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The Luminosity from the Collisions of Two Unequal and Not-Round Beams

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Introduction

In this paper, we derive a general formula for the luminosity from two colliding beams. The two beams are assumed to cross at some small angle with unequal cross sections. Also, both beams are taken to be not necessarily round. Gaussian distribution is assumed in any direction. The following cases: (a) bunched beam vs. unbunched beam, and (b) two unbunched beams, are given special consideration.

An application is made to the case where the cross section of the beams increases quadratically with the distance. The effect of the dispersion is also taken into account for one special case. The behavior of the luminosity is finally discussed. The main parameters in the discussion are: the interaction length, the crossing angle and the ellipticity of the interaction cross section.

General Analysis

The basic formula for the luminosity per crossing is[1]

$$L = N_+ N_- fF \tag{1}$$

where F is the overlapping integral

$$F = \int dt\,dx\,dy\,dz\; d\vec{v}_+ d\vec{v}_- |\vec{v}_+ - \vec{v}_-|\, g_+(\vec{r},\vec{v}_+,t)\, g_-(\vec{r},\vec{v}_-,t). \tag{2}$$

dt is the element of time, dxdydz is the element of volume, $\vec{r}$ and $\vec{v}$ are the position and velocity vectors of a particle. The signs + and - are used to distinguish between the two beams. The distribution functions are normalized to unity, i.e.

$$\int g_\pm(\vec{r},\vec{v}_\pm,t)\, d\vec{r}\, d\vec{v}_\pm = 1.$$

We have three cases:

(a) Both beams are unbunched. The total number of particles in each beam is $N_\pm$. The frequency of encounter f is the lowest revolution frequency.

(b) One beam is unbunched (+) and one bunched (-). N_+ is the total number of particles in the unbunched beam, N_- the number of particles in each bunch of the bunched beam. f is the frequency of encounters between the unbunched beam and the bunches of the other beam.

(c) Both beams are bunched. The number of particles in each bunch is $N_\pm$. f is the frequency of encounters between bunches.

Each beam (±) is moving in the direction $s_\pm$ (see Fig. 1). The angle between the two directions is $\alpha \ll 1$. All the particles are assumed to have the speed of light c. The transverse coordinates are $x_\pm$ and $z_\pm$. We chose $x = x_+ = x_-$ to be the direction perpendicular to the plane of crossing. It is also assumed that around the crossing point there is a space of total length ℓ free of any magnet.

Let us take gaussian distributions of the particles in the transverse planes, i.e. in $x_\pm$, $z_\pm$, $\dot{x}_\pm$ and $\dot{z}_\pm$. We have

$$g_\pm = \frac{\beta_{x\pm}\beta_{z\pm}\delta(\dot{s}_\pm \mp c)}{4\pi^2\sigma_{x\pm}^2\sigma_{z\pm}^2 c^2} H_\pm(s_\pm \mp ct) \quad .$$

$$\cdot \exp\left\{-\frac{1}{2}\left[\frac{x_\pm^2+(\beta_{x\pm}\frac{\dot{x}_\pm}{c})^2}{\sigma_{x\pm}^2} + \frac{z_\pm^2+(\beta_{x\pm}\frac{\dot{z}_\pm}{c})^2}{\sigma_{z\pm}^2}\right]\right\} \qquad (3)$$

where the $\sigma_\pm$'s are the standard deviations of the gaussian distributions. They are functions of $s_\pm$.

The relation between $x_\pm$, $z_\pm$, $s_\pm$ and the main reference frame coordinates x, z and y are

$$x_+ = x_- = x$$

$$s_\pm = y \cos\frac{\alpha}{2} \pm z \sin\frac{\alpha}{2}$$

$$z_\pm = z \cos\frac{\alpha}{2} \mp y \sin\frac{\alpha}{2} .$$

In the following, since we are assuming the crossing angle α is very small, we shall approximate $\cos\frac{\alpha}{2} \sim 1$ and $\sin\frac{\alpha}{2} \sim \frac{\alpha}{2}$.

$H_\pm$ is the longitudinal distribution function. We have two cases:

(a) Unbunched beam, for which we take a uniform distribution. $H_\pm$ is a constant equal to the inverse of the main orbit circumference $2\pi R_\pm$.

(b) Bunched beam. We take a gaussian distribution

$$H_\pm(s_\pm \mp ct) = \frac{\exp\left[-\frac{1}{2}\,\frac{(s_\pm \mp ct)^2}{\sigma_{\ell\pm}^2}\right]}{(2\pi)^{\frac{1}{2}}\,\sigma_{\ell\pm}}$$

where $\sigma_{\ell\pm}$ is, obviously, taken to be a constant.

