2}(l+f)\ln(l+f) - \frac{1+4f+2f^2}{3f(1+f)^2}\right](Z^2+Z)$$

Where

$$f = (m_a^2/m_e^2)(1-x)/x^2 \qquad \textit{and} \qquad r_o = \alpha/m_e^2$$

His formula corresponds with the results obtained from the analytical and numerical integrals to within 30%.

Table 3

x	Tsai/Analytic	x	Tsai/Analytic
.00041	*	.75	71.30 %
.05	*	.80	72.27
.10	86.29 %	.85	73.50
.15	74.37	.90	75.21
.20	70.19	.95	78.05
.25	68.49	.955	78.47
.30	67.80	.96	78.93
.35	67.61	.965	79.45
.40	67.68	.97	80.04
.45	67.92	.975	80.71
.50	68.26	.98	81.51
.55	68.70	.985	82.47
.60	69.21	.99	83.70
.65	69.81	.995	85.43
.70	70.50	.99999578	346,676.53**

* The first two values are missing since most of the integral for those values occurs outside of 1 milliradian, and therefore the ratio is meaningless.
** This number is large because of the assumption that $a^2 t_{min} << 1$.

The assumption that $a^2 t_{min} << 1$ is not a valid assumption for all x with $0 < \theta < .001$ radians. However, most of the integral occurs before the assumption becomes invalid, causing no significant difference. For $x \sim 1$

and $x \sim 0$, the value of $a^2 t_{min} >> 1$ and the value of the integral is drastically different. Making the assumption that $a^2 t_{min} << 1$, and integrating equation 1 as before, results in

$$\frac{d\sigma}{dx} = 2\alpha_a \alpha^2 \left\{ \left[\frac{Z^2}{2} \left(\ln(a^2 f_2^2) - 1 \right) + \frac{Z}{2} \left(\ln(a'^2 f_2{}^2) - 1 \right) \right] \right. \tag{4}$$
$$\cdot \left[-\frac{x^2}{f_1} + \frac{f_5}{f_1^2} - \frac{f_6}{f_1^3} \right] + \left[\frac{Z^2}{2} \left(\ln(a^2 m_e^2) - 1 \right) + \frac{Z}{2} \left(\ln(a'^2 m_e{}^2) - 1 \right) \right]$$
$$\cdot \left[\frac{x^2}{f_4} - \frac{f_5}{f_4^2} + \frac{f_6}{f_4^3} \right] + \left[\ln \left(\frac{f_2 f_4}{f_1 m_e} \right) \right] \cdot \left[x^2 f_3 - f_3^2 f_5 + f_6 f_3^3 \right] (Z^2 + Z)$$
$$\left. + \left[\frac{1}{f_1} - \frac{1}{f_4} \right] \cdot \left[(Z^2 + Z) \left(f_6 - f_3^2 - f_5 f_3 \right) \right] + \frac{f_6 f_3}{2} \left[\frac{1}{f_1^2} - \frac{1}{f_4^2} \right] (Z^2 + Z) \right\}$$

Comparing the values of $d\sigma/dx$ with and without the assumption shows very little difference except for $x \sim 0$ and $x \sim 1$.

