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NOTE ON THE COURANT AND SNYDER INVARIANT

Ken Takayama

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Fermi National Accelerator Laboratory, Batavia, Ill 60510, USA

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1. Introduction

Over the past quarter century, a considerable amount of work has been devoted to the study of the time-dependent linear oscillator

$$\ddot{x} + K(s)x = 0, \tag{1-1}$$

which represents betatron oscillations in accelerators and storage rings. Courant and Snyder/1/ first found that a conserved quantity for Eq.(1-1) is

$$I = \frac{1}{2\beta(s)}\left[x^2 + \left(\frac{\dot{\beta}(s)}{2}x - \beta(s)\dot{x}\right)^2\right], \tag{1-2}$$

where x(s) satisfies Eq.(1-1) and β(s) satisfies the auxiliary equation

$$\frac{1}{2}\beta\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + K(s)\beta^2 = 1. \tag{1-3}$$

Several derivations of the dynamical invariant(1-2) have been given in the literature: The exact invariant was derived by Lewis and Riesenfeld/2/ on the assumption of quadratic invariance. Lutzky/3/ derived the invariant(1-2) from Noether's theorem and recently Korsch/4/ presented a proof of the dynamical invariance of (1-2) , using the method of dynamical algebra. An early discussion about the general interrelation between the differential equation(1-1) and (1-2) can be found in an article by Milne/5/. In addition, a physical meaning of the origin of the invariant was presented by Eliezer and Gray/6/, with the help of auxiliary plane motion.

It is the aim of the present note to review the three different methods for deriving the dynamical invariant (1-2) and to investigate possibilities of applying them to the nonlinear betatron oscillation

$$\ddot{x} + K(s)x + K'(s)x^2 = 0, \tag{1-4}$$

which is derivable from the Hamiltonian

$$H(x, p; s) = \frac{1}{2}(p^2 + K(s)x^2) + \frac{1}{3}K'(s)x^3. \tag{1-5}$$

2. Derivation of Invariant

2-a Time-Dependent Linear Canonical Transformation

We shall show explicitly that a time-dependent Hamiltonian

$$H(x, p; s) = \frac{1}{2}(p^2 + K(s)x^2), \tag{2-1}$$

can be converted to time-independent form with the help of a time-dependent linear canonical transformation and a change of time scale.

The canonical equations of motion obtained from (2-1) are

$$\dot{x} = \frac{\partial H}{\partial p} = p, \tag{2-2-a}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -K(s)x. \tag{2-2-b}$$

First we require that the Hamiltonian in Eq.(2-1) is transformed into the form

$$H'(X, P; s) = \frac{f(s)}{2}(P^2 + X^2), \tag{2-3}$$

with a time-dependent function f(s) which is determined later, by means of the time-dependent linear transformations

$$X = \Lambda_1^1(s)\, x + \Lambda_2^1(s)\, p \,, \qquad (2\text{-}4\text{-}a)$$

$$P = \Lambda_1^2(s)\, x + \Lambda_2^2(s)\, p \,. \qquad (2\text{-}4\text{-}b)$$

Because we assume the canonical transformation, the time-dependent coefficients $\Lambda_1^1(s)$, $\Lambda_2^1(s)$, $\Lambda_1^2(s)$, and $\Lambda_2^2(s)$ in Eq.(2-4) must satisfy the relation

$$\Lambda_1^1(s)\Lambda_2^2(s) - \Lambda_2^1(s)\Lambda_1^2(s) = 1 \,. \qquad (2\text{-}5)$$

The canonical equations of motion obtained from Eq.(2-3) are

$$\dot{X} = \frac{\partial H'}{\partial P} = f(s)\, P \,, \qquad (2\text{-}6\text{-}a)$$

$$\dot{P} = -\frac{\partial H'}{\partial X} = -f(s)\, X \,. \qquad (2\text{-}6\text{-}b)$$

