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Longitudinal Phase Space Tracking With Space Charge and Wall Coupling Impedance

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Introduction

The development of some of the more intricate r.f. procedures in Fermilab's Tevatron I project(1) such as bunch rotation,(2) bunch coalescing,(3) etc. has been greatly aided by computer simulations, i.e., tracking of representative particles in longitudinal phase space on a turn-by-turn basis throughout the process. The program developed for these tracking calculations(4) has been tested subsequently by several users in a variety of applications.(5) One of the more ambitious calculations has been the modeling of transition crossing in the Fermilab 8 GeV booster and exploration of γ_T-jump schemes. These calculations have required augmenting the code to include effects depending on the beam current which I will refer to as collective effects. Specifically I have included the effect of the beam space charge and the coupling impedance between beam current and vacuum chamber on each particle. This note discusses how the collective effects have been introduced into the tracking calculation and gives a sample result from the booster modeling effort. The conclusions from the booster calculation and new program documentation will be provided elsewhere.(6,7)

Calculation of the Space Charge and Wall Impedance Energy Correction

The beam is modeled as a cylindrical charge distribution of constant radius **a** centered in a cylindrical vacuum chamber of radius **b**. The arguments are equally applicable to a rectangular vacuum chamber by using an effective **b/a** ratio.(8) The charge density is assumed to be constant out to **a** for fixed longitudinal coordinate **s'** in the beam rest frame:

$$\rho(r,s')=\begin{cases}\rho_0(s,t) & (0\leq r\leq a)\\ 0 & (a\leq r\leq b)\end{cases} \quad ; \qquad s' = \gamma\,(s-\beta ct) \tag{1}$$

The beam is supposed to be bunched into **h** identical bunches of **N** particles each with bunch length much greater than **a**. If the **s** dependence of ρ_0 is also slow, i.e. if $a(\partial\rho/\partial s)\ll\rho$ then the radial electric field resulting from the beam is

$$\varepsilon_r(r,s,t)=\begin{cases}\dfrac{\lambda(s,t)}{2\pi\varepsilon_0}\dfrac{r}{a^2} & (0\leq a)\\[2ex] \dfrac{\lambda(s,t)}{2\pi\varepsilon_0}\dfrac{1}{r} & (a\leq r\leq b)\end{cases} \tag{2}$$

where the linear charge density is

$$\lambda(s,t) = \frac{\partial q}{\partial s} = \pi a^2\rho_0\ (s,t) \quad . \tag{3}$$

The charge distribution is assumed to be little changed over the time τ_R during which it makes a single turn around the accelerator, i.e. $\tau_R\ (\partial\rho/\partial t)\ll\rho$. The beam current is

$$I_b=\beta c\lambda \quad . \tag{4}$$

This current results in a toroidal magnetic field

$$B_\phi(r,s,t) = \begin{cases}\dfrac{\mu_0\lambda(s,t)}{2\pi}\beta c\,\dfrac{r}{a^2} & (0\leq r\leq a)\\[2ex] \dfrac{\mu_0\lambda(s,t)}{2\pi}\beta c\,\dfrac{1}{r} & (a\leq r\leq b)\end{cases} \tag{5}$$

From the curl-E Maxwell equation

$$\oint \vec{E}\cdot d\vec{s} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{\sigma} \quad ; \tag{6}$$

with the integration contour indicated in Fig. 1 one finds

$$E_S\Delta s + \int_0^b E_r(s+\Delta s)dr - E_W\Delta s - \int_0^b E_r(s)dr = -\Delta s\int_0^b B_\phi dr \tag{7}$$

where E_S and E_W are the longitudinal electric field on-axis and along the beam pipe wall respectively. Defining in the conventional way the geometric parameter

$$g = 2\left[\int_0^a \frac{r}{a^2}dr + \int_a^b \frac{dr}{r}\right] = 1 + 2\,\ln\,(b/a) \tag{8}$$

Eq. 7 becomes

$$E_S + \frac{g}{4\pi\varepsilon_0}\frac{\partial\lambda}{\partial s} - E_W = -\frac{\mu_0\beta c g}{4\pi}\frac{\partial\lambda}{\partial t} \quad . \tag{9}$$

From Eq. 1 and 3 one has

$$\frac{\partial\lambda}{\partial t} = -\beta c\frac{\partial\lambda}{\partial s} \tag{10}$$

and because $\mu_0\varepsilon_0 = c^{-2}$ and $1-\beta^2 = \gamma^{-2}$ one can combine electric and magnetic terms

$$E_S = -\frac{g}{4\pi\varepsilon_0\gamma^2}\frac{\partial\lambda}{\partial s} + E_W \quad . \tag{11}$$

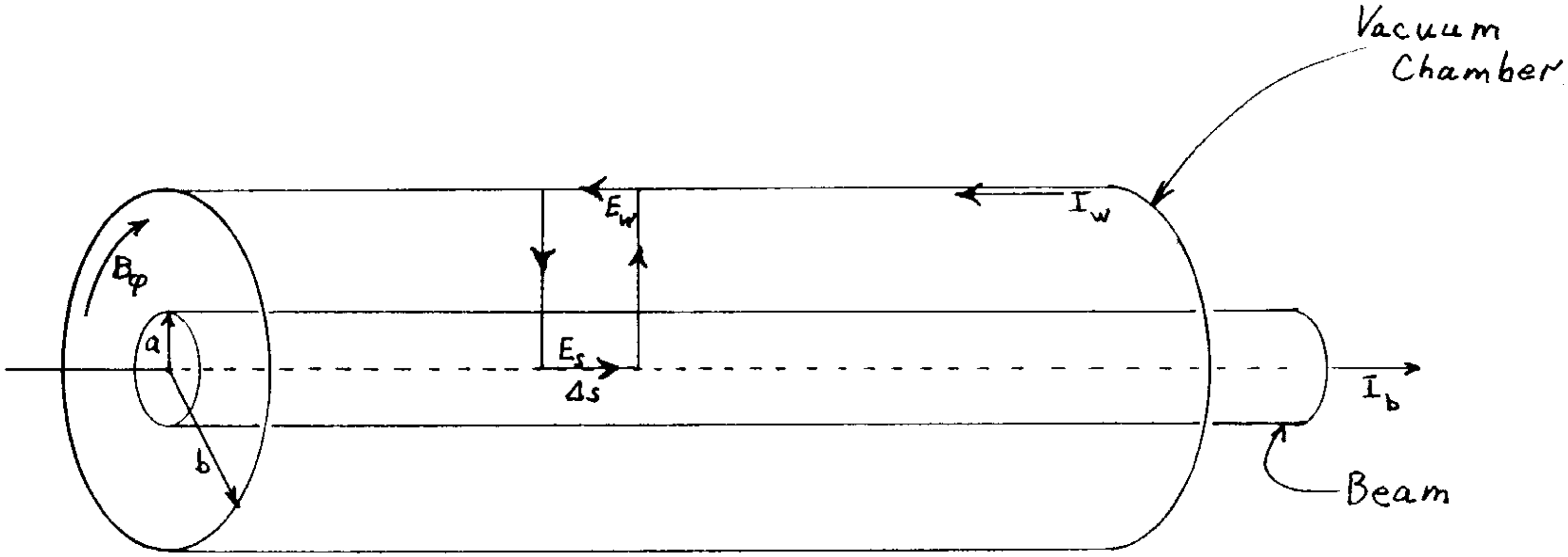

Figure 1: Beam Current Model for the Calculation of the Longitudinal Electric Field

Relating the space charge field to the gradient of the linear charge density in this way indicates how the field arises but does not directly relate the field to the beam current. One would like to combine as much as possible the calculation of the space charge and wall coupling contribution to E_s. This combination is made most easily by expressing both contributions as function of beam current. The constant of proportionality between beam current and the accelerating or decelerating voltage that a beam particle experiences will be an impedance, generally complex and frequency dependent. Because the beam current is not a simple alternating current it must be fourier analyzed and the voltage arising from each component calculated separately. The voltage that a particular particle in the distribution experiences is dependent on its location in the bunch. Therefore, one must account for three phases, namely, the particle's phase relative to the bunch center, the phases of each fourier component of the beam current, and the reactive shift of the voltage components relative to the currents producing them.

