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EFFECTS OF CORRECTION SEXTUPOLES IN SYNCHROTRONS*

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ABSTRACT

In order to have a usable acceptance of the beam momentum spread ($\Delta p/p$), synchrotrons must have at least two families of correction sextupoles which are generally placed at locations where the momentum dispersion function is large. These sextupoles are often called chromaticity correction sextupoles since their function is to eliminate (or to change in the most desirable manner) the dependence of the tunes of betatron oscillation on ($\Delta p/p$) caused by the chromatic aberration of quadrupoles. This report describes various harmful effects arising from these sextupoles and suggests possible cures. In particular, it is emphasized here that the second-order effects be minimized in order to prevent a potentially disastrous reduction in the transverse acceptance of the ring even when the tunes are considered to be safely away from the lowest-order (i.e., third-integer) resonances. The terms responsible for the dependence of tunes on the betatron oscillation amplitudes are explicitely given as a function of the five integrals representing the resonance-driving terms of the lowest-order Hamiltonian. The expression involves sums of infinite number of Fourier components but they can all be computed analytically.

INTRODUCTION

As the energy range of interest in nuclear physics is extended beyond $\sim$ GeV level, many nuclear physicists are seriously considering synchrotrons as their main tool of research in the near future. The importance of synchrotrons is also obvious as a storage device for the beam cooling and as a collider which are certainly popular topics now for nuclear physicists. There are already several plans for designing and building synchrotrons by those accelerator builders whose practical experiences have been limited so far to the world of cyclrotrons. The basic equations describing the particle motion in an external electromagnetic field are of course identical for any type of accelerators but the actual procedures used to design a machine and also the emphasis on specific aspects of beam dynamics are not always shared by the cyclotron builders and the synchrotron builders. One can probably argue that the beam dynamics for cyclotrons is more "involved" than the one for synchrotrons. Certainly there exist situations in which synchrotron designers (at least some of them) may look incred-

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ibly naive to cycltron builders. One example of this may be seen in the calculation of tunes for relatively small synchrotrons (such as rings for the synchrotron light source or for the storage and cooling of antiprotons) where the bend angle of each dipole is of the order of a few degrees instead of a few milliradians and the momentum acceptance is measured in percents instead of 10^{-4}. The standard computer programs for the synchrotron design such as SYNCH or PATRICIA may or may not give the right answer depending on how the field behaves near the edge of magnets. In the same way, there are a few tricks of the trade in the design of synchrotrons which must be kept in mind by cyclotron builders but the habit of many accelerator physicists to publish their important works only as an inromal technical memo (which often remains unaccessible outside the author's own institution) is a definite hindrance to the mutual learnings. One purpose of this note is to ameliorate this situation albeit to a very limited extent.

Unlike the magnetic field in cyclotrons with its flutters and spirals on top of the isochronous requirement, the ideal field for synchrotrons is extremely simple consisting of dipole and quadrupole types only. Moreover, it is more fashionable nowadays to have dipoles and quadrupoles as separate elements (Fermilab main ring, CERN SPS) than to combine them as a gradient-type bending magnet (Brookhaven AGS, CERN PS). In reality, of course, both dipoles and quadrupoles have higher-order multipoles (i.e., sextupoles, octupoles, etc.) and for some machines such as the Fermilab main ring and Tevatron, a very elaborate system of correction magnets is essential in order to make them work properly. Nonlinear fields are especially hard to avoid in superconducting magnets for which the field shape is almost totally dependent on the conductor placement (in contrast to the pole shape for conventional iron magnets). At low excitation level, the sextupole field created by the persistent current in the superconducting filament is particularly nasty and this more than anything else determins the injection energy of a large superconducting machine such as the proposed 20 TeV ring SSC. Although it is an importnat issue for designing synchrotrons, the subject of this report is not something associated with these unintended nonlinear fields but rather difficulties caused by sextupoles that must be installed in any synchrotron as its essential elements. The necessity of them arises from the chromatic aberration of quadrupoles whose effective focusing strength is naturally weaker(stronger) for particles with positive (negative) values of $\Delta p/p$. The effect is usually measured in terms of "chromaticity" defined as $\xi \equiv \Delta\nu/(\Delta p/p)$ where $\Delta\nu$ is the deviation of the tune for particles with the momentum deviation $(\Delta p/p)$ from the tune corresponding to $(\Delta p/p)=0$. Generally speaking, ξ is not too far from the value of tune itself (with the negative sign) but its magnitude can be substantially larger for colliders with special low-β insertions or for rings with unusual lattice structures. Since the action of a chromaticity-correcting sextupole is proportional to the dispersion function at that location, the strength required can be large when it is difficult to find a suitable location in the ring with a reasonably large value of dispersion function. For such cases, one must certainly be aware of the second-order effects in addition to the standard precaution of avoiding the lowest-order (i.e., third-integer) resonances.