In order to determine the unknown time-dependent coefficients in (2-3),(2-4), the relations in (2-2),(2-4),and (2-6) are combined in such a manner that the new canonical variables are replaced by the old ones. This is effected by taking the time derivatives of the relations in (2-4), replacing X and P by the expressions (2-6) , and then substituting x and p by the quantities given in (2-2). Finally we equate the coefficients of like powers of x, p, x^2, and p^2 from both sides of the equations and obtain the relations among the coefficients

$$\dot{\Lambda}_1^1 = K(s)\Lambda_2^1 + f(s)\Lambda_1^2 \,, \qquad (2\text{-}7\text{-}a)$$

$$\dot{\Lambda}_2^1 = -\Lambda_1^1 + f(s)\Lambda_2^2, \qquad (2\text{-}7\text{-}b)$$

$$\dot{\Lambda}_1^2 = -f(s)\Lambda_1^1 + K(s)\Lambda_2^2, \qquad (2\text{-}7\text{-}c)$$

$$\dot{\Lambda}_2^2 = -f(s)\Lambda_2^1 - \Lambda_1^2. \qquad (2\text{-}7\text{-}d)$$

The coupled equations (2-7) may be solved by the well-known matrix method. However we show a set of particular solutions satisfying Eqs.(2-5),(2-7). Taking $\Lambda_2^1=0$ and replacing Λ_2^2 with $\rho(s)$, it is then trivial to obtain the solution for Λ_1^1 from (2-5). The solution is

$$\Lambda_1^1(s) = \rho^{-1}(s). \qquad (2\text{-}8)$$

Substituting $\Lambda_2^1=0$, $\Lambda_2^2=\rho(s)$, and (2-8) into (2-7-b), we have

$$f(s) = \rho^{-2}(s). \qquad (2\text{-}9)$$

Also substituting the time derivative of Λ_1^1, $\Lambda_2^1=0$, and (2-9) into (2-7-a), we have

$$\Lambda_1^2 = -\dot{\rho}(s). \qquad (2\text{-}10)$$

Eq.(2-10) is equivalent to Eq.(2-7-d). Next, substituting (2-8), (2-9) and (2-10) into Eq.(2-7-a), we obtain the differential equation satisfied by $\rho(s)$,

$$\ddot{\rho} + K(s)\rho = \rho^{-3}. \qquad (2\text{-}11)$$

If we replace $\rho(s)$ with $\sqrt{\beta(s)}$, the differential equation for the so- called betatron amplitude function $\beta(s)$ will be easily written down

$$\frac{1}{2}\beta\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + \beta^2 K(s) = 1. \tag{2-12}$$

Furthermore, if the change of independent variable

$$\phi(s) = \int^s f(s')\,ds', \tag{2-13}$$

is made, the Hamiltonian H' becomes

$$H''(X, P; \phi) = \frac{1}{2}(P^2 + X^2). \tag{2-14}$$

Evidently the new Hamiltonian H" is a constant of motion in the coordinate system of $(X,P;\phi)$. It is apparent that Eq.(2-4) is invariant in the old system (x,p;s);

$$\frac{dH''}{ds} = \frac{dH''}{d\phi}\frac{d\phi}{ds} = f(s)\frac{\partial H''}{\partial\phi} = 0.$$

Next let us show (2-14) as a function of $\beta(s)$, x and $\dot{x}$. Using (2-8) and (2-10), we write the s-dependent coefficients of $\Lambda_1^1(s)$, $\Lambda_2^1(s)$, $\Lambda_1^2(s)$, and $\Lambda_2^2(s)$ in (2-4-a),(2-4-b) with the function $\beta(s)$,

$$\begin{aligned}
\Lambda_1^1(s) &= \beta^{-\frac{1}{2}}(s), \\
\Lambda_2^1(s) &= 0, \\
\Lambda_1^2(s) &= -\frac{1}{2}\beta^{-\frac{1}{2}}(s)\dot{\beta}(s), \\
\Lambda_2^2(s) &= \beta^{\frac{1}{2}}(s).
\end{aligned} \tag{2-15}$$