At this point, the following approximation, valid in the limit of small crossing angle, is made

$$|v_+ - v_-| \simeq 2c .$$

This approximation allowed the integration over $\vec{v}_+$ and $\vec{v}_-$. We obtain

$$F = \frac{c}{2\pi^2}\int \frac{H_+(s_+ - ct)H_-(s_- + ct)}{\sigma_{x+}\sigma_{x-}\sigma_{z+}\sigma_{z-}}\,dtdxdydz\;\cdot$$

$$\cdot\;\exp\left\{-\frac{1}{2}\left[\frac{x_+^2}{\sigma_{x+}^2} + \frac{x_-^2}{\sigma_{x-}^2} + \frac{z_-^2}{\sigma_{z+}^2} + \frac{z_-^2}{\sigma_{z-}^2}\right]\right\}.$$

The integration over x is also easily done

$$F = \frac{2c}{(2\pi)^{\frac{3}{2}}} \int \frac{(\sigma_{x+}^2 + \sigma_{x-}^2)^{-\frac{1}{2}}}{\sigma_{z+}\,\sigma_{z-}} H_+(s_+-ct)H_-(s_-+ct)dtdydz \cdot$$

$$\cdot \exp\left\{-\frac{1}{2}\left[\frac{z_+^2}{\sigma_{z+}^2} + \frac{z_-^2}{\sigma_{z-}^2}\right]\right\}.$$

When we are performing the integration over the time t, we again have to consider the three cases.

(a) Two unbunched beams. The integration is rather trivial. We obtain

$$F = \frac{2}{(2\pi)^{\frac{5}{2}} R_o} \int \frac{(\sigma_{x+}^2 + \sigma_{x-}^2)^{-\frac{1}{2}}}{\sigma_{z+}\,\sigma_{z-}} e^{-\frac{1}{2}\left(\frac{z_+^2}{\sigma_{z+}^2} + \frac{z_-^2}{\sigma_{z-}^2}\right)} dydz \qquad (4)$$

where R_o is smallest between R_+ and R_-.

(b) One bunched beam and one unbunched beam. Also in this case, the integration is obvious. In the limit the bunch length $2\sigma_{\ell-}$ is very small compared to the circumference $2\pi R_+$, we have the same result shown by (4) with $R_o = R_+$.

(c) Two bunched beams. The integration can be done also in this case, though it is more complicated. We leave this case out of the present analysis since it has already received enough attention[1].

We shall concentrate on the first two cases (a) and (b), which, as we have seen, have the same overlapping integral which is given by (4). Observe that the dependence of σ_x and σ_z on y and z prevents further integration in general. Nevertheless, in the limit of very small angle crossing, we can assume they depend only on y. In this case the integration over z can be done as shown in Appendix A.

We have

$$F = \frac{1}{2\pi^2 R_o} \int_{-\ell/2}^{+\ell/2} \frac{dy\ e^{-\frac{\alpha^2 y^2}{2(\sigma_{z+}^2+\sigma_{z-}^2)}}}{(\sigma_{z+}^2+\sigma_{z-}^2)^{\frac{1}{2}}\ (\sigma_{x+}^2+\sigma_{x-}^2)^{\frac{1}{2}}} . \tag{5}$$

A Special Case

Let us apply (5) to the following special case. Introduce the beta-function $\beta(y)$ and the dispersion function $D(y)$ and let us write

$$\sigma^2 = \frac{\varepsilon}{\sigma}\ \beta(y) + [\delta \cdot D(y)]^2$$

which applies for either x or z and for either + or −. $\pi\varepsilon$ is the emittance which includes 95% of all the beam (see Appendix B), and δ is the standard deviation of the relative momentum ($\Delta p/p$) distribution, which is assumed to be gaussian.

Take the following expressions for $\beta(y)$ and $D(y)$

$$\beta = \beta^* + \frac{y^2}{\beta^*}$$

$$D = D'\ y$$

where β^* is the value at the crossing point and D' is a constant. Introduce the average beam current, $I = Nec/2\pi R$, and the integral

$$K(\xi,\eta,\omega) = \int_o^{\xi} \frac{e^{-\frac{\eta^2 u^2}{4(1+u^2)}}\ du}{(1+u^2)^{\frac{1}{2}}\ (1+\omega^2 u^2)^{\frac{1}{2}}} \tag{6}$$

then, finally, we have for the luminosity, from (1),

$$L = 2\frac{I_+ I_-}{\pi e^2 c} \frac{K(\xi, \eta, \omega)}{\sqrt{A_x B_z}} \tag{7}$$

where

$$\eta = \alpha\sqrt{\frac{2}{B_z}}$$

$$\omega = \sqrt{\frac{B_x A_z}{A_x B_z}}$$

$$\xi \quad \frac{\ell}{2}\sqrt{\frac{B_z}{A_z}}$$

and

$$A = \frac{\varepsilon_+\beta_+^* + \varepsilon_-\beta_-^*}{6}$$

$$B = \frac{\varepsilon_+/\beta_+^* + \varepsilon_-/\beta_-^*}{6} + (D_+'^2 + D_-'^2)\delta^2.$$