Table 4

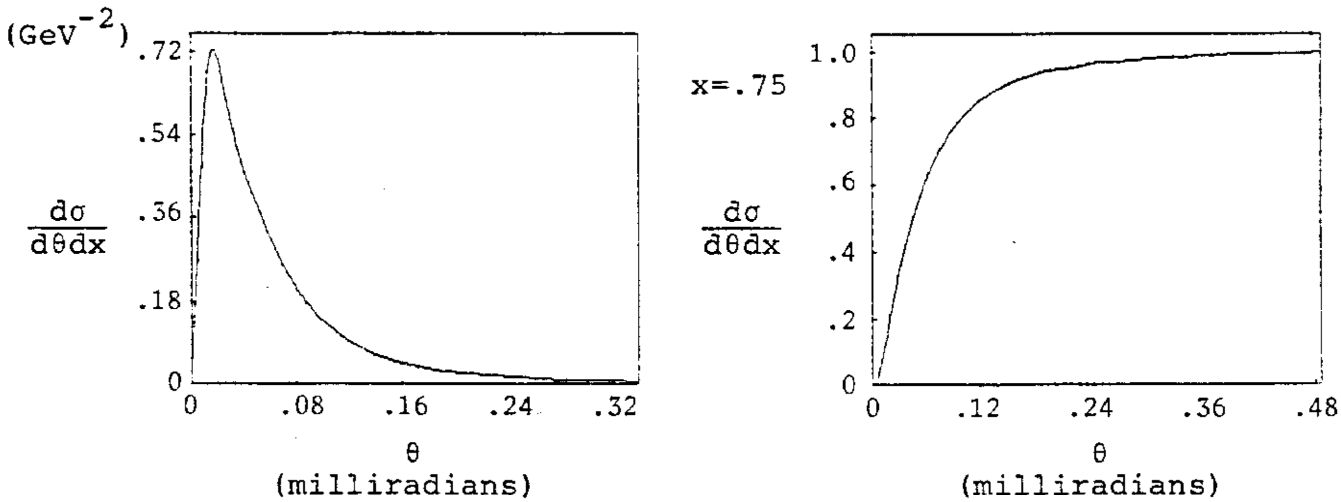
x	Assumption / Analytic	x	Assumption / Analytic
0.00041	128.97 %	0.750	100.23 %
0.050	100.14	0.800	100.24
0.100	100.10	0.850	100.25
0.150	100.11	0.900	100.28
0.200	100.13	0.950	100.33
0.250	100.14	0.955	100.34
0.300	100.15	0.960	100.35
0.350	100.16	0.965	100.36
0.400	100.17	0.970	100.38
0.450	100.18	0.975	100.40
0.500	100.19	0.980	100.43
0.550	100.20	0.985	100.48
0.600	100.20	0.990	100.58
0.650	100.21	0.995	100.87
0.700	100.22	.99999578	399396.75

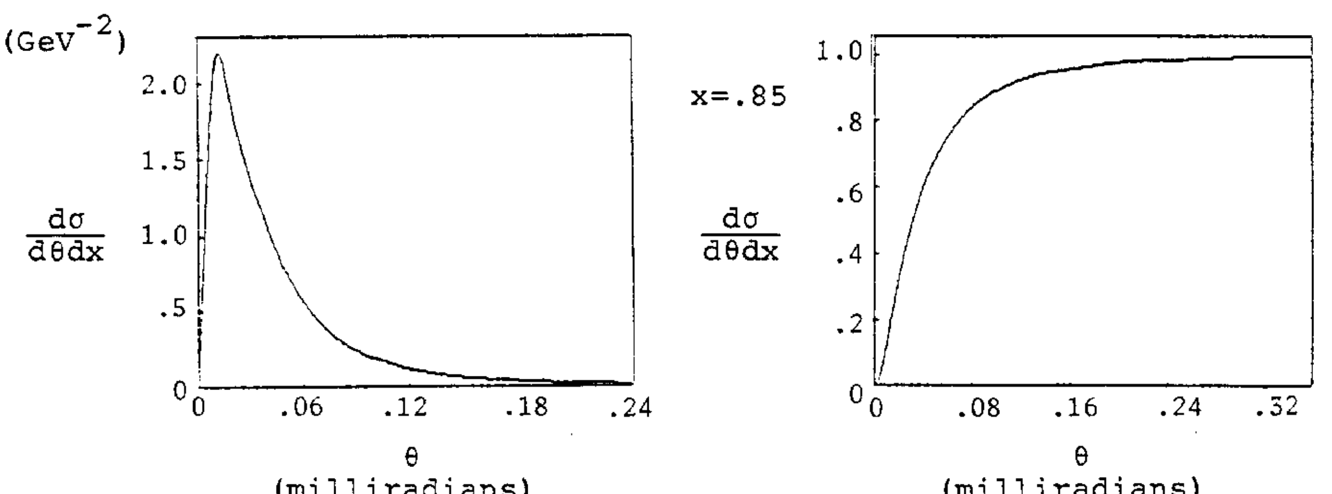
4 Angular Distribution of $d\sigma/d\theta dx$