Setting these values in (2-4-a),(2-4-b) and substituting them into (2-14), we obtain the invariant

$$H'' = \frac{1}{2}\left[\left(-\frac{1}{2}\beta^{-\frac{1}{2}}\dot{\beta}x + \beta^{\frac{1}{2}}p\right)^2 + \left(\beta^{-\frac{1}{2}}x\right)^2\right]. \tag{2-16}$$

Setting $p=\dot{x}$ and H"= I in (2-16), we write the dynamical invariant in the form

$$I = \frac{1}{2\beta(s)}\left[x^2 + \left(\frac{\dot\beta(s)}{2}x - \beta(s)\dot x\right)^2\right]. \tag{2-17}$$

In addition for reference, we show that the generating function for the canonical transformation (2-4-a),(2-4-b) is easily derived.

Using a generating function $F^2(x,P;s)$ of the second type, we write

$$p = \frac{\partial F^2}{\partial x} = \frac{1}{\Lambda_2^2}(P - \Lambda_1^2 x), \tag{2-18-a}$$

$$X = \frac{\partial F^2}{\partial P} = \Lambda_1^1 x. \tag{2-18-b}$$

From (2-18-b), we assume

$$F^2(x,P;s) = \Lambda_1^1 P + g(x;s), \tag{2-19}$$

where g(x) is an arbitrary function of x. Substituting (2-19) into (2-18-a) and equating the term of P and x on both sides, we obtain

$$\frac{\partial g}{\partial x} = -\frac{\Lambda_1^2}{\Lambda_2^2}x. \tag{2-20}$$

From (2-20), g(x;s) becomes

$$g(x;s) = -\frac{\Lambda_1^2}{2\Lambda_2^2}x^2 + h(s). \tag{2-21}$$

Furthermore, setting h(s)=0 in (2-21), we have the generating function

$$F^2(x,P;s) = \Lambda_1^1 P - \frac{\Lambda_1^2}{2\Lambda_2^2}x^2. \tag{2-22}$$

2-b Dynamical Algebra

We can construct easily the dynamical algebra for the Hamiltonian

$$H(x, p; s) = \sum_{n=1}^{3} h_n(s)\Gamma_n(x, p), \tag{2-23}$$

following the usual procedure. Here the dynamical algebra is the Lie algebra of the phase-space functions Γ_n, which are closed under the action of the Poisson bracket $[\ ,\ ]$:

$$[\Gamma_n, \Gamma_m] = \sum_{r=1}^{3} C_{n.m}^{r}\Gamma_r, \tag{2-24}$$

where the $C_{n.m}^{r}$ are the structure constants of the algebra. For the Hamiltonian (2-23), Γ_n has a set of Poisson brackets

$$[\Gamma_1, \Gamma_2] = -2\Gamma_1, \quad [\Gamma_2, \Gamma_3] = -2\Gamma_3, \quad [\Gamma_3, \Gamma_1] = \Gamma_2. \tag{2-25}$$

From Eq.(2-25) we see easily that the algebra is closed.

The structure constants $C_{n.m}^{r}$ are described by the matrices

$$C_{n.m}^{1} = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_{n.m}^{2} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_{n.m}^{3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}. \tag{2-26}$$

The time development of a phase-space function I is given by

$$\frac{dI}{ds} = \frac{\partial I}{\partial s} + [I, H], \tag{2-27}$$

and the dynamical invariant I is characterized by

$$\frac{dI}{ds} = 0, \quad i.e., \quad \frac{\partial I}{\partial s} = -[I, H]. \tag{2-28}$$