If in Eq. 9 one uses Eq. 10 to replace $\partial\lambda/\partial s$ instead of $\partial\lambda/\partial t$ one gets in place of Eq. 11 for E_s

$$E_s = \frac{g}{4\pi\varepsilon_0\gamma^2\beta c}\ \frac{\partial\lambda}{\partial t} + E_w \ . \tag{12}$$

Using Eq. 4 to introduce the beam current and $Z_0 = (\varepsilon_0 c)^{-1} = 377\Omega$ to introduce impedance units

$$E_s = \frac{Z_0 g}{4\pi(\beta\gamma)^2\beta c}\ \frac{\partial I_b}{\partial t} + E_w \ . \tag{13}$$

E_w arises from the beam current image flowing in the wall and therefore,

$$E_w = I_w Z_w/2\pi R = -\ I_b Z_w/2\pi R \tag{14}$$

where Z_W is the total wall impedance for the beampipe. The accelerating or decelerating voltage generated by the beam current is

$$V = -\int E_s ds = -2\pi RE_s - 2\pi RE_W = -\frac{Z_0 gR}{2(\beta\gamma)^2 c}\frac{\partial I_b}{\partial t} + I_b Z_W \quad . \tag{15}$$

Consider the voltage component produced by a single fourier component of the beam current $I_\omega = I_0 e^{+i\omega t}$. The impedance seen by the beam at frequency $\omega/2\pi$ is

$$Z_\omega = \frac{V_\omega}{I_\omega} = -\frac{i\omega Z_0 gR}{2(\beta\gamma)^2 c} + Z_W \tag{16}$$

Because of the assumption that there are **h** identical beam bunches, the relation between ω and the beam circulation frequency is

$$\omega = nh\omega_R = nh\beta c/R \tag{17}$$

so that

$$\frac{Z_\omega}{nh} = -i\frac{Z_0 g}{2\beta\gamma^2} + \frac{Z_W}{nh} \tag{18}$$

From this we find that impedance corresponding to the space charge has the phase but not the frequency dependance resulting from a physical capacitor. It is possible for an inductive wall to cancel the space charge effect for all fourier components but only at one energy.

To find how the collectively generated voltage in turn affects the evolution of the distribution over many turns around an accelerator or storage ring one can proceed iteratively starting from some plausible distribution, say for example a measured one, and calculate before each turn the voltage generated by the existing distribution. The chosen starting distribution may not be consistent with the space charge or the Z_W for the particular problem; that question can be investigated by tracking a selection of distributions with fixed accelerator parameters. Indeed, there is not much a priori guidance for arbitrary Z_W, but when space charge is dominant one may reasonably start from an elliptical distribution which gives a parabolic current distribution.[9] The voltage acting on a particular particle is of course a function of its position in the bunch. The situation described by the initial assumptions is a quasi-stationary one in which there are **h** identical bunches in the beam which make up a current which affects all bunches the same way on a given turn. The change in the distribution during one turn, which by the proceeding assumption must be small, is incorporated in calculating the voltage acting on each particle on the following turn.

To find the frequency spectrum of the beam current we fourier analize the charge distribution in its rest frame with respect to the variable

$$\phi = hs'/R \quad . \tag{19}$$

Introducing a normalized periodic distribution function $f(\phi)$ such that

$$\int_{-\pi}^{\pi} f\,(\phi)d\phi = 1 \tag{20}$$

and a real fourier expansion to exhibit the phases explicitly,

$$f\,(\phi) = \Sigma\, a_n \cos\,(n\phi - \phi_n) \tag{21}$$

the component of the beam current with $\omega = hn\omega_R$ is

$$I_n = \frac{ehN\beta c}{R}\, a_n \cos\,(n\phi - \phi_n) = ehN\omega_R\, a_n \cos\,(n\phi - \phi_n) \quad . \tag{22}$$

The net impedance at each harmonic (Eq. 18) is also represented in real form by a magnitude Z_n and phase χ_n. Therefore, the energy increment for the i-th particle produced by the beam current (subscripted "c" for collective effects) is

$$eV_c^{(i)} = ehN\omega_R \sum_{n=1} a_n Z_n \cos(n\phi^{(i)} - \phi_n + \chi_n) \quad . \tag{23}$$

The lower limit for the sum is n=1 because there is no steady d.c. current flowing in a pure resistance, the lowest frequency present being the rf fundamental $h\omega_R/2\pi$. The n=0 term represents the current averaged over all frequencies.

For numerical calculation one constructs $f(\phi)$ by binning the ϕ-projection of the particle distribution. The fourier series can be obtained by a fast discrete transform (FFT) of the bin occupation numbers normalized so that the n=0 term gives the correct average current. The resulting coefficients are then combined to give real amplitudes and phases a_n and ϕ_n. The amount of computing is reduced by tabulating eV_c at the same interval used in calculating $f(\phi)$. By this means one need evaluate Eq. 23 once per bin on each turn rather than once for every particle on every turn.

The choice of the bin width for $f(\phi)$ involves both numerical and physical considerations. Clearly too few bins results in a poor representation of **f** whereas many bins will produce a good representation only if there are a sufficient number of particles per bin. The space charge contribution to eV_c is often the dominant one and is generally largest in the wings of the distribution, where the occupancy is low, because there $\partial\lambda/\partial s$ is greatest. To track a number of particles giving an adequate representation of the wings maybe very costly. It may be a satisfactory resolution of this dilemma to divide the initial phase space distribution into several classes distinguished by how far from the center of the distribution they lie. By assigning few particles with high weight to the center of the distribution and many particles with lesser weight to the outer partitions one can treat the problem very satisfactorily until the mixing of the separate partitions becomes large. This strategy has not been needed or tested in the work done so far, but the tracking program is constructed to use it efficiently.

According to the initial assumptions of smooth longitudinal variation, $a(\partial\rho/\partial s) \ll \rho$, amplitudes a_n corresponding to wavelengths $\lambda_n \lesssim b$ should be negligible. Were they not negligible it would be wrong to include them in Eq. 23 for $eV_c^{(i)}$ because waves with angular frequency $\omega > \omega_c$ propagate freely in the beam pipe at a velocity different from the beam. Therefore, the assumption of quasi-static coherence of the components during a beam circulation period τ_R would not be satisfied. The microwave cutoff consideration leads to an upper limit on the sum. The cutoff for the lowest made (TE_{11}) of a circular pipe of radius **b** is

$$\lambda_c = 3.4126b \quad . \tag{24}$$

Note that the fourier analysis of the charge distribution has been performed in the beam frame whereas λ_c is calculated in the lab frame where electromagnetic waves are propagating in a stationary beampipe. The length λ_c is Lorentz contracted by a factor γ^{-1} when transformed to the beam frame. The fundamental interval of the fourier analysis is one wavelength of the accelerator rf, λ_{rf}. Therefore, the maximum number of fourier components consistent with the assumptions of the calculation is

$$n_c = \lambda_{rf}/\lambda_c = \gamma\lambda_{rf}/3.4126b \quad . \tag{25}$$

The distribution of particles must be divided into $2n_c$ bins to determine n_c components.

Lee and Teng[10] have compared exact calculations of longitudinal space charge field with calculations from the local gradient of λ for distributions appropriate to their investigation of negative mass instability in the Fermilab booster. They calculate the gradient by differencing the population of adjacent bins constructed with the field point on the common boundary. They find the most faithful approximation is given using a bin size $w_b \sim 3b/4$. This value should be divided by γ following the same basic argument as above to get the correct energy dependance. Therefore, the number of fourier components to choose for optimum precision in the evaluation of the space charge term is

$$n_Q = \frac{1}{2}\,\frac{2\pi R}{hw_b} = \frac{1}{2}\,\frac{2\pi R\gamma}{h(3b/4)} = \frac{4\pi R\gamma}{3bh} = \frac{2}{3}\,\frac{\lambda}{b}\,\frac{\lambda_{rf}}{b}\,\gamma \tag{26}$$

This number is almost twice n_c (Eq. 25) given by the microwave cutoff and therefore it should probably be used, if at all, only for calculations in which the wall impedance term is small or absent. Both numbers will be unacceptably large in many applications. If the distribution is changing sufficiently slowly one can save some computing time by computing the fourier transform every few turns rather than every turn, but I know of no test other than trials with different frequency of recalculating the transform which will show when this technique is justified. Practically, the limit on the number of fourier components is likely to be set by computing resources. One will doubtless track what he can reasonably afford, say $\sim 10^4$ particles once the mechanics have been checked out in test runs. Because statistical fluctuation of the bin population should be of the order of the square root of the population, one will probably try to have $\sim 10^2$ particles in the bins where λ is changing most rapidly; in general these will be bins with far less than the typical number of particles. Depending on the particular accelerator, either Eq. 25 or 26 may easily require $\sim 10^3$ bins. Thus one would infer that the practical limit in particles may fail by an order of magnitude or more to satisfy the requirements for a realistic simulation. However, if the initial distribution is reasonably smooth one can get excellent fits to it with far fewer components. Suppose, for example one is investigating negative mass instability in a proton synchrotron. The initial distribution is presumably elliptical and bunched well inside the bucket boundaries. In such a case 64 or even 32 components may give a superb representation of the starting point. Furthermore, because the fourier analysis spans 2π of the rf phase, many of the bins are empty; the expression for eV_c does not be evaluated for the empty bins. If one does not need to pursue the calculation in the regime where the bunch breaks up into small clusters but is satisfied to track only to the onset of the instability, far fewer bins and particles than indicated by Eq. 25 or Eq. 26 may suffice. Clearly these are qualitative considerations and results of such calculations must be checked by testing the sensitivity of the features of interest to changes in bin width and/or number of particles tracked.