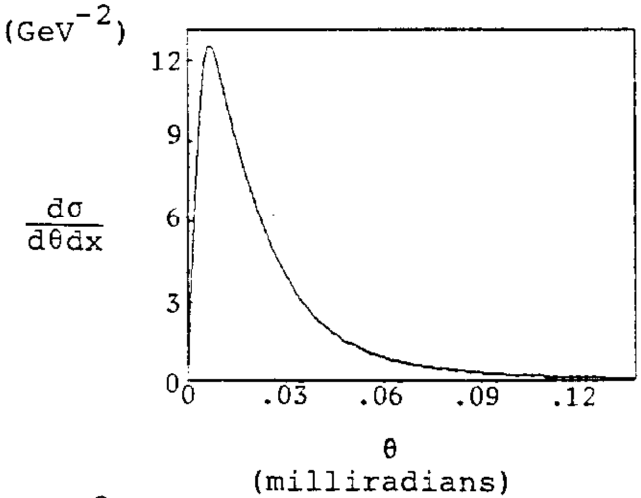
The angular distribution of $d\sigma/d\theta dx$ is concentrated at very small angles. When $x > .45$, more than 98.4% of the entire integral is included in the first milliradian.

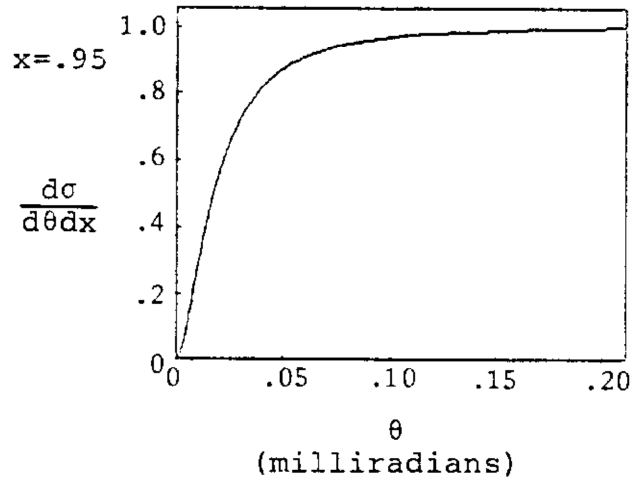
$$\frac{d\sigma}{d\theta dx}$$

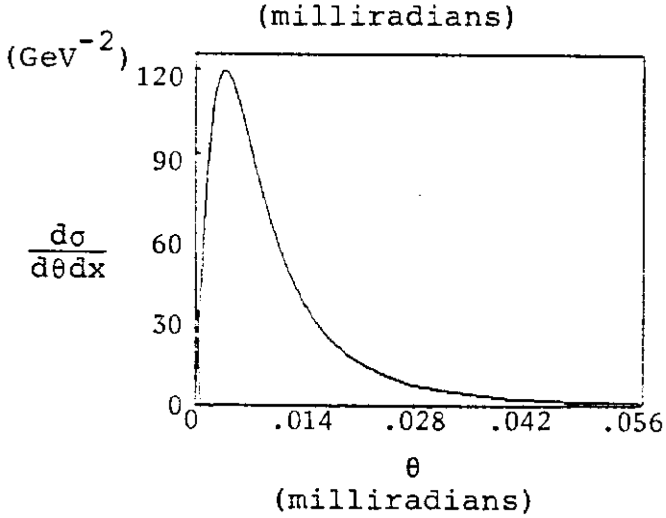
Fraction of Cross Section $< \theta$

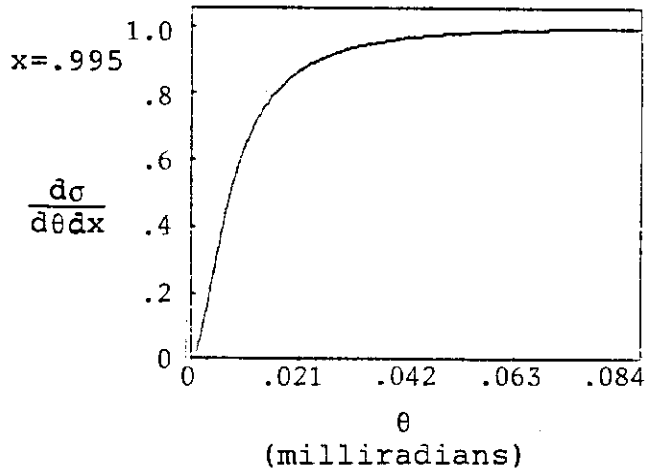