We now look for an invariant that is a member of the dynamical algebra

$$I = \sum_{n=1}^{3} \lambda_n(s)\Gamma_n, \tag{2-29}$$

which gives, with (2-28),

$$\sum_{r=1}^{3} [\, \dot{\lambda}_r + \sum_{n,m}^{3} C^r_{n,m} h_m(s)\lambda_n(s)]\,\Gamma_r = 0, \tag{2-30}$$

and therefore the system of linear first-order equations

$$\dot{\lambda}_r + \sum_{n}^{3} [\, \sum_{m}^{3} C^r_{n,m} h_m(s)]\,\lambda_n = 0, \tag{2-31}$$

with $h_1(s)=1$, $h_2(s)=0$, $h_3(s)=K(s)$. The coefficients $\lambda_n(s)$ of the dynamical invariant

$$I = \frac{1}{2}\lambda_1(s)p^2 + \lambda_2(s)\,px + \frac{1}{2}\lambda_3(s)x^2, \tag{2-32}$$

are solutions of the differential equations

$$\frac{d}{ds}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} = \begin{pmatrix}0 & -2 & 0\\ K(s) & 0 & -1\\ 0 & 2K(s) & 0\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}. \tag{2-33}$$

Setting $\lambda_1 = \beta_c(s)$, we find

$$\lambda_2 = -\dot{\beta}_c/2, \tag{2-34-a}$$

$$\dot{\lambda}_3 = -K(s)\dot{\beta}_c, \tag{2-34-b}$$

$$\lambda_3 = \ddot{\beta}_c/2 + K(s)\beta_c. \tag{2-34-c}$$

Equating the derivative of (2-34-c) with (2-34-b), we finally obtain

$$\dddot{\beta}_c + 4K(s)\dot{\beta}_c + 2\dot{K}(s)\beta_c = 0, \tag{2-35}$$

which has the integral

$$\frac{1}{2}\beta_c \ddot{\beta}_c - \frac{1}{4}\dot{\beta}_c^2 + K(s)\beta_c^2 = C, \tag{2-36}$$

with integration constant C. The solution (2-36) determines the $\lambda_n(s)$ and the dynamical invariant (2-33) is therefore expressed in the form

$$I = \frac{1}{2\beta_c}\left[C x^2 + \left(\frac{\dot{\beta}_c}{2} x - \beta_c P\right)^2\right]. \tag{2-37}$$

The arbitrariness implied by the presence of the constant C is illusory, as may be verified by making the scale transformation

$$\beta(s) = C^{-\frac{1}{2}}\beta_c, \tag{2-38}$$

$\beta(s)$ being a new auxiliary function of s. The auxiliary equation which $\beta(s)$ satisfies is

$$\frac{1}{2}\beta\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + K(s)\beta^2 = 1. \tag{2-39}$$

After discarding a constant multiplicative facter $C^{\frac{1}{2}}$ and setting $P=\dot{x}$, we write Eq.(2-37) in the form

$$I = \frac{1}{2\beta}\left[x^2 + \left(\frac{\dot{\beta}}{2} x - \beta\dot{x}\right)^2\right]. \tag{2-40}$$

2-c Noether's Theorem

The formulation of Noether's theorem used is the one given by Lutzky. If the transformation

$$G = \xi(x,s)\frac{\partial}{\partial s} + n(x,s)\frac{\partial}{\partial x},$$

leaves the action integral $\int L(x,\dot{x};s)\,ds$ invariant,

$$\xi\frac{\partial h}{\partial s} + n\frac{\partial h}{\partial x} + (\dot{n} - \dot{x}\dot{\xi})\frac{\partial h}{\partial \dot{x}} + \dot{\xi} h = \dot{f}, \tag{2-41}$$

where f=f(s,t), and

$$\dot{\xi} = \frac{\partial \xi}{\partial s} + \dot{x}\frac{\partial \xi}{\partial x}, \quad \dot{n} = \frac{\partial n}{\partial s} + \dot{x}\frac{\partial n}{\partial x}, \quad \dot{f} = \frac{\partial f}{\partial s} + \dot{x}\frac{\partial f}{\partial x},$$

then a constant of the motion for the system is given by

$$\Phi = (\xi\dot{x} - n)\frac{\partial h}{\partial \dot{x}} - \xi h + f. \tag{2-42}$$