Example of a Simulation Including Both Space Charge and Wall Impedance

As a sample of what may come out of a turn-by-turn tracking when the beam induced voltage is included in the energy change per turn I show some results[11] for the Fermilab Booster when the beam current is at its all time record value.[12] The simulation is "realistic" with respect to the phase feedback included and the frequency dependence of the wall impedance[13] but "unrealistic" in that the bunch is started out part way into the cycle with an emittance (.02eVs) which has not been blown-up for space charge effects during injection. The E-ϕ distribution of the initial bunch is given in Fig. 2a, Fig. 2b shows the azimuthal or phase projection over one bucket. The beam induced voltage is plotted versus azimuth in Fig. 2c. The range of +65 to -155 kV may be compared to the accelerating potential of 883 kV to see that at this current the beam induced voltage is not a "small perturbation". After 1000 turns we see in Figs. 3 a,b,c the effects of longitudinal instability leading to some breakup of the bunch. Because this instability begins below transition it is not negative mass instability arising from the space charge term but is instead caused by the wall impedance Z_W. Fig. 4 shows the distribution very nearly at the time of transition, where the usual bucket does not exist. Some particles are lost from the bucket between turns 1000 and 2000. Fig. 5 gives the developments 1300 turns later. The asymmetry of the filamentation results from the asymmetry of the bucket immediately after transition. Note that even 1300 turns later the bucket extends noticeably further below the synchronous energy than above. Particles were preferentially lost from the high energy side the distribution of Fig. 4 after transition.

The foregoing sample is intended only to be illustrative of the kinds of phenomena that arise as one introduces the effect of the beam current on the synchrotron motion of beam particles. Analytical techniques can establish the existence and often the threshold for the collective longitudinal instabilities. To determine the effect of the instability on the evolution of the distribution and to asses the efficacy of remedial measures there may be no simpler approach than a reasonably realistic particle tracking of the type described in this note. Indeed, there may be no other approach available. Many other aspects of accelerator operation may also be more clear with the aid of complete phase space distributions respesenting the effect on the beam of the known and suspected