Some necessary considerations in arranging sextupoles in storage rings were discussed by D. Edwards and others[1] and examples were given when the phase advance per cell is 90^{o}. Subsequently, H. Wiedemann presented an extensive discussion on the undesirable side effects of correction sextupoles in connection with the design of PEP.[2] For the second-order effects of sextupoles, Wiedemann followed the earlier work by F. Cole[3] but the formulars were limited to one degree of freedom only. More recently, there have been renewed interests in the analytical understanding of the reduction in transverse acceptance[4] although it is still too early to say unequivocally that something original has been accomplished. In what follows, the main results of these works will be summarized and, when it is felt to be necessary to do so, further discussions will be presented.

EFFECTS ON THE LINEAR LATTICE PARAMETERS

An elementary but nevertheless practical advice should be given here before going into the discussion of main subjects. Whenever strong nonlinear elements are present in a ring, it is very important to minimize the deviation of closed orbit in the two transverse directions in order to prevent the "feeding-down" effects. For example, a horizontal closed-orbit deviation within a sextupole element will effectively create a quadrupole (as well as a dipole) field and, depending on the spatial distribution (i.e., Fourier components) of relevant quantities, one may experience totally unexpected beam behaviors. When the deviation is in the vertical direction, the quadrupole field created is of the skew type and this may often confuse the beam diagonosis even when the beam quality itself is not affected.

When one talks about the change (and ways to minimize it) in the linear lattice parameters, β's and α's of Courant and Snyder[5] and the momentum dispersion function X_p caused by a group of sextupoles, it is useful to distinguish a situation which may be called *global* from the one called *local*. In the *global* situation, sextupoles are distributed around the ring so that one cannot really talk about the "inside" and "outside" of the sextupole group. Furthermore, one is interested in minimizing the change in lattice parameters everywhere in the ring. The report by Wiedemann[2] treats a situation of this type. Compared to the *global* case, the *local* situation is in principle easier to handle since the goal is to reduce (or often completely eliminate) the change only at a specific location in the ring. When the group of sextupoles under consideration is localized and a nontrivial fraction of the ring circumference is "outside" the group, the perturbation caused by the group can be totally confined so that the "specific location" is now any point outside. This happens in large colliders when a group of strong sextupoles must be installed near low-β insertions in order to cancel particularly nasty chromatic effects created by the nearby quadrupoles.

Almost all the formulas that are needed to calculate the changes in linear lattice parameters can be found in the classical paper by Courant and Snyder[5] even though they do not mention the effect of non-linear field explicitely. Because of the page limitation imposed on this report, it is necessary to assume here that the reader is familiar with their work, at least Sections 4. (a) and (b). More specifically, the formulas are

change in the momentum dispersion function

Eq. (4. 7), p.18 for *local*,

Eq. (4.12), p.19 for *global*,

cahnge in the betatron oscillation parameters β and α

Eq. (4.50)*, p.26 for *local*,

Eq. (4.53), p.26 for *global*.

Throughout this report, the integrated strength of each sextupole (divided by the particle rigidity $B\rho$) will be called S, $S \equiv (B''\ell/B\rho)$.