The Lagrangian $h = \frac{1}{2}(\dot{x}^2 - K(s)\cdot x^2)$ gives the equation of motion (1); using this lagrangian in (2-41) and equating coefficients of powers of $\dot{x}$ to zero, we obtain a set of equations for ξ, n, f

$$\frac{\partial \xi}{\partial x} = 0, \tag{2-43-a}$$

$$\frac{\partial n}{\partial x} - \frac{1}{2}\frac{\partial \xi}{\partial s} = 0, \tag{2-43-b}$$

$$\frac{\partial n}{\partial s} - \frac{1}{2}K(s)\frac{\partial \xi}{\partial x} - \frac{\partial f}{\partial x} = 0, \tag{2-43-c}$$

$$-\frac{1}{2}\xi\dot{K}(s)x^2 - nK(s)x - \frac{1}{2}K(s)\frac{\partial \xi}{\partial s}x^2 - \frac{\partial f}{\partial s} = 0. \tag{2-43-d}$$

Eq.(2-43-a) implies that ξ is a function of s alone. From (2-43-b) and (2-43-c), we obtain the results

$$n(x, s) = \frac{1}{2}\dot{\xi}x + \psi(s), \tag{2-44}$$

$$f(x, s) = \frac{1}{4}\ddot{\xi}x^2 + \dot{\psi}(s)x + C(s), \tag{2-45}$$

where $\ddot{\psi}(s) + K(s)\cdot\psi(s) = 0$ and C(s) is an arbitrary function of s alone. Choosing C(s)=0, ψ(s)=0 and substituting (2-44),(2-45) into (2-43-d), we find

$$\dddot{\xi} + 4K(s)\dot{\xi} + 2\dot{K}(s)\xi = 0. \tag{2-46}$$

Eq.(2-46) has the integral

$$\frac{1}{2}\xi\ddot{\xi} - \frac{1}{4}\dot{\xi}^2 + K(s)\xi^2 = C, \tag{2-47}$$

where C is an integration constant. Replacing ξ with $\beta_c(s)$ in (2-44) , (2-45), and (2-47), we have

$$n(x,s) = \frac{1}{2}\dot{\beta}_c(s)x, \tag{2-48-a}$$

$$f(x,s) = \frac{1}{4}\ddot{\beta}_c(s)x^2, \tag{2-48-b}$$

$$\frac{1}{2}\beta_c\ddot{\beta}_c - \frac{1}{4}\dot{\beta}_c^2 + K(s)\beta_c^2 = C. \tag{2-48-c}$$

Further using (2-48-c), we obtain

$$f(x,s) = \frac{1}{2}\left[\frac{C}{\beta_c} + \frac{\dot{\beta}_c^2}{4\beta_c} - K(s)\beta_c\right]x^2. \tag{2-49}$$

Finally setting $\xi = \beta_c$ in (2-42) and substituting (2-48-a),(2-49) into (2-32), we write the invariant

$$\Phi = \frac{1}{2\beta_c}\left[Cx^2 + \left(\frac{\dot{\beta}_c}{2}x - \beta_c\dot{x}\right)^2\right]. \tag{2-50}$$

The arbitrariness by the presence of the constant C can be removed in the same way as in the previous subsection.

