national accelerator laboratory

FN-267
0100
November 1974

"DEPOLARIZATION" OF A POLARIZED PROTON BEAM IN A CIRCULAR ACCELERATOR

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This effect has long been studied by many authors. Hence, we will only give a general review of the theory and offer a few handy formulas for quick estimates of the effect.

In the rest-frame (rotating) of the proton the classical spin ($\vec{s}$) equation is simply

$$\dot{\vec{s}} = \frac{ge}{2mc}\,\vec{s} \times \vec{B} \qquad (1)$$

where $\vec{B}$ is the magnetic field and $g = 5.58$ is the gyromagnetic ratio of the proton.

I. Several qualitative observations are interesting:

A. If the proton travels entirely in the midplane (no closed-orbit error and no vertical betatron oscillation) $\vec{B}$ is purely vertical. The spin $\vec{s}$ will simply precess about the vertical $\vec{B}$ and the vertical polarization will be conserved.

B. "Depolarization" is caused by a precession about a horizontal field component $\vec{B}_h$ which exists for protons with non-zero vertical oscillations or with nonvanishing closed-orbit errors. We shall consider here only "intrinsic" effects due to vertical oscillations. In this case, $\vec{B}_h$ is oscillatory with frequency components

$$\omega = (k \pm \nu + \gamma)\omega_o \qquad (2)$$

where k = integer, ν = vertical tune, ω_o = revolution frequency in the rest-frame. The term $k\omega_o$ arises from the magnet lattice structure, $\pm\nu\omega_o$ comes from the vertical oscillation, and $\gamma\omega_o$ corresponds to the Thomas precession term in the rotating rest-frame. $\vec{B}_h$ may be analyzed into oppositely rotating components. Only the component which rotates in the same sense and at approximately the same angular speed as the vertical prescession of $\vec{s}$ can produce a large secular horizontal precession or "depolarization".

C. The frequency of precession about the vertical field component B_v is

$$\Omega \equiv \frac{geB_v}{2mc} = \frac{g\gamma}{2}\frac{eB_v}{m\gamma c} = \frac{g\gamma}{2}\omega_o . \tag{3}$$

As the proton is accelerated Ω sweeps across component frequencies of $\vec{B}_h$ and "depolarization" occurs. These resonances occur at $\Omega = \omega$ or

$$\frac{g-2}{2}\gamma = 1.79\gamma = k \pm \nu \qquad (k = \text{integer}). \tag{4}$$

They must be crossed rapidly to avoid sizable "depolarization".

D. Eq. (1) is identical to a rigid rotation with angular velocity $\frac{ge}{2mc}\vec{B}$. Thus, for a unique $\vec{B}$ the distribution of $\vec{s}$ is retained throughout the motion. If, before a resonance, the spins of all protons with the same amplitude and phase of vertical oscillation, hence same amplitude and phase of $\vec{B}_h$(hence, a unique $\vec{B}$) are distributed over a vertical cone C (Fig. 1) with semi-vertex angle θ_o (initial polarization $P_o = \cos\theta_o$), after crossing the resonance the spins must be distributed over a similar cone, say, C' which may be inclined from the vertical. Of course, cone C' must be presessing about the vertical (z) direction, thereby sweeping out a ring sector between two vertical cones with semi-vertex angles $\Delta\theta \pm \theta_o$. Because of the cylindrical symmetry $\Delta\theta$ must be independent of the phase of

$\vec{B}_h$ which gives only the zero-point of the azimuthal angle of $\vec{B}_h$ in the xy-plane. (x along direction of motion, y in radially outward normal direction). Hence for all phases but a unique amplitude of the vertical betatron oscillation the final polarization P_f is

$$P_f = \cos\theta_o \cos\Delta\theta = P_o \cos\Delta\theta \tag{5}$$

where $\Delta\theta$ depends only on the vertical amplitude. Therefore, for a given amplitude P_f will have a definite value between P_o and $-P_o$. The only "smearing" or true depolarization is caused by the spread of the vertical amplitudes in the beam.

E. Suppose we start with a beam having a spread of vertical amplitude but the same polarization P_o. After crossing a strong resonance, protons with zero amplitude will still have polarization P_o. Protons with some definite amplitude A_o will have $\Delta\theta = \frac{\pi}{2}$ and zero polarization; protons with amplitude > $2A_o$ will have $\Delta\theta \sim \pi$ and polarization $\sim -P_o$. If, now, we slow-extract the beam by a scheme which extracts beam with various vertical amplitudes sequentially in time (such as a vertical resonant extraction system) protons with different polarizations between P_o and $-P_o$ will be sorted out in time during the spill. One has to make sure, however, that the polarization versus time distribution of the slow-spilled beam is not convoluted by the momentum spread in the beam or the extraction mechanism.