(a) change in the momentum dispersion function X_p

The function X_p is simply the closed-orbit deviation for the unit value of $(\Delta p/p)$. Therefore, the change in X_p can be found from the formula for closed-orbit deviations. When the lowest-order change in X_p is written as $(\Delta X_p)(\Delta p/p)$, the driving term of the equation for (ΔX_p) is $-\frac{1}{2}SX_p^2$. It is convenient to use the normalized complex quantity ΔZ,

$$(\Delta Z)_o \equiv (\Delta X_p)_o/\sqrt{\beta_o} + i\sqrt{\beta_o}(\Delta X_p' + \frac{\alpha}{\beta}\Delta X_p)_o \qquad (1)$$

where the prime denotes the derivative with respect to the path-length coordinate. The subscript o indicates that one is considering the change at the origin of the path-length coordinate from which, for example, the phase advances are measured. For the *local* situation, this choice of origin does not lead to any loss of generality. From Eq.(4. 7) of Courant-Snyder, one finds for sextupoles $k=1,2,\ldots.N$,

$$(\Delta Z)_o = \frac{e^{-i\pi\nu}}{2\sin(\pi\nu)}\sum_1^N f_k\sqrt{\beta_k}\; e^{i\psi_k} \qquad \text{"local"} \qquad (2)$$

where $f_k \equiv -\frac{1}{2}SX_p^2$ at the k-th sextupole, ψ_k is the phase advance from the origin to the k-th sextupole and ν is the tune. It is often possible to place a group of sextupoles at equivalent locations in the ring, that is, locations where both β and X_p take the same values. The condition that the change (ΔZ) be zero outside the sextupoles is then simplified to

$$\sum_1^N S_k\, e^{i\psi_k} = 0 \qquad \text{"local"} \qquad (3)$$

* Here a minus sign is missing in front of the integral and the argument of cosine function should be $2\nu(\pi+\phi_1-\phi)$ instead of $2\nu(\pi+\phi-\phi_1)$.

If there are more than one group, this condition should be satisfied by each group separately. For *global* cases, one obviously cannot limit the consideration to the change at the orign alone. According to Eq.(4.12) of Courant-Snyder,

$$(\Delta X_p) \text{ at } \phi = \sqrt{\beta(\phi)}\ \nu^2 \sum_n \frac{a_n}{\nu^2 - n^2}\, e^{in\phi} \qquad \textit{"global"} \qquad (4)$$

where $\phi \equiv \psi/\nu$ and a_n ($n=-\infty$ to $+\infty$) is the n-th Fourier component of the driving term,

$$a_n = (1/2\pi\nu) \sum_k f_k \sqrt{\beta_k}\, e^{-in\phi_k} \qquad (a_{-n} = a_n^*) \qquad (5)$$

In order to reduce ΔX_p substantially at many locations around the ring, it may be desirable to control several a_n's with n near ν although no more than two are usually important.

(b) change in the betatron oscillation parameters

Since the change in α is related to the change in β,

$$\Delta\alpha = -\frac{1}{2(\nu\beta)} \frac{d}{d\phi} (\Delta\beta), \qquad (6)$$

one can generalize Eq.(4.50) of Courant-Snyder for $(\Delta\beta)$ to an equation for the complex quantity

$$(\Delta Q)_o \equiv (\Delta\beta/\beta)_o - i\ (\Delta\alpha - \frac{\alpha}{\beta}\Delta\beta)_o \qquad (7)$$

$$(\Delta Q)_o = -\frac{e^{2i\pi\nu}}{2\sin(2\pi\nu)} \sum_k (S X_p)_k\ \beta_k\ e^{2i\psi_k}\cdot(\Delta p/p) \qquad \textit{"local"} \quad (8)$$