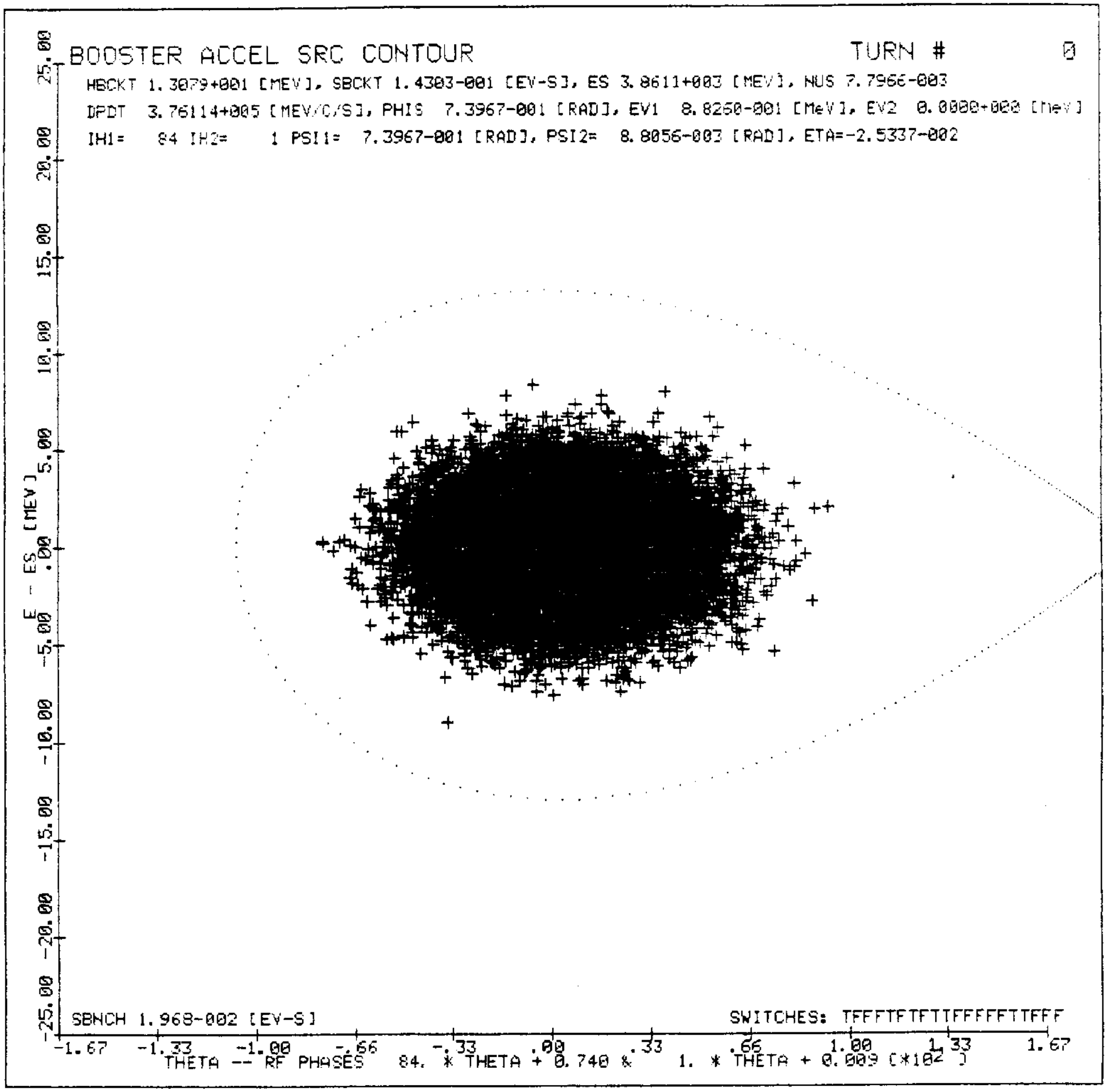

Figure 2a: Initial Booster Bunch at 2.88 GeV, η = -0.025

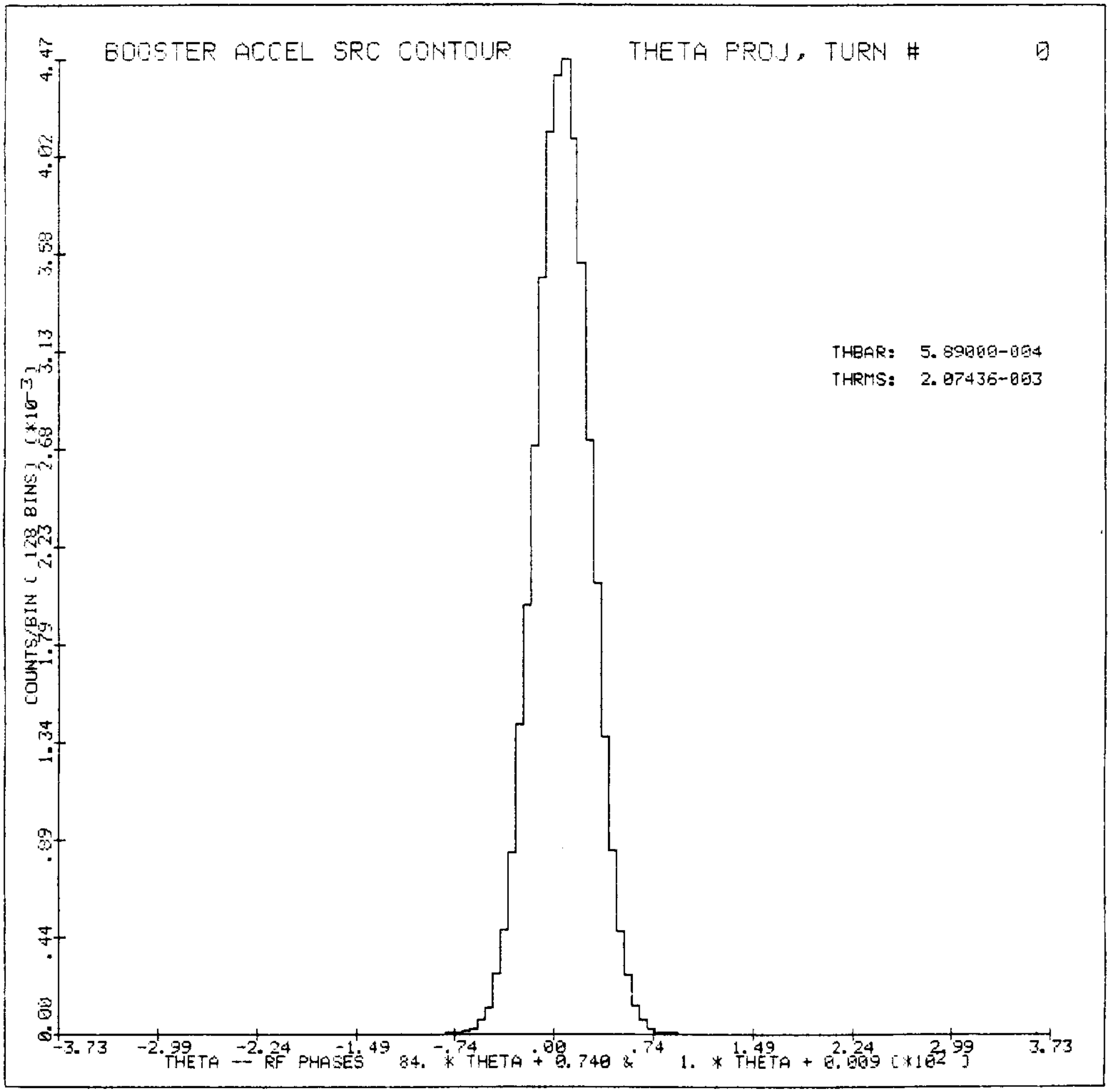

Figure 2b: Azimuthal Charge Distribution in the Initial Bunch

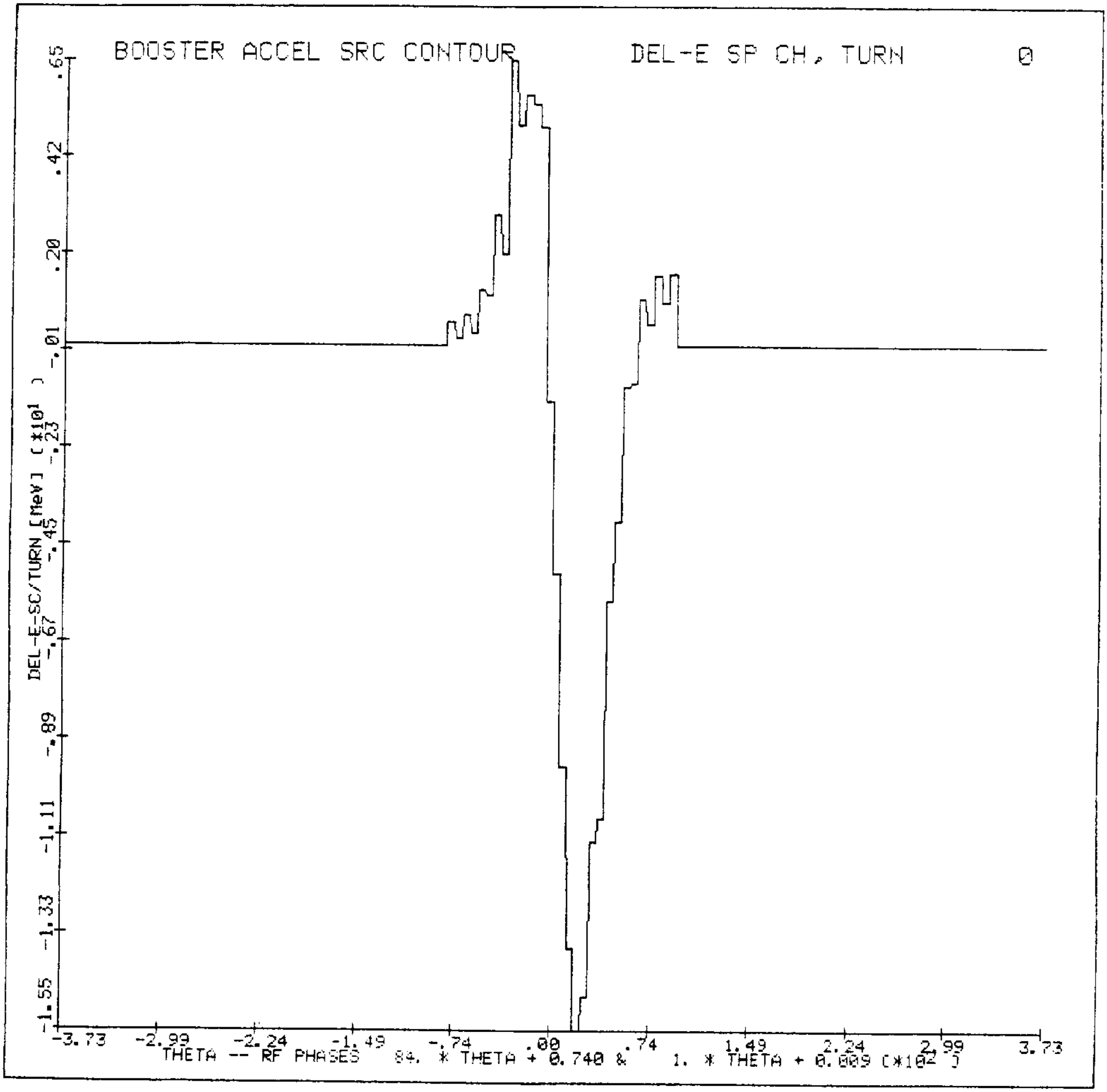

Figure 2c: Spacecharge and Wall Impedance Energy Increment (per turn)
as a Function of Azimuth for the Initial Bunch