3. Cosiderations of Arbitrariness Appearing in Each Method

The unknown variables, integration constants, equations of condition, and arbitrariness finally left as their result can be summarized as follows:

unknown variables & integration constants	equations of condition	arbitrariness
2 – a $f(s)$ $IC(\Lambda^1_1)$ $\Lambda^1_1(s)$ $IC(\Lambda^1_2)$ $\Lambda^1_2(s)$ $IC(\Lambda^2_1)$ $\Lambda^2_1(s)$ $IC(\Lambda^2_2)$ $\Lambda^2_2(s)$ 9	$\Lambda^1_1\Lambda^2_2 - \Lambda^1_2\Lambda^2_1 = 1,$ $\frac{d\mathbf{\Lambda}}{ds} = M_a\mathbf{\Lambda},\ \mathbf{\Lambda} \equiv \begin{pmatrix}\Lambda^1_1\\ \Lambda^1_2\\ \Lambda^2_1\\ \Lambda^2_2\end{pmatrix}$ $\Lambda^1_2(s) = 0,$ $IC(\Lambda^1_2) = 0.$ 7	integration constants $c_1,\ c_2,$ coming from diff. eq. $\frac{1}{2}\beta\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + \beta^2 K(s) = 1.$ 2
2 – b $\lambda_1(s)$ $IC(\lambda_1)$ $\lambda_2(s)$ $IC(\lambda_2)$ $\lambda_3(s)$ $IC(\lambda_3)$ 6	$\frac{d\boldsymbol{\lambda}}{ds} = M_b\boldsymbol{\lambda},\ \boldsymbol{\lambda} \equiv \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}$ 3	integration constants $c,\ c_1,\ c_2$ coming from diff. eq. $\frac{1}{2}\beta\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + \beta^2 K(s) = c.$ 3
2 – c $IC_x(\xi)$ $\xi(s)$ $IC_s(\xi)$ $IC_x(n)$ $n(s)$ $IC_s(n)$ $f(s)$ $IC_x(f)$ $IC_s(f)$ 9	$\frac{\partial\xi}{\partial x} = 0,$ $\frac{\partial n}{\partial x} - \frac{1}{2}\frac{\partial\xi}{\partial s} = 0,$ $\frac{\partial n}{\partial s} - \frac{1}{2}K(s)\frac{\partial\xi}{\partial x} - \frac{\partial f}{\partial x} = 0,$ $-\frac{1}{2}\xi\dot{K}x^2 - nKx - \frac{K}{2}\frac{\partial\xi}{\partial s}x^2 - \frac{\partial f}{\partial s} = 0,$ $IC_s(n) = 0,$ $IC_s(f) = 0.$ 6	integration constants $c,\ c_1,\ c_2$ coming from diff. eq. $\frac{1}{2}\beta\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + \beta^2 K(s) = c.$ 3

where IC(A) is the integration constant appearing when the differential equation for A where A symbolizes the variables Λ_i^j, λ_i, ξ, n, and f is solved and $IC_s(A)$, $IC_x(A)$ are integration constants appearing when the partial differential equations for A are solved.

We already know that the arbitrariness coming from C can be removed. Therefore, adopting the periodic solution of the auxiliary equation

$$\frac{1}{2}\beta\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + K(s)\beta^2 = 1,$$

with K(s+L)=K(s), where L is the circumference of a ring, we can decide uniquely the dynamical invariant I

$$I = \frac{1}{2\beta(s)}\left[x^2 + \left(\frac{\dot{\beta}}{2}x - \beta\dot{x}\right)^2\right],$$

with $\beta(s) = \beta(s+L)$.

4. Towards an Invariant for Time-Dependent Nonlinear Betatron Oscillation

The existence of invariants for non-harmonic systems was recently demonstrated in several articles. Ray and Reid/7/ derived the invariant for the nonlinear equation of motion

$$\ddot{x} + K(s)x + \frac{g}{x^3} = 0, \tag{4-1}$$

using Noether's theorem. Kaushal and Korsh also presented the invariant of (4-1) with the help of dynamical algebra. The invariant derived by them is written as

$$I = \frac{1}{2}\left[2g\left(\frac{\rho(s)}{x}\right)^2 + c\left(\frac{x}{\rho(s)}\right)^2 + (\dot{\rho}(s)x - \rho(s)\dot{x})^2\right], \quad (4\text{-}2)$$

where $\rho(s)$ satisfies the auxiliary equation

$$\ddot{\rho} + K(s)\rho = \frac{C}{\rho^3}, \quad (4\text{-}3)$$

with the integration constant C.