Under the condition that reduces the relation (2) to the simpler one of (3), one can again eliminate the change ΔQ outside the sextupoles by simply satisfying the relation

$$\sum S_k\, e^{2i\psi_k} = 0 \qquad \textit{"local"} \qquad (9)$$

If the change caused by sextupoles is to cancel out the change created by nearby quadrupoles, one must use the condition

$$(X_p\cdot\beta) \sum_k^N S_k\, e^{2i\psi_k} = \sum_j^M (B'\ell/B\rho)_j\ \beta_j\ e^{2i\psi_j} \qquad \textit{"local"} \quad (10)$$

where $(X_p\cdot\beta)$ is assumed to be common to all sextupoles $k = 1,2,\ldots N$ but β_j's are generally quite different at different quadrupole locations $j = 1,2,\ldots M$. Such a situation arises in a collider with low-β insertions with a number of strong insertion quadrupoles. Note that this condition must hold in the two transverse directions separately, that is, one with ($\beta = \beta_x$, $\psi = \psi_x$) and another with ($\beta = \beta_y$, $\psi = \psi_y$) while other quantities including the dispersion function X_p are kept the same in two conditions. For the *global* situation, one must express $(\Delta\beta/\beta)$ at general locations in terms of the Fourier components J_n ($n = -\infty$ to $+\infty$),

$$J_n = \sum_k (SX_p)_k \, \beta_k \, e^{-in\phi_k} , \tag{11}$$

$$(\Delta\beta/\beta) \text{ at } \phi = -\frac{\nu}{\pi} (\Delta p/p) \sum_n \frac{J_n}{4\nu^2 - n^2} e^{in\phi} \qquad \textit{"global"} \tag{12}$$

How many J_n's should be controled depends on many factors but it is unusual to see more than four families of sextupoles in a ring. The minimum number of family is two since one must control J_o for ν_x and ν_y independently:

$$\Delta\nu_{x \text{ or } y} = \frac{\pm 1}{4\pi} \sum_k (SX_p)_k \cdot (\beta_k)_{x \text{ or } y} \cdot (\Delta p/p) \tag{13}$$

(plus sign for x, minus sign for y)

RESONANCES AND THE SECOND-ORDER EFFECTS

It is well-known that the normal sextupole field $B_y = \frac{1}{2}B''(x^2 - y^2)$ and $B_x = B''xy$ can drive third-integer resonances of the form

$$3 \cdot \nu_x = n \qquad \text{and} \qquad \nu_x \pm 2 \cdot \nu_y = n \tag{14}$$

If the sextupole field is periodic around the ring with the period N (for example, the lattice period), n must be integer multiples of N and one usually tries to choose tunes such that any of these resonances is at least one-quater unit away from the design operating point. This however is not always possible when special correction sextupoles are needed and one must never take it for granted that a particular arrangement of sextupoles in the ring is "safe" simply because the resonance width computed from the standard formula[6] is small compared to the distance to the resonance. Since this issue has been discussed throughly elsewhere,[7] only the main conclusion will be restated here: Even when the operating point is believed to be safely away from all third-integer resonances, one must try to minimize certain first-order expressions in order to prevent undesirable sextupole effects such as distortions of the beam shape in phase space and, for some instances, even non-negligible reductions in the transverse acceptance. One customarily relies on time-consuming numerical beam trackings by computers for studying the effects but they are often hard to use effectively as a design guidance. A simple summary of the analytical approach to this problem[3,8] is therefore presented here as something complementary to the purely numerical approach.

In terms of the action-angle variables ($I_{x,y};a_{x,y}$) and the independent variable $\theta \equiv$ s/average machine radius, one writes ($z = x$ or y)

$$z = \sqrt{2I_z\beta_z}\ \sin F_z; \quad dz/ds = \frac{\sqrt{2I_z}}{\sqrt{\beta_z}}\ (\cos F_z - \alpha_z \sin F_z) \tag{15}$$

where $F_z \equiv Q_z + a_z = (\psi_z - \nu_z\theta) + a_z$. The quantity Q_z, which is periodic in θ with the period 2π, is a measure of the alternating-gradient focusing. In the presence of sextupole fields $S_k=(B''\ell/B\rho)_k$, the Hamiltonian to be used for the equations of motion