Observe that A is the quadratic sum of the beam sizes at the crossing point, i.e.

$$A = \sigma_+^{*2} + \sigma_-^{*2} \quad .$$

Also, in the case $D_+' = D_-' = 0$, B is the quadratic sum of the divergences ψ (see Appendix B) at the crossing point, i.e.

$$B = \psi_+^{*2} + \psi_-^{*2} .$$

In the case both beams are round but not necessarily with the same cross section, then $\omega = 1$. If, in addition, the two beams have also the same cross section and there is no dispersion ($D' = 0$), then

$$\eta = \alpha\frac{\beta^*}{\sigma^*} \qquad , \qquad \xi = \frac{\ell}{2\beta^*}$$

and

$$\sqrt{A_x B_z} = 2\sigma^{*2}/\beta^* = \varepsilon/3.$$

In this very special case, our expression for the luminosity reduces to the one obtained by E. Keil[2].

Observe also that with our notation the interaction length ℓ enters the expression of the luminosity (7) only at the upper limit of the normalized overlapping integral (6), and that the crossing angle α enters the same expression at the shoulder of the exponential inside the normalized overlapping integral.

We have already seen that, in the case of round and equal beams, the quantity $\sqrt{A_x B_z}$ at the demominator of the right hand side of (7) is the beam emittance and, henceforth, an invariant. This is closely true also for unequal and not-round beams. Thus the behavior of the luminosity is entirely described by the normalized overlapping integral (6) with the three normalized parameters ξ, η and ω. The first of these parameters, ξ, is the normalized distance from the crossing point; the second one, η, is the normalized crossing angle, and ω is a measure of the "ellipticity" of the interaction. We have seen that $\omega = 1$ for round beams; also $\omega > 1$ when the cross section of the interaction is wider on the plane of crossing than it is on the mid-plane, and vice versa.

The "saturated" luminosity is obtained by setting $\xi = \infty$. Let us call

$$K_\infty = K(\xi = \infty, \eta, \omega).$$

This parameter is shown in Fig. 2, versus η for some values of ω. The luminosity decreases monotonically with η and ω.

The ration K/K_∞ is shown in Figs. 3, 4 and 5, versus ξ and for some values of ω.

From the experimental apparatus point of view, an important parameter is the actual length where almost all the luminosity is concentrated. We define ξ_c to be the normalized full length around the crossing point including 99.0% of all the luminosity. ξ_c is shown in Fig. 6, versus η and again for some values of ω. The actual interaction length decreases monotonically with η and ω.

References

1. L. Smith, PEP Note-20, April 1972
2. E. Keil, Nucl. Instr. and Methods 113 (1973), pages 333-339

Appendix A

We assume that σ_x and σ_z depend only on y. We want to perform the integration over z at the right hand side of (4)

$$\int e^{-\frac{1}{2}\left(\frac{z_+^2}{\sigma_{z+}^2} + \frac{z_-^2}{\sigma_{z-}^2}\right)} dz =$$

$$= \int e^{-\frac{1}{2}\left\{\frac{(z-y\alpha/2)^2}{\sigma_{z+}^2} + \frac{(z+y\alpha/2)^2}{\sigma_{z-}^2}\right\}} dz$$

$$= \int e^{-\frac{1}{2}\left\{(Az+By)^2 + C^2y^2\right\}} dz$$

where

$$A = \left(\frac{1}{\sigma_{z+}^2} + \frac{1}{\sigma_{z-}^2}\right)^{\frac{1}{2}}$$

$$B = \frac{\alpha}{2A}\left(\frac{1}{\sigma_{z-}^2} - \frac{1}{\sigma_{z+}^2}\right)$$

$$C = \frac{\alpha}{(\sigma_{z+}^2 + \sigma_{z-}^2)^{\frac{1}{2}}}$$

as it is easy to verify.

The same integral then becomes

$$\frac{\sqrt{2\pi}}{A} e^{-\frac{1}{2} C^2y^2}$$

which leads to (5).

Appendix B

Consider a particle which has an upright ellipse described by the equation

$$\beta x'^2 + \frac{x^2}{\beta} = \varepsilon$$

as trajectory in the (x,x')-phase plane. ε is a constant, actually the invariant action of the trajectory. The trajectory is closed and the area of the ellipse is $\pi\varepsilon$.

Consider a beam which has gaussian distribution in either direction x and x', centered to $x = x' = 0$. The distribution in the invariant ε then must be

$$f(\varepsilon) = \frac{e^{-\varepsilon/\varepsilon_o}}{\varepsilon_o}$$

where $\int f(\varepsilon)d\varepsilon = 1$ and ε_o is a measure of the width of the distribution.