II. Solution of Equation (1)

A. To calculate $\Delta\theta$ across a given resonance, let us write (see Fig. 2 for the coordinates used)

$$\begin{cases} \vec{s} = \left[\hat{i}\sin\left(\int\Omega dt+\phi\right) + \hat{j}\cos\left(\int\Omega dt+\phi\right)\right]\sin\theta + k\cos\theta \\ \vec{B} = B_h\left(\hat{i}\cos\int\omega dt - \hat{j}\sin\int\omega dt\right) + B_v\hat{k} \end{cases} \tag{6}$$

where we have taken only the relevant rotating component of $\vec{B}_h$. Substituting Eq. (6) in Eq. (1) we get for the three components

$$\begin{cases} \dot{\theta}\sin\left(\int\Omega dt+\phi\right) + \dot{\phi}\tan\theta\cos\left(\int\Omega dt+\phi\right) = \Omega r\sin\int\omega dt \\ \dot{\theta}\cos\left(\int\Omega dt+\phi\right) - \dot{\phi}\tan\theta\sin\left(\int\Omega dt+\phi\right) = \Omega r\cos\int\omega dt \\ \dot{\theta} = \Omega r\cos\left[\int(\Omega-\omega)dt+\phi\right] \end{cases} \tag{7}$$

where $r \equiv B_h/B_v$ measures the "strength" of the resonance. Eqs. (7) are equivalent to the two equations

$$\begin{cases} \dot{\theta} = \Omega r\cos\left[\int(\Omega-\omega)dt+\phi\right] \\ \dot{\phi}\tan\theta = -\,\Omega r\sin\left[\int(\Omega-\omega)dt+\phi\right] \end{cases} \tag{8}$$

or the single complex equation

$$\frac{d}{dt}\left(\sin\theta\, e^{-i\phi}\right) = \Omega r\cos\theta\, e^{i\int(\Omega-\omega)dt}. \tag{9}$$

Note that $\sin\theta\, e^{-i\phi}$ is the projection of the spin on the equitorial plane referred to coordinates rotating with the vertical precession. When $r = 0$ both θ and ϕ are constants.

B. For a simple case we assume that in the neighborhood of the resonance

$$\begin{cases} \Omega-\omega = a^2 t \qquad \text{where} \\ a^2 \equiv \frac{d}{dt}(\Omega-\omega) = \text{constant which measures the "speed" of crossing,} \end{cases} \tag{10}$$

and define

$$\tau \equiv at, \qquad K \equiv \frac{\Omega r}{a}. \tag{11}$$

Then Eq. (9) becomes

$$\frac{d}{d\tau}(\sin\theta\ e^{-i\phi}) = K\cos\theta\ e^{i\frac{\tau^2}{2}}. \tag{12}$$

This equation is difficult to integrate in general, but the quantity of interest, $\cos\Delta\theta$ ($\theta=0$ at $\tau=-\infty$, and $\theta=\Delta\theta$ at $\tau=+\infty$), can be obtained in closed form:

$$\cos\frac{\Delta\theta}{2} = e^{-\frac{\pi}{4}K^2} \quad \text{or} \quad \cos\Delta\theta = 2e^{-\frac{\pi}{2}K^2}-1. \tag{13}$$

For $K = 0.664$ $\cos\Delta\theta = 0$ and the beam is totally "depolarized". For $K > 0.664$ $\cos\Delta\theta$ is negative and the polarization is flipped. To get $\cos\Delta\theta < -0.9$ we need $K > 1.381$.

C. We can estimate the order-of-magnitude of r for an azimuthally uniform gradient machine ($k=0$). In this case the horizontal field is purely radial; hence transverse to the motion, and identical in Lorentz transformation and in the additional Thomas precession term to the transverse vertical field. In the laboratory frame, in a quarter of a vertical oscillation the vertical angle turned is $\frac{A\nu}{R}$ where A = vertical amplitude and R = ring radius, and the horizontal turning angle around the ring is $\frac{\pi}{2\nu}$. Therefore, remembering that B_h is the amplitude of one rotating component, we have

$$r = \frac{B_h}{B_v} = \frac{1}{2}\left(\frac{\pi}{2}\,\frac{\text{vertical angle}}{\text{horizontal angle}}\right) = \frac{A\nu^2}{2R}. \tag{14}$$

In general we can write

$$r = \frac{A\nu^2}{2R}\,G_k \tag{15}$$

where G_k is a geometrical factor depending on the ring-magnet lattice. The quantity ν^2/R, hence the resonance strength r, is generally an order of magnitude larger for strong focusing machines than for weak focusing machines.