$$(da_z/d\theta) = (\partial H/\partial I_z), \qquad (dI_z/d\theta) = -(\partial H/\partial a_z)$$

can be written as

$$\begin{aligned} H(a_x,a_y,I_x,I_y;\theta) &= \nu_x I_x + \nu_y I_y + (2I_x)^{3/2}\{\Sigma\ A_{3m}\cos(3a_x-m\theta+\alpha_{3m}) \\ &- 3\ \Sigma A_{1m}\cos(a_x-m\theta+\alpha_{1m})\} + (2I_y)\sqrt{2I_x}\{\Sigma 2\cdot B_{1m}\cos(a_x-m\theta+\beta_{1m}) \\ &- \Sigma B_{+m}\cos(a_x+2a_y-m\theta+\beta_{+m}) - \Sigma B_{-m}\cos(a_x-2a_y-m\theta+\beta_{-m})\} \end{aligned} \tag{16}$$

in which Σ's are all from $m=-\infty$ to $+\infty$. Fourier components of the five driving terms of the third-integer resonances are

$$A_{3m}e^{i\alpha_{3m}} = (i/48\pi)\sum_k (S\beta_x^{3/2})_k\ e^{i(3Q_x+m\theta)_k} \tag{17.1}$$

$$A_{1m}e^{i\alpha_{1m}} = (i/48\pi)\sum_k (S\beta_x^{3/2})_k\ e^{i(Q_x+m\theta)_k} \tag{17.2}$$

$$B_{1m}e^{i\beta_{1m}} = (i/16\pi)\sum_k (S\beta_y\sqrt{\beta_x})_k\ e^{i(Q_x+m\theta)_k} \tag{17.3}$$

$$B_{\pm m}e^{i\beta_{\pm m}} = (i/16\pi)\sum_k (S\beta_y\sqrt{\beta_x})_k\ e^{i(Q_x\pm 2Q_y+m\theta)_k} \tag{17.4\&5}$$

Based on this Hamiltonian and the equations of motion, one can develop the standard treatment of third-integer resonances $3\nu_x=n$ and $\nu_x \pm 2\nu_y=n$. When the operating point is far away from these resonances, as indeed it should be unless one is dealing with special situations such as the resonance extraction, this first-order treatment gives a rather meaningless result since one of its fundamental assumptions is not valid.[7] The second-order Hamiltonian is obtained by the canonical transformation $(a_{x,y};I_{x,y}) \to (b_{x,y};J_{x,y})$ with the generating function

$$\begin{aligned} S(a_x,J_x,a_y,J_y) &= a_xJ_x + a_yJ_y + (2J_x)^{3/2}\{\Sigma\frac{A_{3m}}{m-3\nu_x}\sin(q_{3m}) \\ &- 3\Sigma\frac{A_{1m}}{m-\nu_x}\sin(q_{1m})\} + (2J_y)\sqrt{2J_x}\{\Sigma\frac{2B_{1m}}{m-\nu_x}\sin(p_{1m}) \\ &- \Sigma\frac{B_{+m}}{m-\nu_+}\sin(p_{+m}) - \Sigma\frac{B_{-m}}{m-\nu_-}\sin(p_{-m})\} \end{aligned} \tag{18}$$

where $\nu_\pm = \nu_x \pm 2\nu_y$, $q_{3m}=3a_x-m\theta+\alpha_{3m}$, $q_{1m}=a_x-m\theta+\alpha_{1m}$, $p_{1m}=a_x-m\theta+\beta_{1m}$,

$p_{\pm m}=a_x\pm 2a_y-m\theta+\beta_{\pm m}$. The new and the old variables are related in the usual manner,

$$b_z = \partial S/\partial J_z \quad \text{and} \quad I_z = \partial S/\partial a_z \tag{19}$$