We want to define the emittance $\pi\varepsilon_{max}$ which includes only the fraction $1 - p$, $(p < 1)$, of the beam. We have

$$\frac{1}{\varepsilon_o}\int_o^{\varepsilon_{max}} e^{-\varepsilon/\varepsilon_o}\, d\varepsilon = 1 - p$$

from which

$$\varepsilon_{max} = -\varepsilon_o \log p.$$

The distribution in x and x' is

$$f(x,x') = \frac{|\log p|}{\varepsilon_{max}}\, e^{-\frac{|\log p|}{\varepsilon_{max}}\left(\frac{x^2}{\beta} + \beta x'^2\right)}.$$

Perform the integration over x' to get distribution over x only, and vice versa

$$f(x) = \frac{|\log p|}{\varepsilon_{max}} e^{-\frac{|\log p|}{\varepsilon_{max}} \frac{x^2}{\beta}} \left\{ \int e^{-\frac{|\log p|}{\varepsilon_{max}} \beta x'^2} dx' \right\}$$

$$f(x') = \frac{|\log p|}{\varepsilon_{max}} e^{-\frac{|\log p|}{\varepsilon_{max}} \beta x'^2} \left\{ \int e^{-\frac{|\log p|}{\varepsilon_{max}} \frac{x^2}{\beta}} dx \right\}.$$

From the above distributions we derive the relationships between the standard deviation of the distribution in x, σ, the standard deviation of the distribution in x', ψ, the fraction of excluded beam, p, and the corresponding beam emittance $\pi\ \varepsilon_{max}$

$$\pi\ \varepsilon_{max} = 2\frac{\sigma^2}{\beta}\ |\log p|\ \pi$$

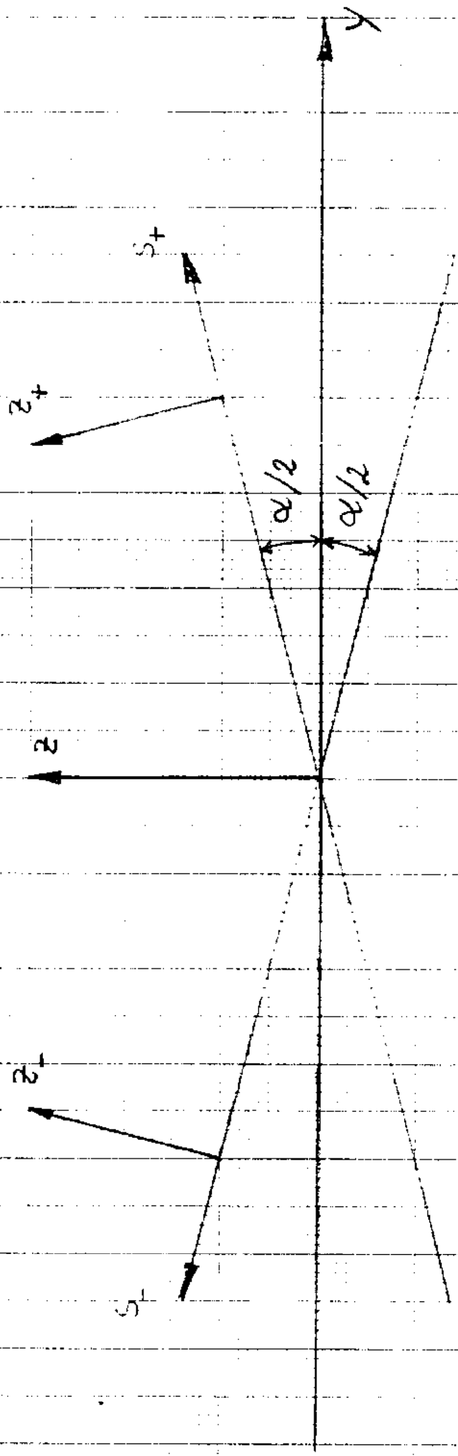
$$= 2\psi^2\beta\ |\log p|\ \pi\ .$$

The relation between σ and ψ is

$$\sigma = \beta\psi\ .$$

For instance, the emittance which includes 95% of the beam ($p = 0.05$) is

$$\pi\ \varepsilon_{max} = 6\frac{\pi^2}{\beta}\ \pi\ .$$



(The x-direction is perpendicular to the plane)