Leach/9/ attempted to construct the invariant for time-dependent nonlinear harmonic oscillator of more interest to accelerator physicists,

$$\ddot{x} + K(s)x + K'(s)x^2 = 0, \quad (4\text{-}4)$$

using the so-called time-dependent nonlinear canonical transformation. However, such transformations must be considered as infinite series; if we use a generating function of the second type $F^2(x,P;s)$ so that

$$p = \frac{\partial F^2}{\partial x}, \quad X = \frac{\partial F^2}{\partial P}, \quad (4\text{-}5)$$

we write

$$F^2(x, P; s) = xP + \sum_{r=3}^{\infty}\sum_{j=0}^{r} A_j^r(s)\, x^j P^{r-j}. \quad (4\text{-}6)$$

Difficulties with convergence, therefore, are expected. These difficulties are seen in other methods. For instance, the dynamical algebra for the system (4-4) does not close for finite, but becomes an infinite set. Namely, the system of linear first-order equations (2-31), which determines the invariant I, is infinite in extent. This means that there are questions of

convergence and existence of solutions, which are related to the existence or non-existence of dynamical invariants.
Hence, it seems difficult to describe the invariant of (4-4) in the form of a finite polynomial.

Nevertheless, results of numerical studies for the system (4-4) /10/,/11/,/12/, seem to indicate the existence of the invariant, which is equivalent to the existence of invariant curves on the Poincaré map. For simplicity, we consider betatron oscillations receiving kicks due to a sextupole field located on the orbit. Such a system is described in the term of the Hamiltonian

$$H(x, p; s) = \frac{1}{2}\left(p^2 + K(s)x^2\right) + \frac{\epsilon}{3} x^3 \delta(s), \tag{4-7}$$

where ϵ is the parameter of the sextupole field strength. Using time-dependent linear canonical transformation and time scale change discussed in 2-a,

$$\begin{aligned} x &= \sqrt{\beta(s)}\,\eta, \\ p &= \frac{p_\eta + \sqrt{\beta(s)}\,(\dot{\sqrt{\beta(s)}})\,\eta}{\sqrt{\beta(s)}}, \\ \phi(s) &= \int^s \frac{ds'}{Q\beta(s')}. \end{aligned} \tag{4-8}$$

we can transform the Hamiltonian (4-7) into the form

$$H'(\eta, p_\eta; \phi) = \frac{Q}{2}\left(p_\eta^2 + \eta^2\right) + \epsilon\frac{Q\beta^{\frac{5}{2}}(s)}{3}\eta^3\delta(s), \tag{4-9}$$

with the betatron tune of Q. Further, setting $Q\beta(s)\delta(s)$ to $\delta(\phi)$, we have

$$H'(\eta, p_\eta; \phi) = \frac{Q}{2}\left(p_\eta^2 + \eta^2\right) + \frac{1}{3}\epsilon\beta^{\frac{3}{2}}(s)\eta^3\delta(\phi). \tag{4-10}$$