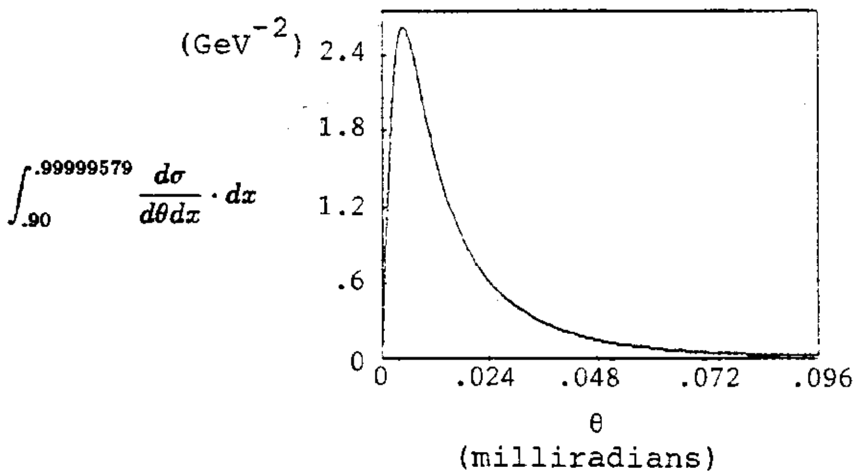








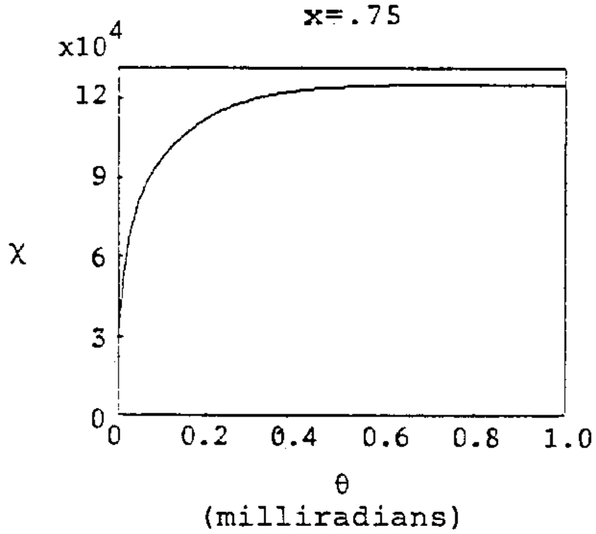


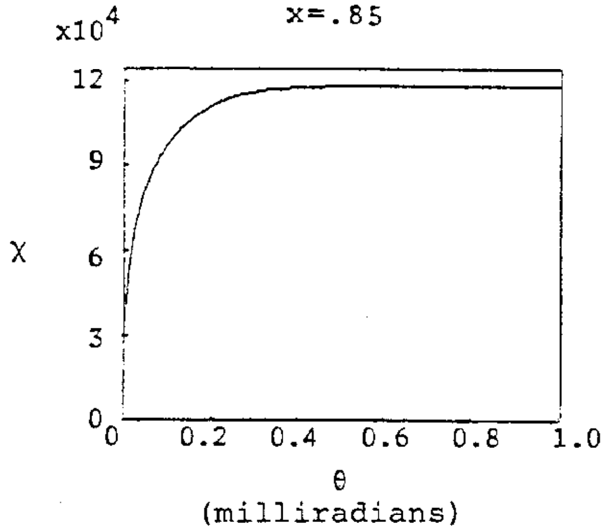


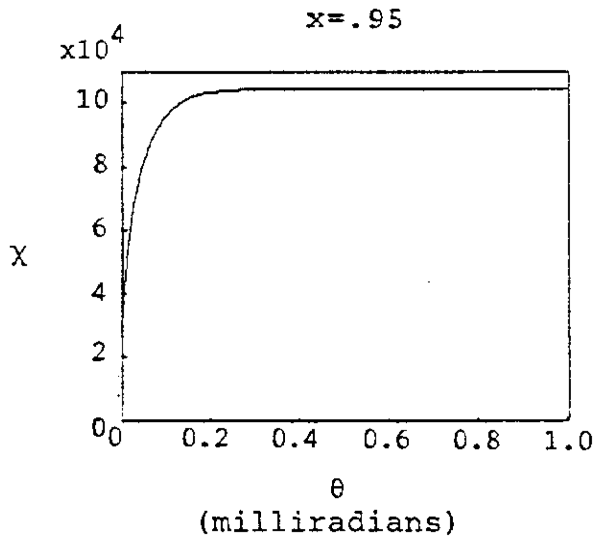
Since the distribution is concentrated at very small angles, the boundary conditions should not make a significant difference whether they go from 0 to .001 radians or 0 to π radians. Therefore, the boundary conditions nor