FIG. 1

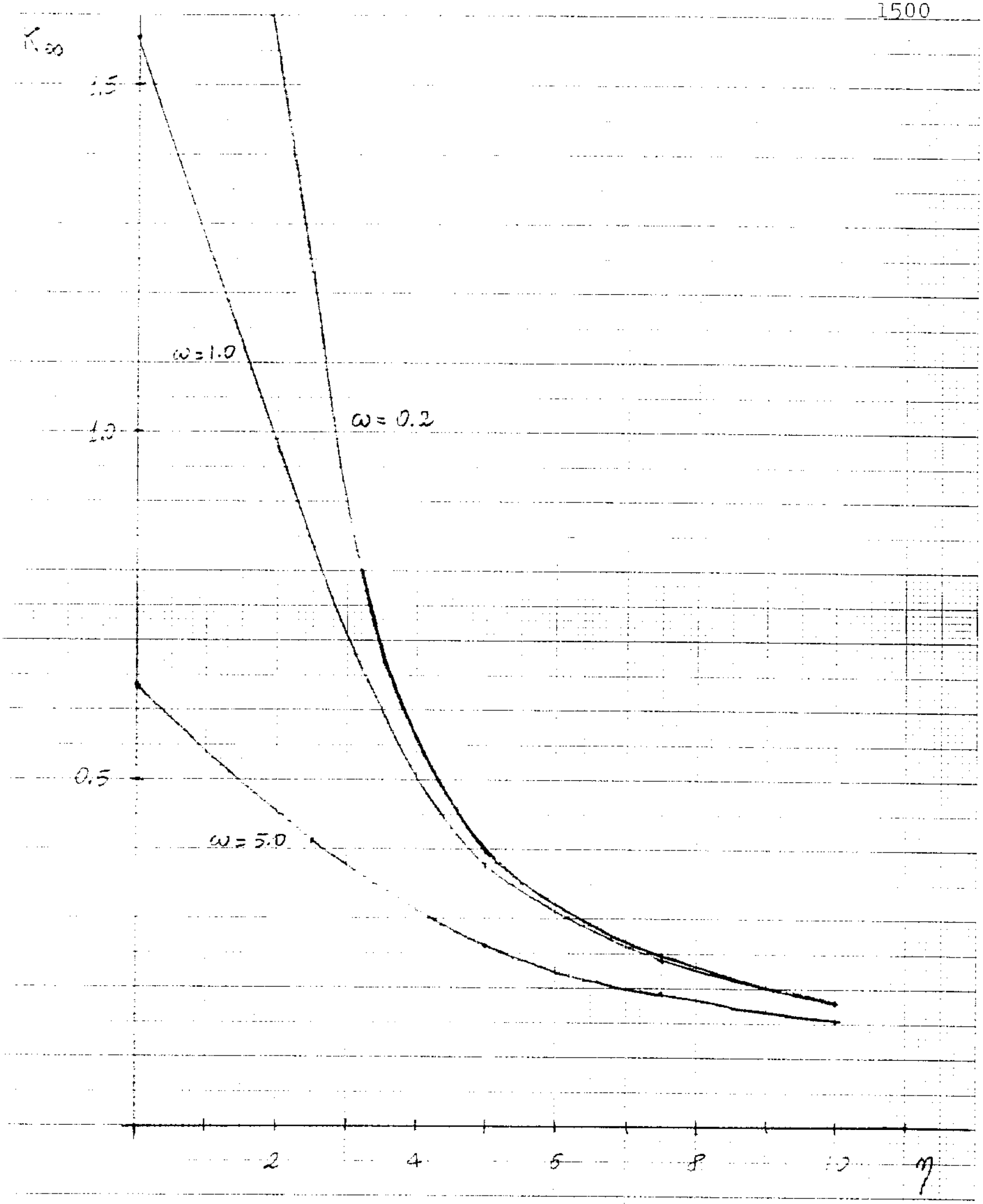


FIG. 2