The new Hamiltonian is

$$\begin{aligned} K(b_x,b_y,J_x,J_y;\theta) &= H + \partial S/\partial\theta \\ &= \nu_x J_x + \nu_y J_y + \{(2I_x)^{3/2}-(2J_x)^{3/2}\}\{\Sigma A_{3m}\cos(q_{3m})-3\Sigma A_{1m}\cos(q_{1m})\} \\ &\quad +\{(2I_y)\sqrt{2I_x}-(2J_y)\sqrt{2J_x}\}\{2\Sigma B_{1m}\cos(p_{1m})-\Sigma B_{\pm m}\cos(p_{\pm m})\} \end{aligned} \tag{20}$$

When this is expanded as a power series in the sextupole strength S, the lowest-order terms are S^2 and they are composed of numerous resonance-driving terms of the sixth-, fourth- and second-integer types as well as terms independent of all angle variables. The derivation is straightforward but certainly very tedious; if written explicitly several pages are needed to list them all. In many cases, the most important terms are the ones independent of angles b_x, b_y or θ since they are responsible for the dependence of tunes on the oscillation amplitudes:

$$\begin{aligned} \Delta K = {} & (2J_x)^2\cdot(\tfrac{9}{2})\{\Sigma\frac{A_{3m}^2}{m-3\nu_x} + 3\Sigma\frac{A_{1m}^2}{m-\nu_x}\} + (2J_y)^2(\tfrac{1}{2})\{\Sigma\frac{4B_{1m}^2}{m-\nu_x} \\ & + \Sigma\frac{B_{\pm m}^2}{m-\nu_\pm}\} + (2J_x)(2J_y)\cdot 2\{\Sigma\frac{B_{+m}^2}{m-\nu_+} - \Sigma\frac{B_{-m}^2}{m-\nu_-} \\ & - 6\,\Sigma\frac{A_{1m}B_{1m}}{m-\nu_x}\cos(\alpha_{1m}-\beta_{1m})\} \end{aligned} \tag{21}$$

In principle, one must find all Fourier components ($m=-\infty$ to $+\infty$) from Eqs.(17.1 to 5) although one often replaces the phase factors with quantities independent of m:

$$\begin{aligned} &(3Q_x+m\theta)_k \rightarrow (3\psi_x)_k, \quad (Q_x+m\theta)_k \rightarrow (\psi_x)_k, \\ &(Q_x \pm 2Q_y+m\theta)_k \rightarrow (\psi_x \pm 2\psi_y)_k \end{aligned} \tag{22}$$

These substitutions are justified in Eqs.(18) and (21) because of the denominators. Even when this is not allowed, one can perform the infinite sum over m analytically using the relation

$$\begin{aligned} \sum_m \frac{e^{i(m\theta+b)}}{m-a} &= -\frac{\pi}{\sin(\pi a)}\, e^{-i\{a(\pi-\theta)-b\}} && \text{for } 0<\theta<2\pi \\ &= -\pi\cot(\pi a)\, e^{ib} && \text{for } \theta=0 \end{aligned} \tag{23}$$

Summations over m in Eq.(18) are then replaced with finite summations over k (index for sextupoles) and, in Eq.(21), they are replaced with double summations over pairs of k's.

When a group of sextupoles are all at equivalent locations (β_x and β_y independent of the index k) and the approximations (22) are valid, one can minimize the sextupole effects by satisfying the conditions

$$\Sigma S_k \, e^{i(3\psi_x)_k} = \Sigma S_k \, e^{i(\psi_x)_k} = \Sigma S_k \, e^{i(\psi_x \pm 2\psi_y)_k} = 0 \qquad (24)$$

These and the similar conditions (3), (9) and (10) are further simplified when the phase advance from the k-th to the (k+1)th sextupoles is constant throughout the entire group. This happens when sextupoles are placed always near the horizontally-focusing or vertically-focusing quadrupoles of the regular cells. Note also that the condition (24) applies to all sextupoles while (3), (9) and (10) are only for those where the dispersion function is not zero.

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