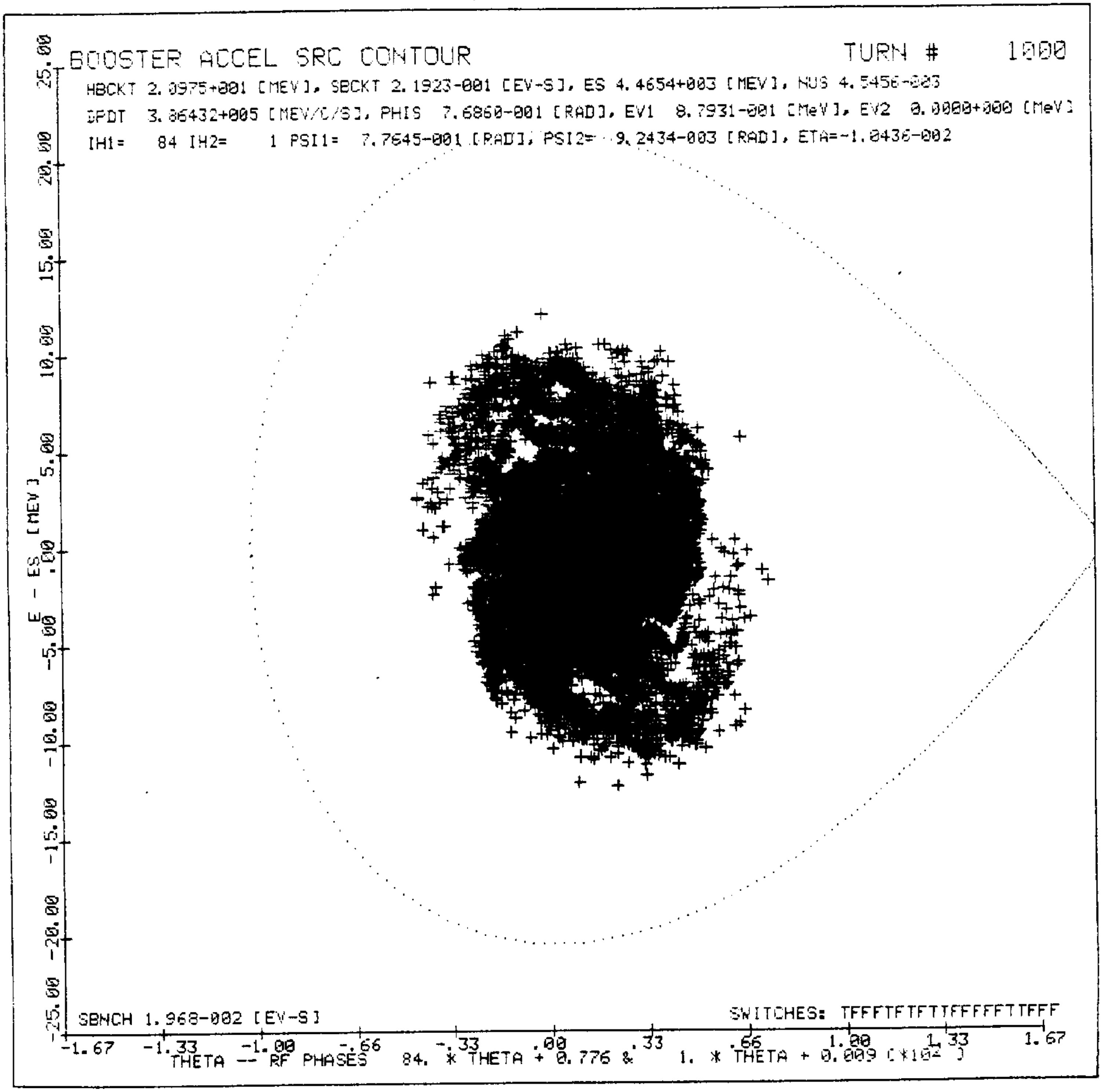

Figure 3a: Booster Bunch After 1000 Turns; 3.49 GeV, η = -0.01

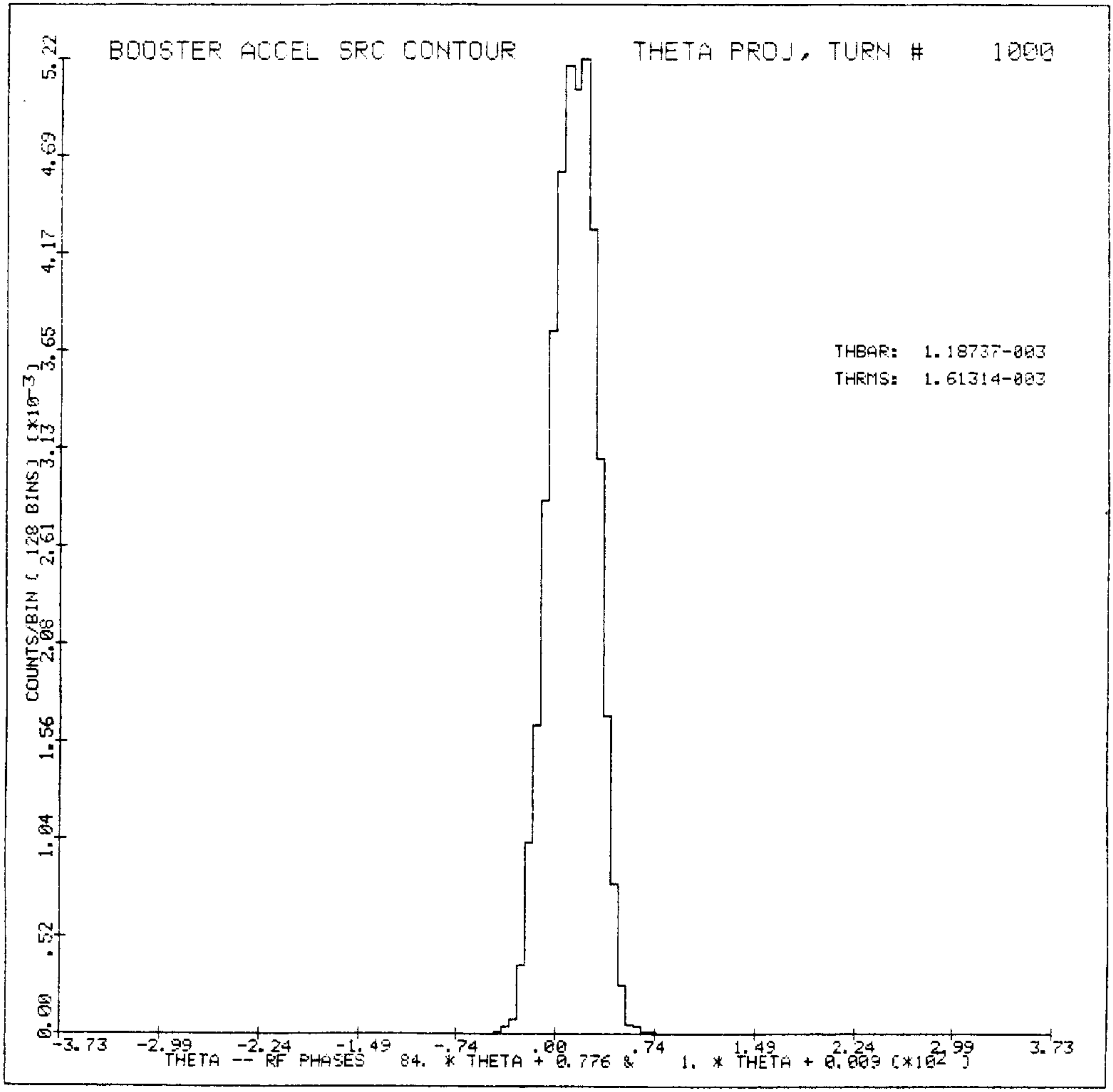

Figure 3b: Azimuthal Charge Distribution at 3.49 GeV

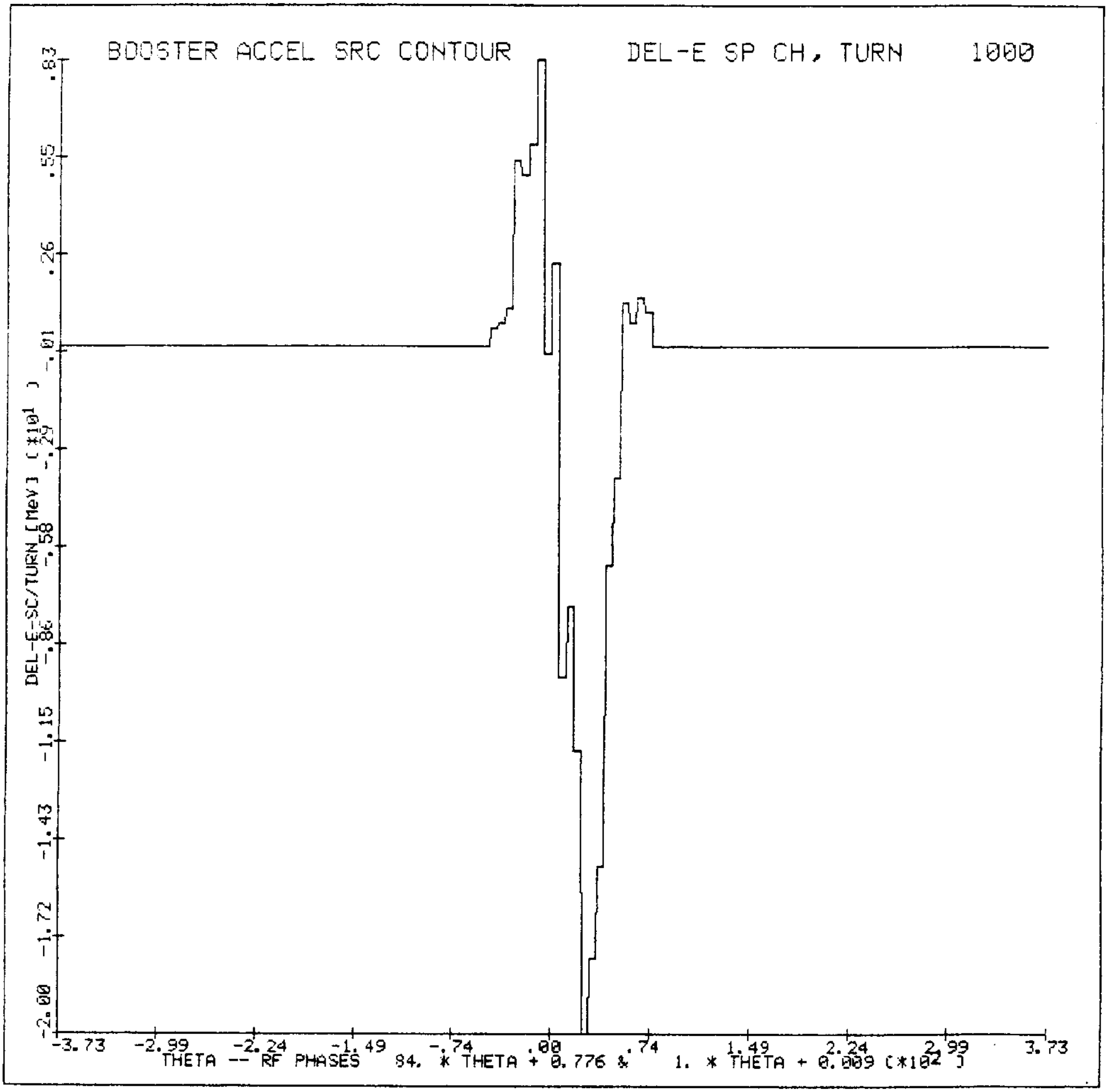

Figure 3c: Spacecharge and Wall Impedance Energy Increment (per turn) as a Function of Azimuth at 3.49 GeV