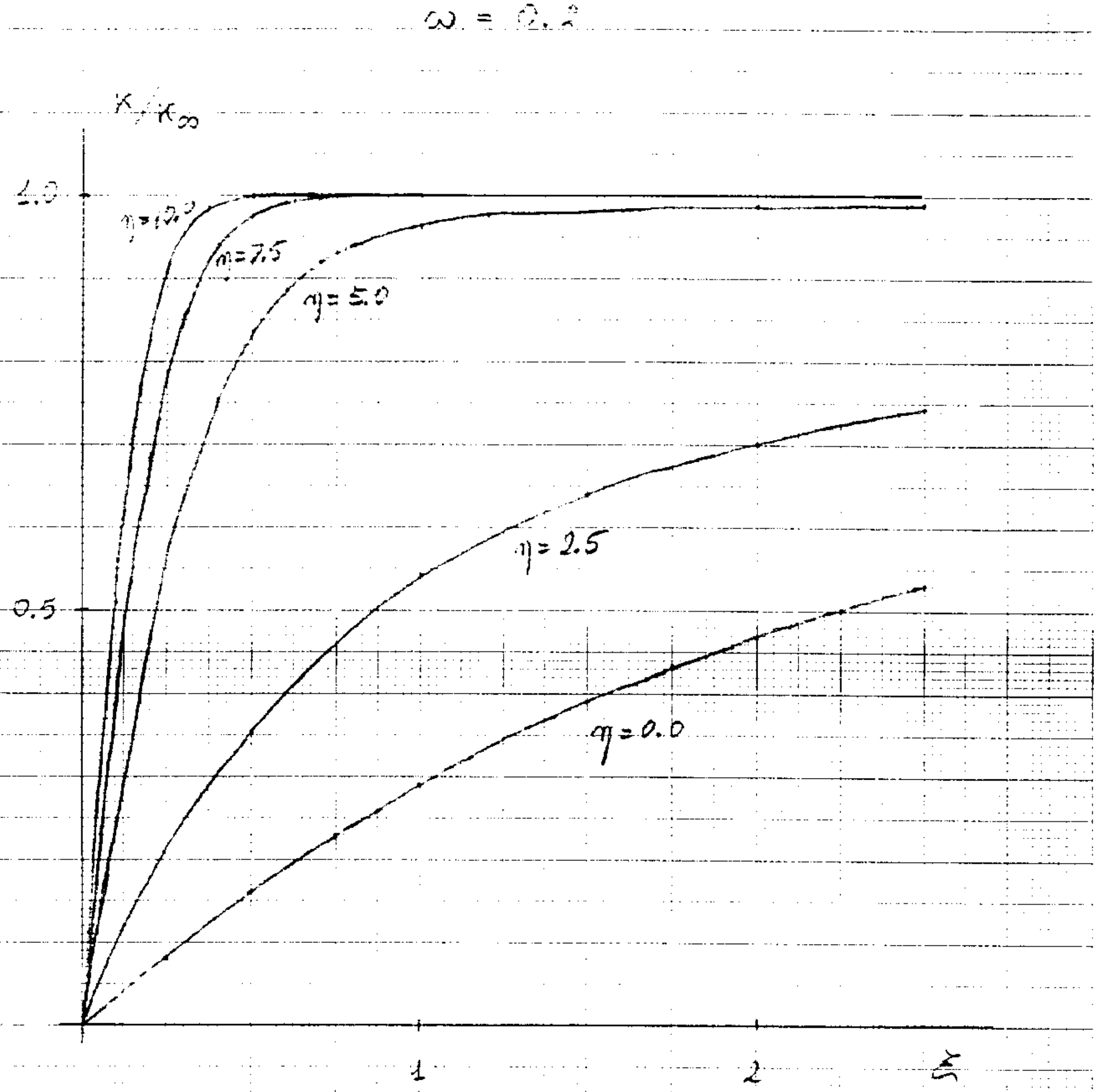


FIG. 3

$\omega = 1.0$