the assumption that $a^2 t_{min} << 1$ make a significant difference in the value of the cross section.

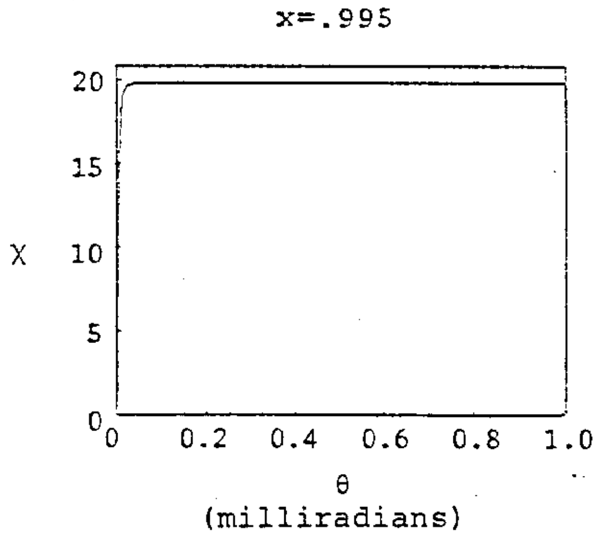
5 Atomic Form Factors

The energy-angle distribution formula is multiplied by the atomic form factors χ. When x is held constant, χ vs. θ will rise quickly for small values of θ and remain constant for larger values of θ. As x becomes larger, χ will rise faster.