The canonical equations derived from the Hamiltonian(4-10) are equivalent to the recursion equations,

$$\begin{pmatrix} P_\eta \\ \eta \end{pmatrix}' = \begin{pmatrix} \cos 2\pi Q & -\sin 2\pi Q \\ \sin 2\pi Q & \cos 2\pi Q \end{pmatrix} \begin{pmatrix} P_\eta + \xi \eta^2 \\ \eta \end{pmatrix}, \tag{4-11}$$

where ξ is $-\epsilon \beta^{\frac{3}{2}}(0)$. If we replace (η , P_η) into (x,y) and assume Q=0.25, the recusion(4-11) become

$$y' = -x \ ,$$
$$x' = y + \xi x^2. \tag{4-12}$$

Furthermore, if the scale change

$$x = \xi^{-1} X,$$
$$y = \xi^{-1} Y, \tag{4-13}$$

is made, the recursion equations (4-12) become

$$Y' = -X \ ,$$
$$X' = Y + X^2. \tag{4-14}$$

Hence, the system (4-7) becomes free of machine parameters. This is desirable to study the universal properties for the system (4-7).

Fig.1 shows the Poincaré map obtained by the recusion equations (4-14). A sequence of mapping points around the origin seems to be located on the closed curves. However, it is apparently impossible to prove strictly this expectation by finite iteration of (4-14).

If there is an invariant for (4-7), then the invariant function $I(X,Y)$ must be invariant under the transformation (4-14):

$$I(Y+X^2,-X) = I(X,Y). \tag{4-15}$$

So far, we do not know such a function $I(X,Y)$. In addition, the question for stability of (4-7) is still left. It can be reduced to the question of the finiteness of the progression, which consists of the sequence of numbers derived from the one-dimensional recursion equation between three terms,

$$X_{n+1} + X_{n-1} = X_n^2 , \tag{4-16}$$

which is obtained by substituting the first equation into the second in Eq.(4-14). But we do not know the mathematical proof for the finiteness of such a progression. All these questions are still open.

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References and Footnotes

/1/ E.D.Courant and H.S.Snyder, " Theory of the Altenating Gradient Synchrotron ", Ann. Phys., Vol.3, 1 (1958).

/2/ H.R.Lewis and W.B.Riesenfeld, " An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field ", J. Math. Phys., Vol.10, No.8, 1458 (1969).

/3/ M.Lutzky, " Nother's Theorem and the Time-Dependent Harmonic Oscillator ", Phys. Lett., 68A, No.1, 3 (1978).

/4/ H.J.Korsh, " Dynamical Invariants and Time-Dependent Harmonic Systems ", Phys. Lett., 74A, No.5, 294 (1979).

/5/ W.E.Milne, " The Numerical Determination of characteristic Numbers ", Phys. Rev., Vol.35, p863 (1930).

/6/ C.J.Eliezer and A.Gray, " a Note on the Time-Dependent Harmonic Oscillator ", SIAM J. Appl. Math., Vol.30, No.3, 463 (1976)

/7/ J.R.Ray and J.L.Reid, " Noether's Theorem, Time-Dependent Invariants and Nonlinear Equations of Motion ", J. Math. Phys. Vol.20, No.10, 2054 (1979).

/8/ R.S.Kaushal and H.J.Korsch, " Dynamical Noether Invariant for Time-Dependent Nonlinear Systems ", J. Math. Phys., Vol.22 No.9, 1904 (1981).

/9/ P.G.Leach, " Towards an Invariant for the Time-Dependent Anharmonic Oscillator ", J. Math. Phys., Vol.20, No.1, 96 (1979).

/10/M.Mcmillan, " a Problem in the Stability of Peridic Systems ", in Topics in Modern Phsics---a Tribute to E.U.Condon, Colorado Asso. Univ. Press, (1971).

/11/E.A.Crosbie, T.K.Khoe, and, R.J.Lari, " the Effect of a Delta Function Sextupole Field on Phase Space Trajectories ", IEEE Trans. on Nucl. Scien., NS-18,1077(1971).

/12/L.J.Laselett, " Stochasticity ", IX Intern. Conf. on High Energy Accel., SLAC, Stanford, 394 (1974).

" Some Illustrations of Stochasticity ", in Topics in Nonlinear Dynamics a Tribute to Sir Edward Bullard, AIP Conf. Proc., No.46 221 (1978).

Fig. 1