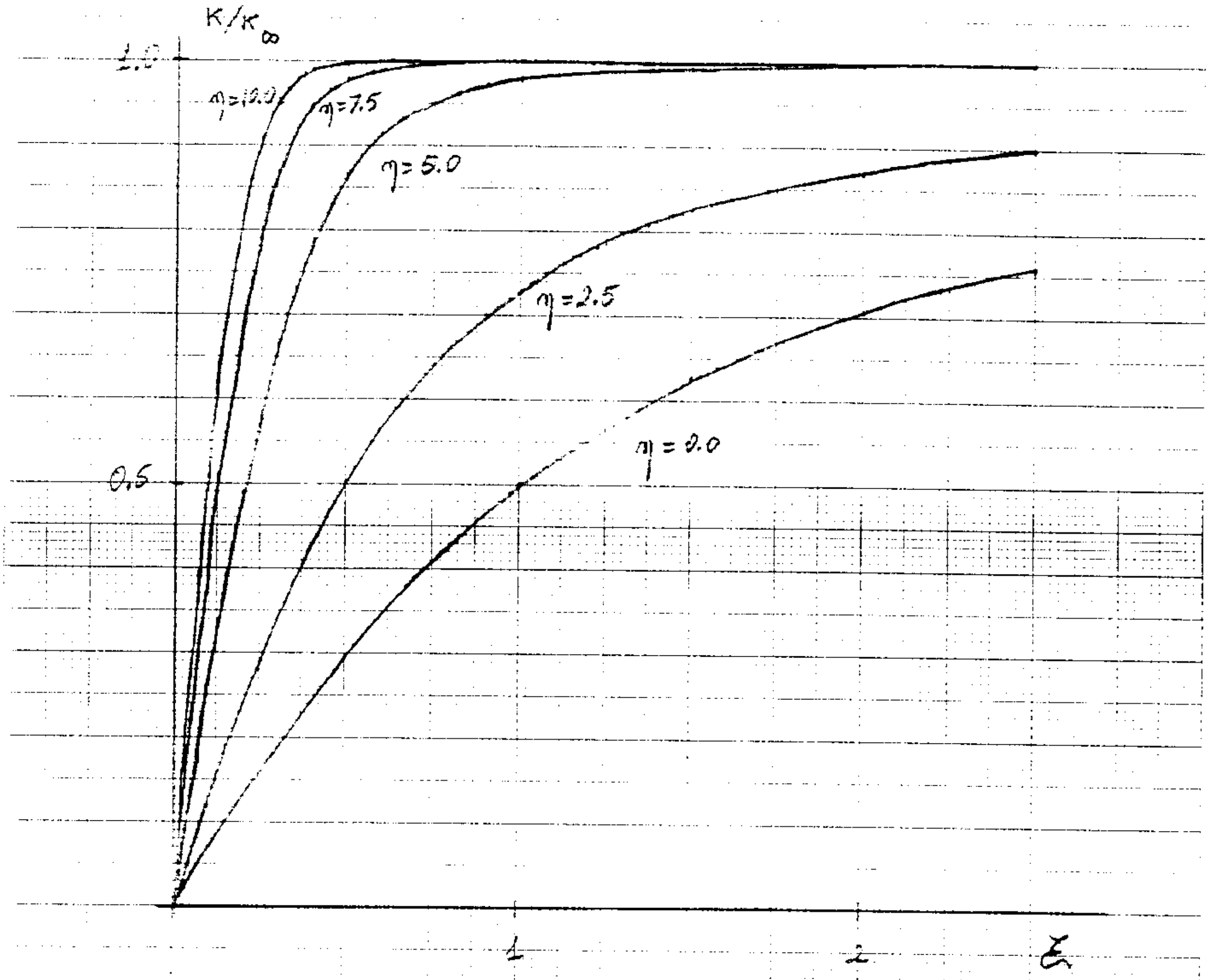


FIG. 4

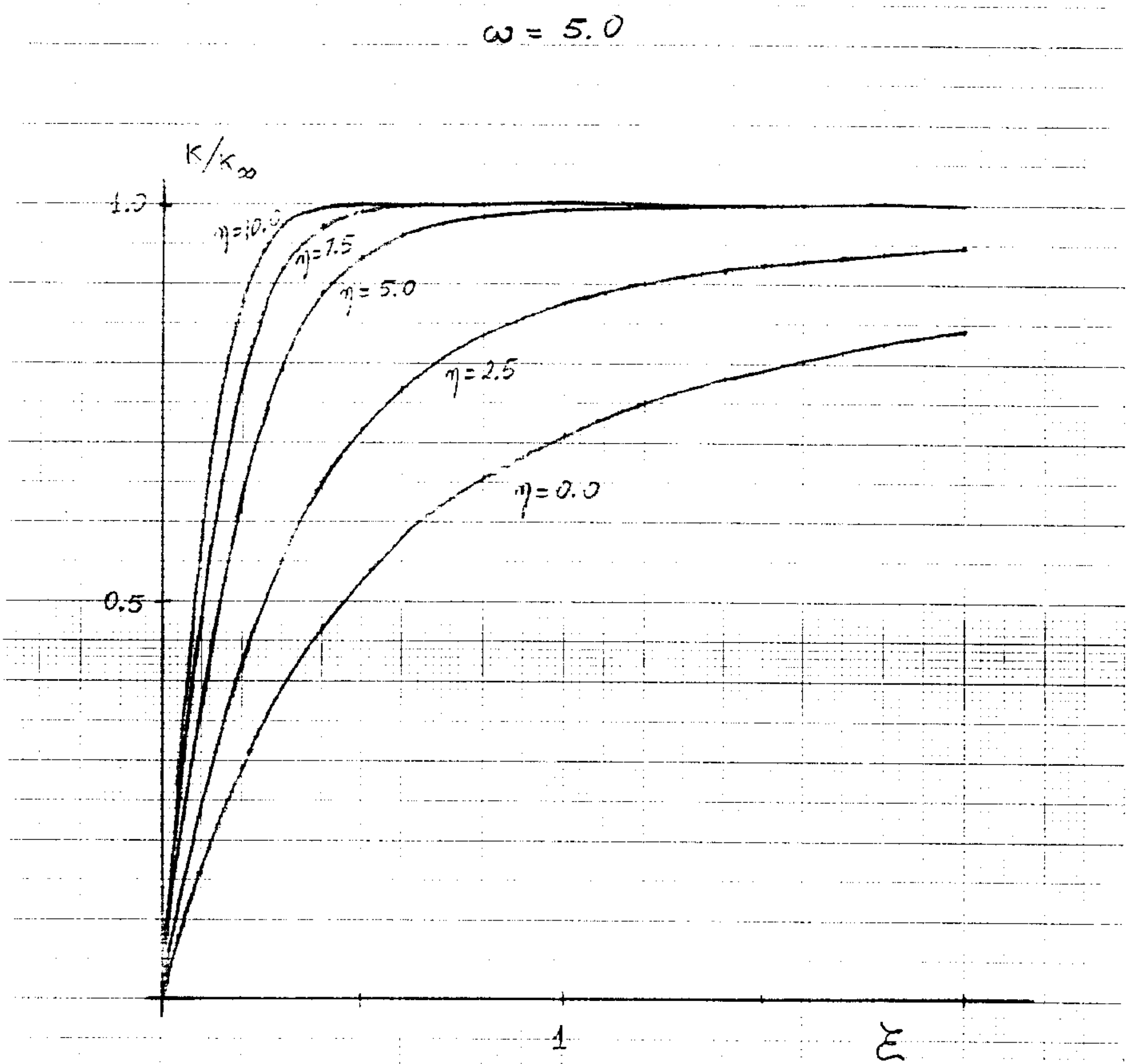