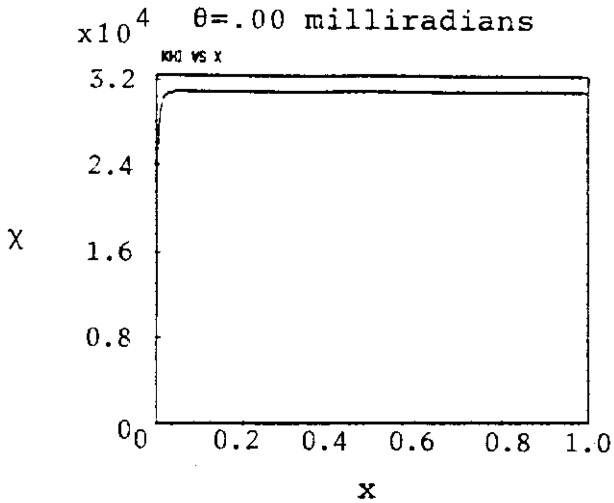


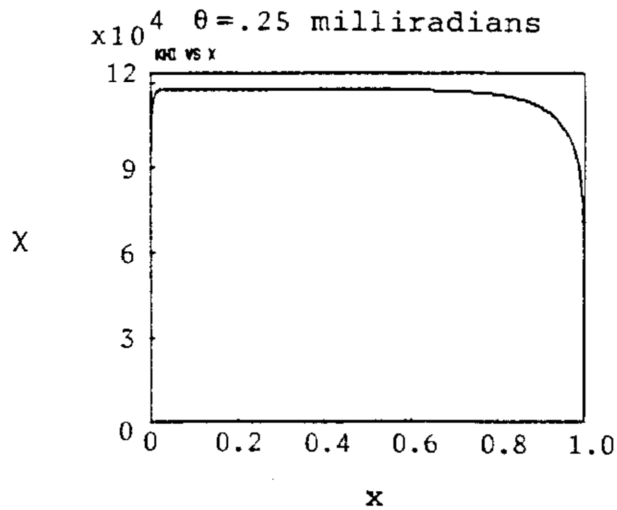


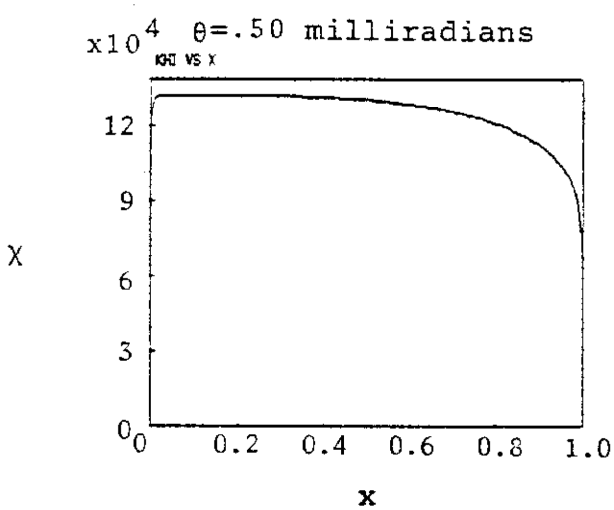


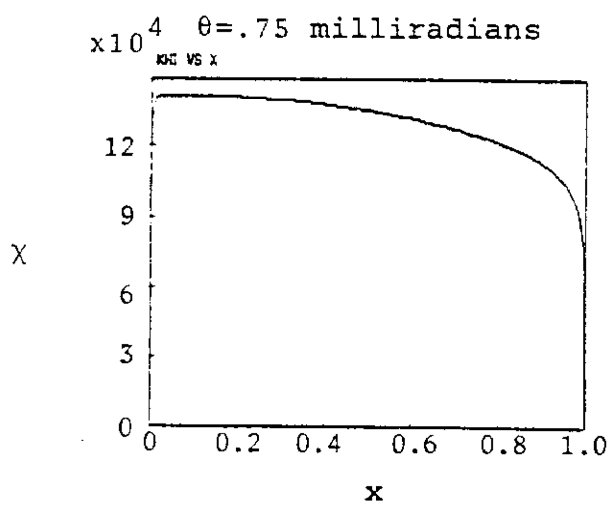


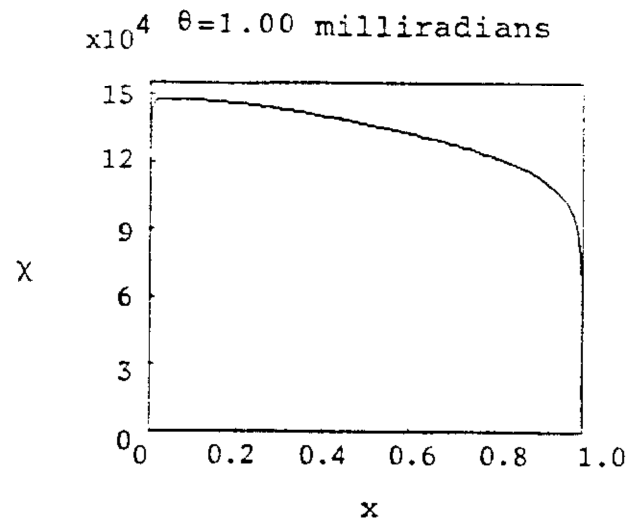
For a constant θ, χ rises quickly and then falls. If $\theta \sim 0$ and $x \sim 1, \chi$ will fall rapidly. As θ becomes larger, the fall begins sooner and becomes less abrupt.