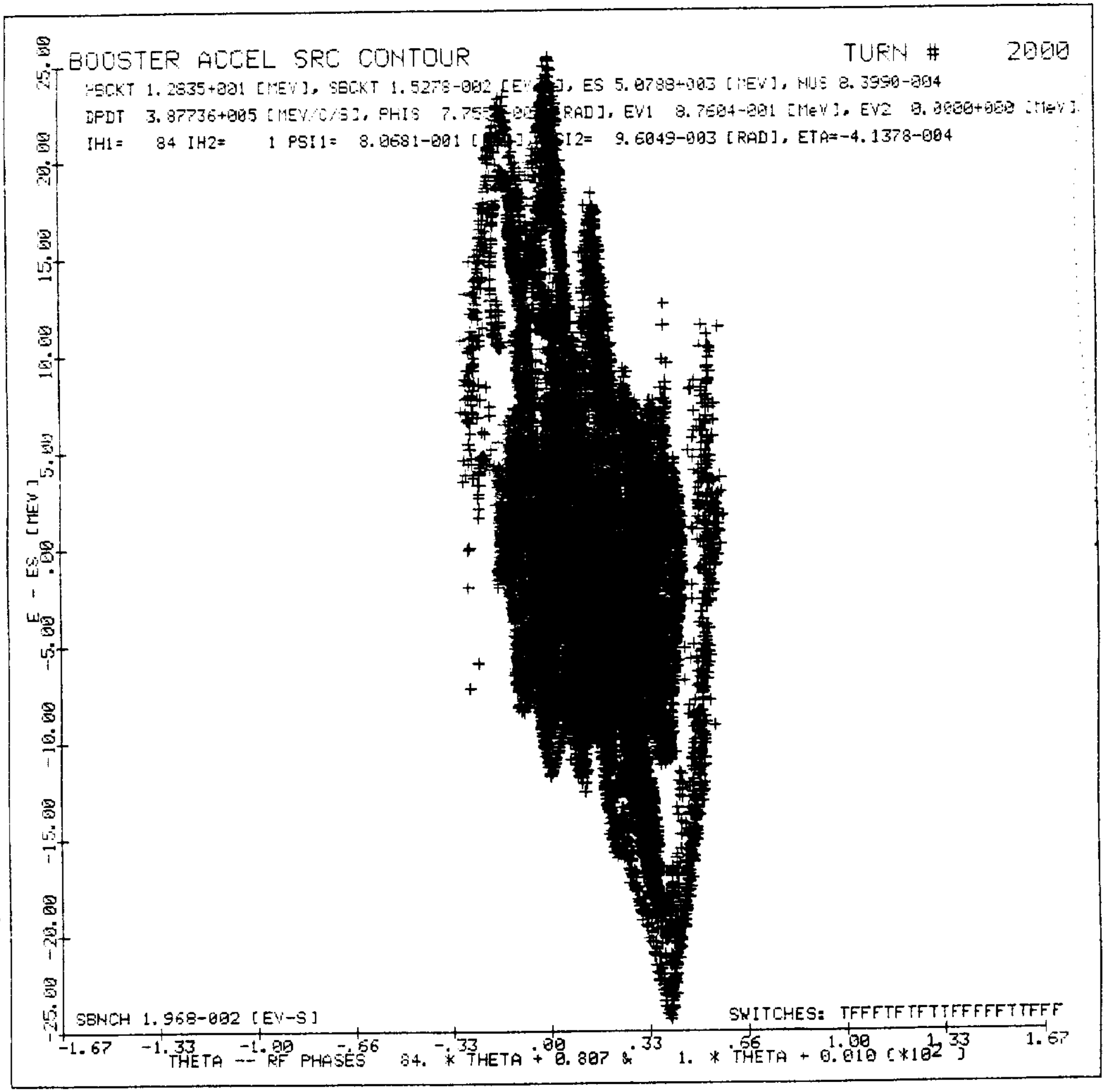

Figure 4a: Booster Bunch at Transition; 2000 Turns, 5.90 GeV, η = 0.00

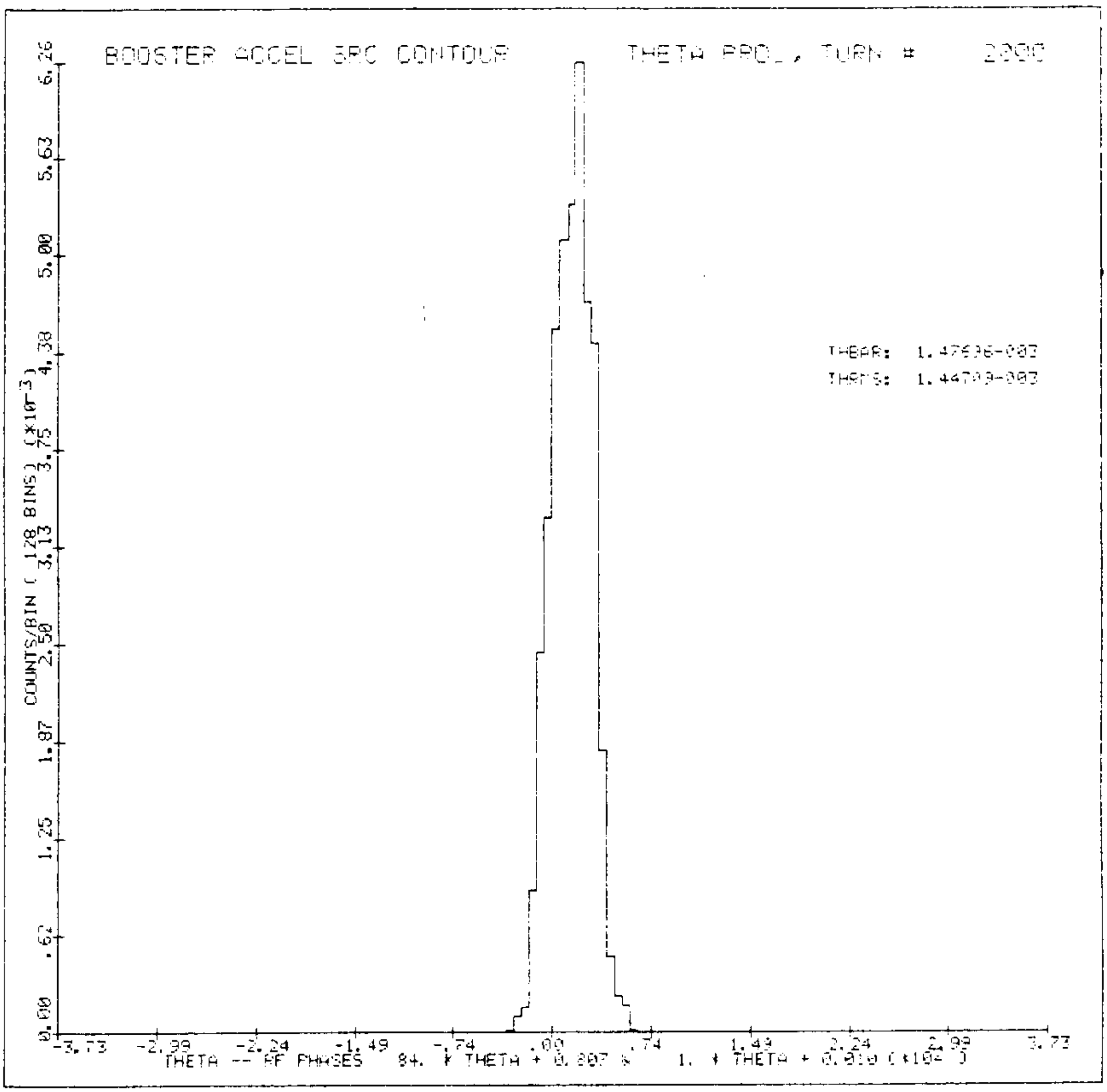

Figure 4b: Azimuthal Charge Distribution at Transition

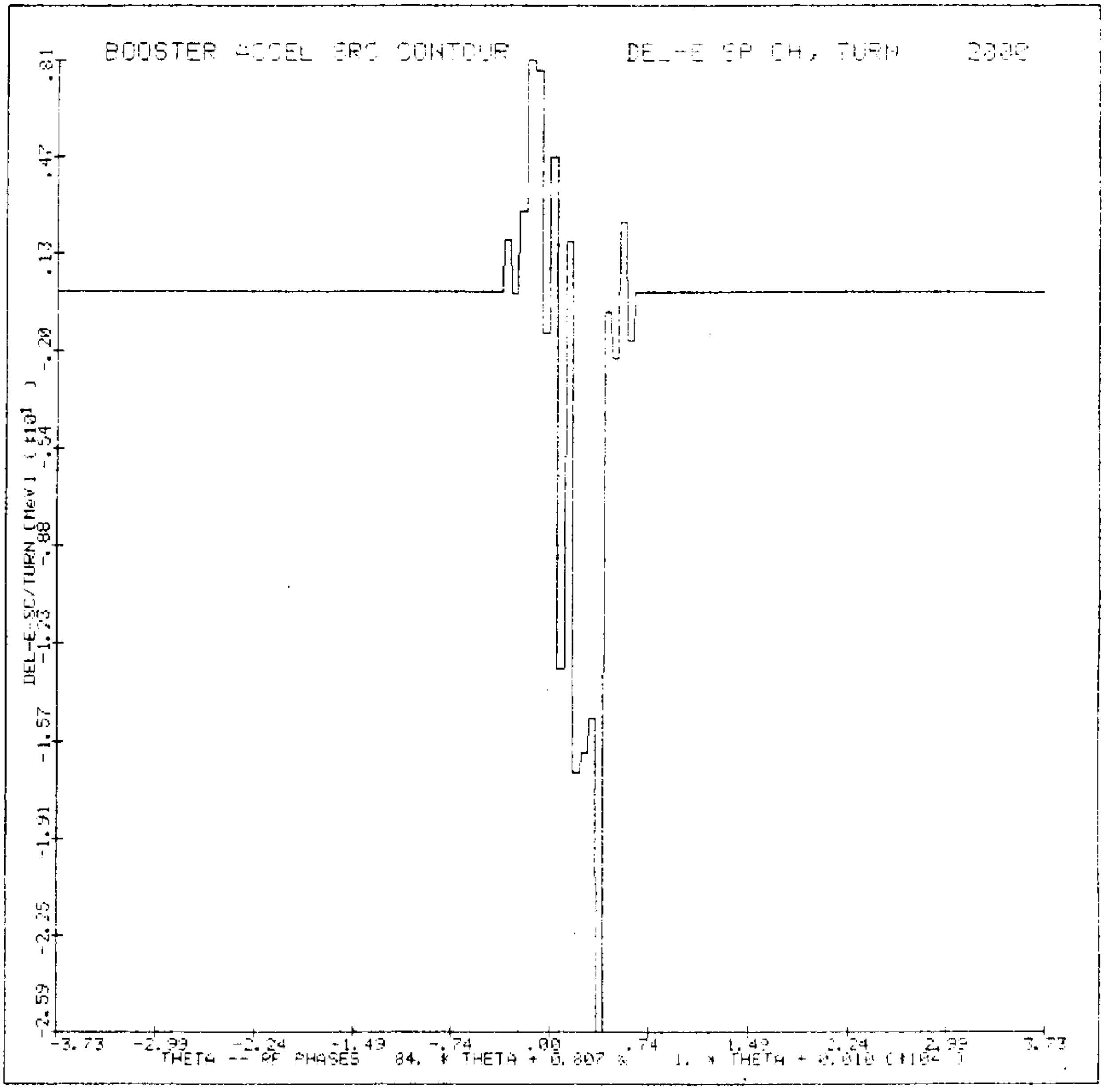

Figure 4c: Spacecharge and Wall Impedance Energy Increment (per turn) as a Function of Azimuth at Transition