FIG. 5

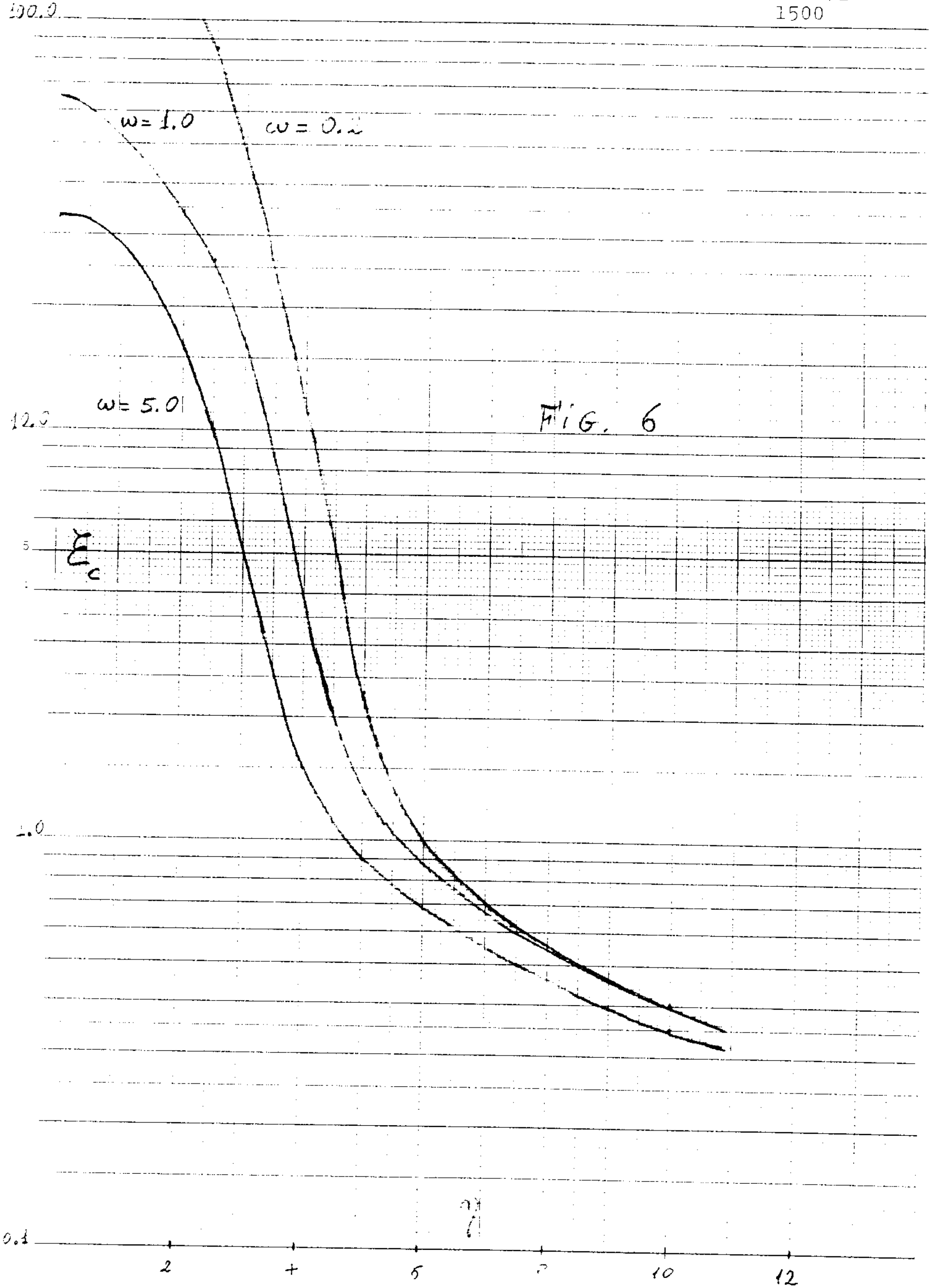


FIG. 6