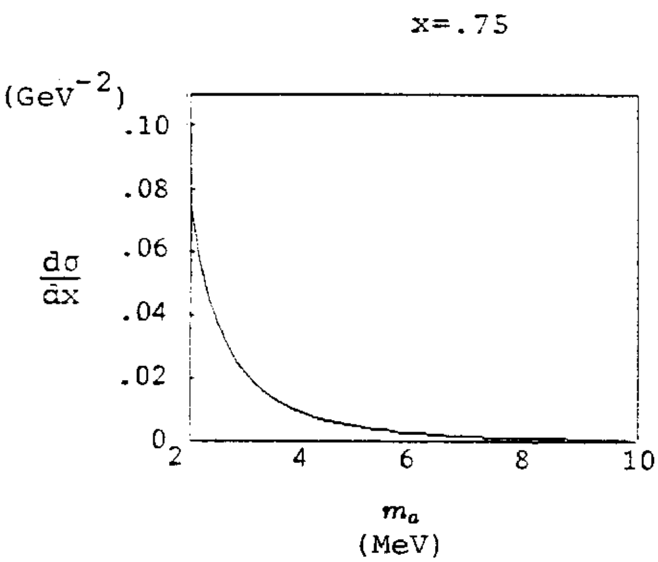


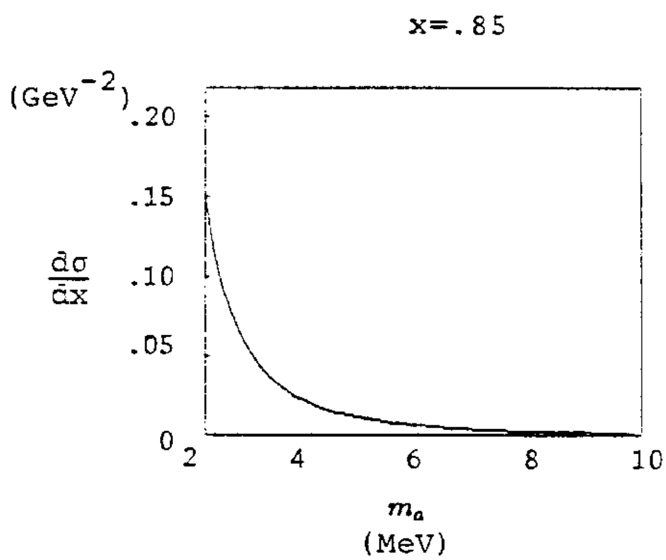


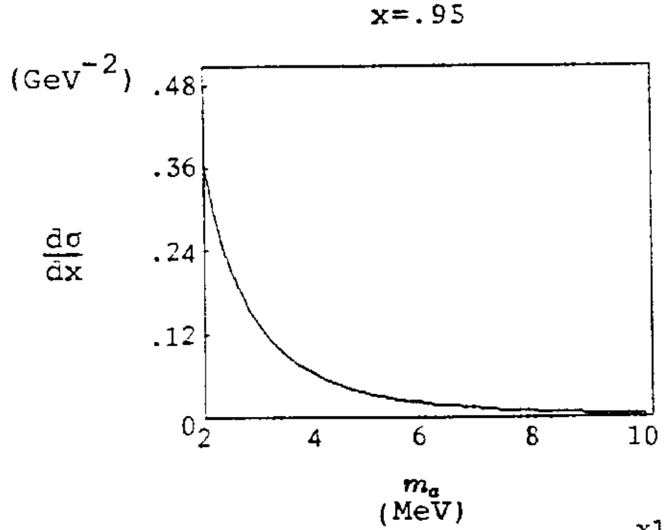


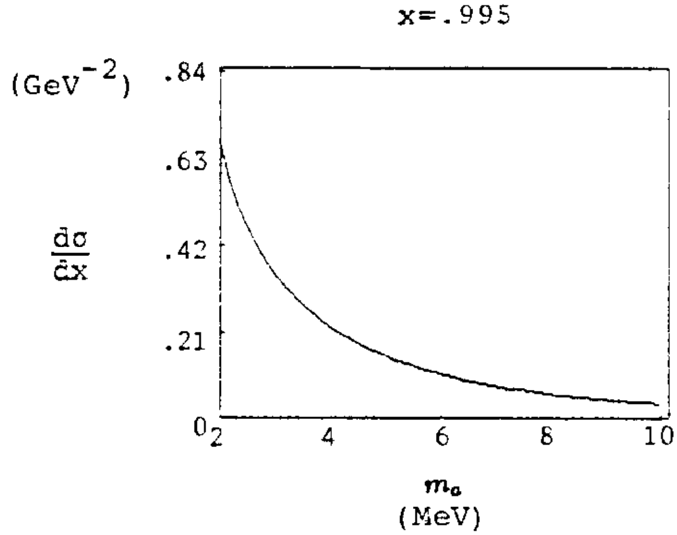
6 Mass of the Axion vs Cross Section

The last relationship studied was the effect of the mass of the axion, with a constant lifetime, on $d\sigma/dx$. The lifetime was held constant at 10^{-14} sec., and the mass was varied from 1.7 MeV to 10 MeV. When x is held constant, a smaller mass will yield a larger energy distribution.









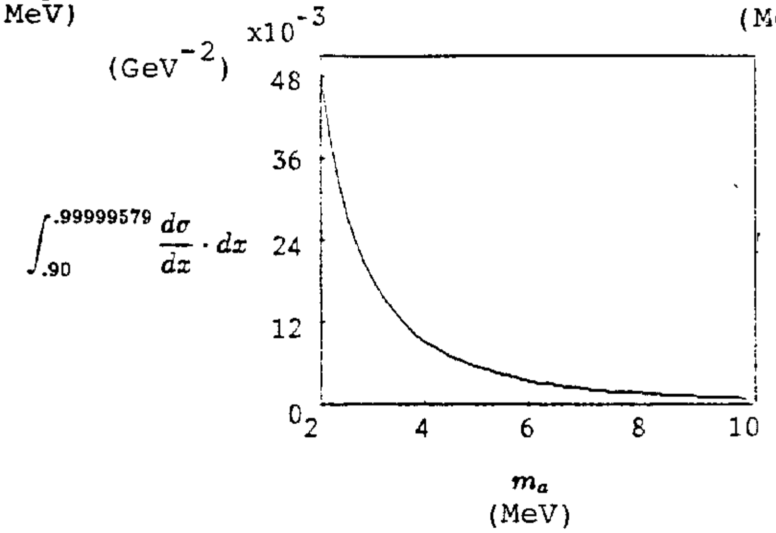


Table 5

m_a	x = .75	x = .85	x = .95	x = .995
1.7 MeV	9.992E-2	.197	.459	.771
3.0 MeV	1.931E-2	4.065E-2	.120	.328
5.0 MeV	4.693E-3	1.014E-2	3.305E-2	.145
8.0 MeV	1.278E-3	2.805E-3	9.551E-3	5.754E-2
10.0 MeV	6.865E-4	1.517E-3	5.239E-3	3.479E-2

7 Conclusion

Tsai's energy-angle distribution formula for axions has been integrated both analytically and numerically. Tsai's integration agrees with both the analytic and numeric integrals to within 30%. Graphs of the energy-angle distribution, along with the atomic form factors help give a little more understanding about the energy-angle distribution and energy distribution formulas. Lastly, it has been seen that as the mass of the axion increases, the cross-section decreases provided the lifetime is held constant.

8 References

1. Dr. Y. S. Tsai, Axion Bremsstrahlung by an Electron Beam, SLAC-Pub-3926.

9 Acknowledgments

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