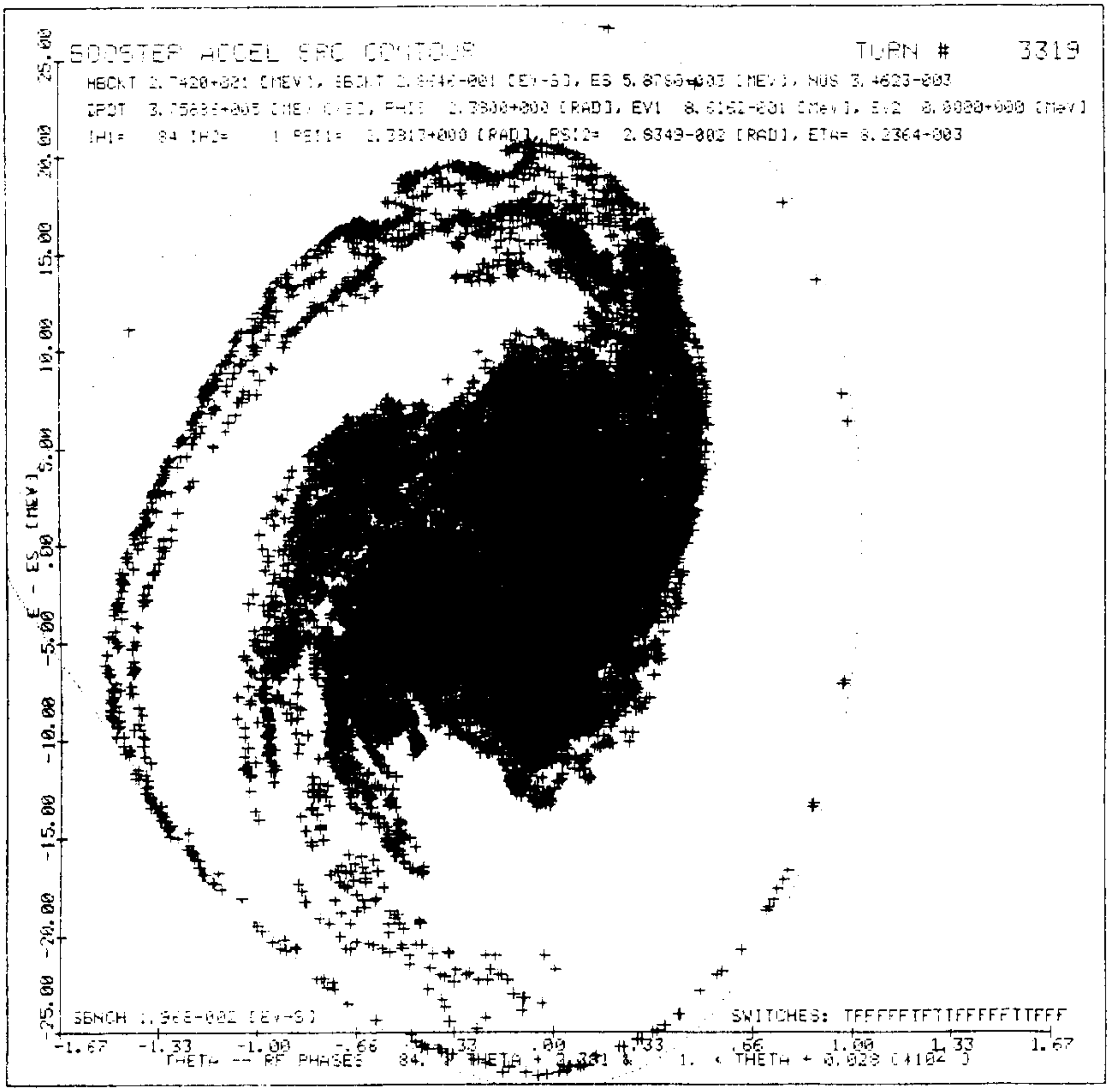

Figure 5a: Booster Bunch After 3319 Turns; 5.90 GeV, η = 0.01

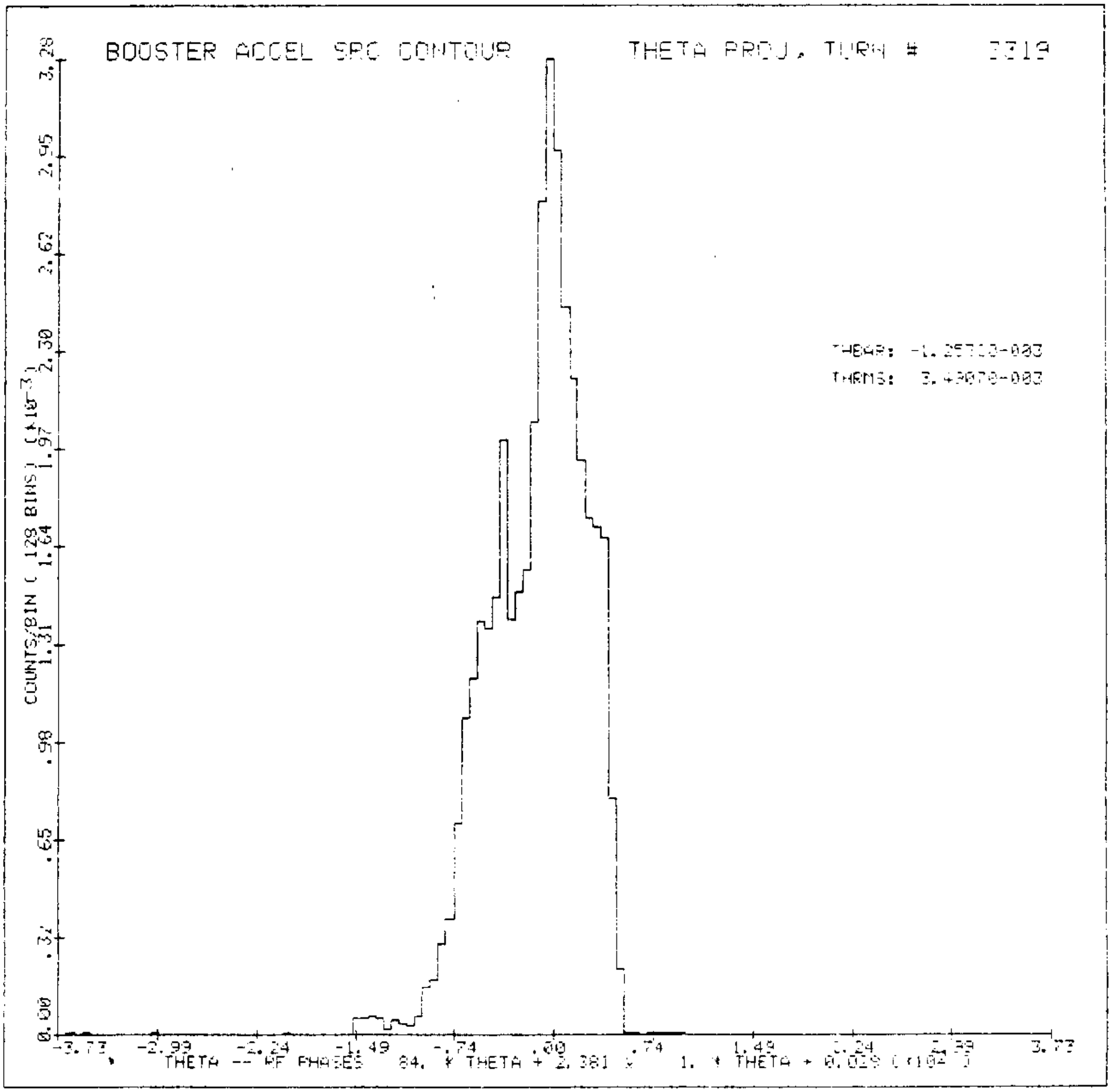

Figure 5b: Azimuthal Charge Distribution at 5.90 GeV

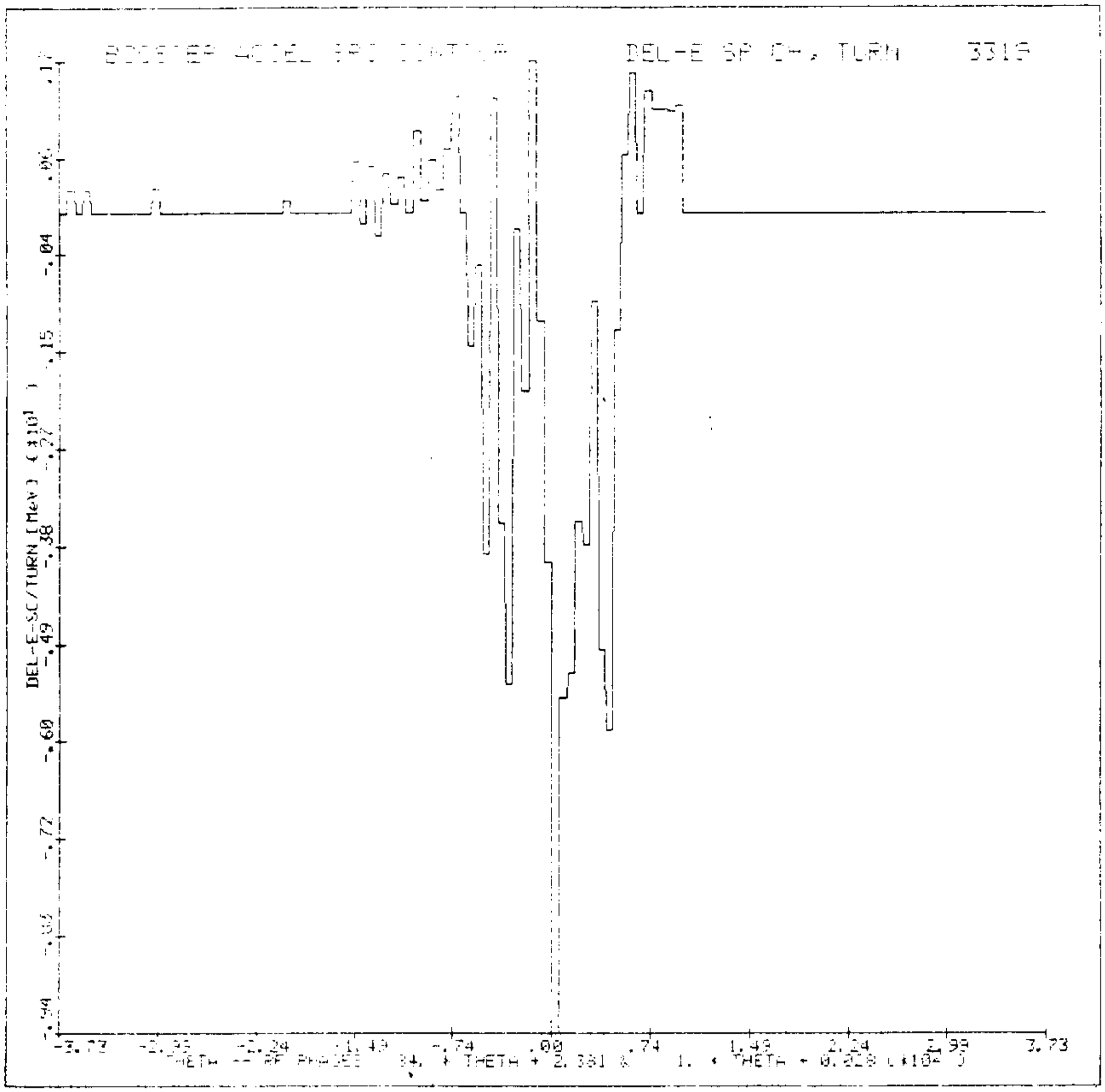

Figure 5c: Spacecharge and Wall Impedance Energy Increment (per turn)
as a Function of Azimuth at 5.90 GeV

departures from those idealized conditions which allow simple characterization and calculation of beam behavior; such simulation results are a valuable bridge between "text book" cases and a useful understanding of the diverse phenomena of real accelerators.

Acknowledgement

I am indebted to Peter Lucas of Fermilab who not only implemented the space charge and wall current features in the FPS 164 version of ESME but who in that process uncovered several problems with code which was being developed at the time. His careful testing and extensive application of the FPS 164 version to transition crossing in the booster were a major contributor to the completion of this work. In several instances his FPS implementation improved upon the CYBER code I was writing; in nearly every such case I have gratefully adopted those improvements.

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