D. The quantum mechanical spin equation in the rest-frame

$$i\hbar\dot{\Psi} = - \vec{\mu}\cdot\vec{B}\Psi = - \frac{ge}{2mc}\frac{\hbar}{2}\vec{\sigma}\cdot\vec{B}\Psi \tag{16}$$

where $\vec{\mu}$ = magnetic moment of proton, $\vec{\sigma}$ = 2x2 spin matrices, and Ψ = 2-component spin state vector; has the solution

$$\Psi = \begin{pmatrix} e^{\frac{i}{2}(\int\Omega dt+\phi)}\cos\frac{\theta}{2} \\ ie^{-\frac{i}{2}(\int\Omega dt+\phi)}\sin\frac{\theta}{2} \end{pmatrix} e^{i\frac{\chi}{2}} \equiv \begin{pmatrix} \text{up} \\ \text{down} \end{pmatrix} \tag{17}$$

where

$$\dot{\chi}\cos\theta + \dot{\phi} = 0 \tag{18}$$

and θ and ϕ are given as before by Eq. (8) or Eq. (9). The polarization defined by

$$P \equiv \frac{|\text{up}|^2 - |\text{down}|^2}{|\text{up}|^2 + |\text{down}|^2} = \cos\theta \tag{19}$$

is identical to that given before. It is interesting to note that the irrelevant phase χ never appeared in the classical treatment.

III. ν-Jump Scheme

A. To avoid excessive "depolarization" one can use ν-jump quadrupoles to cross the resonance rapidly as was done successfully on the Argonne ZGS. Crossing a resonance more rapidly one increases a (defined by Eq. (10)), hence reduces K (defined by Eq. (11)). From Eqs. (2) and (3) we get, with ν-jump

$$K = \frac{\Omega r}{\sqrt{\frac{d}{dt}(\Omega-\omega)}} = \frac{\frac{g}{2}\,\omega_o\,\gamma r}{\sqrt{\omega_o\,\frac{d}{dt}(\frac{g-2}{2}\gamma\mp\nu-k)}}$$

$$= \frac{\frac{g}{2}\,\omega_o\,\gamma r}{\sqrt{\frac{\omega_o^{\,2}}{2\pi}(\frac{g-2}{2}\,\frac{d\gamma}{dn}\mp\frac{d\nu}{dn})}} = \frac{2.79\,\sqrt{2\pi}\,\gamma r}{\sqrt{1.79\frac{d\gamma}{dn}\mp\frac{d\nu}{dn}}} \tag{20}$$

where $\frac{d}{dn}$ denotes change per revolution. Eq. (20) gives the required direction ($\mp\frac{d\nu}{dn} > 0$) and speed ($\mp\frac{d\nu}{dn}$ should be large enough to reduce K, hence $\Delta\theta$, to an acceptably small value.) of the ν-jump.

B. We can estimate the range $\Delta\nu$ of the ν-jump required. From $\Omega-\omega = (1.79\dot{\gamma}\mp\dot{\nu})\omega_o t$ we get

$$\frac{\tau^2}{2} \equiv \int(\Omega-\omega)dt = (1.79\dot{\gamma}\mp\dot{\nu})\omega_o\frac{t^2}{2} = (1.79\dot{\gamma}\mp\dot{\nu})\omega_o\frac{(\delta\nu)^2}{2\dot{\nu}^2}$$

or

$$\delta\nu = \left[2\pi(1.79\frac{d\gamma}{dn}\mp\frac{d\nu}{dn})\right]^{-1/2}\frac{d\nu}{dn}|\tau|\,. \tag{21}$$

The major contribution to $\Delta\theta$ comes from within $-2<\tau<2$. Thus, a total ν-jump range of

$$\Delta\nu = 2(\delta\nu \text{ with } |\tau| = 2) = 4\frac{d\nu}{dn}\left[2\pi(1.79\frac{d\gamma}{dn}\mp\frac{d\nu}{dn})\right]^{-1/2} \tag{22}$$

is adequate.

C. Now we analyze the strongest 8-ν (k=8) resonance of the ZGS.

("Acceleration of Polarized Protons to 8.5 GeV/c", T. Khoe et al, ANL Report, May 1974.) Without ν-jump ($\frac{d\nu}{dn} = 0$) an initial polarization of $P_o = 65\%$ is reduced to $P_f = 20\%$. This gives

$$\cos\Delta\theta = \frac{20}{65} = 0.308 \quad \text{and} \quad K = 0.520 \text{ (from Eq. (13))}.$$

Eqs. (20) and (15) then give

$$r = 0.783\text{x}10^{-4} \quad \text{and} \quad A = 0.776 \text{ cm}$$

where we used the parameters $\nu = 0.8$, $\gamma = 4.016$, $R = 27.4$ m, $\frac{d\gamma}{dn} = 10^{-5}$, and $G_8 = 0.864$* given in the reference mentioned above. This A value corresponds to an average vertical amplitude of ~ 1.0 inch at injection (50 MeV) as expected.

With ν-jump and for the fast crossing $\frac{d\nu}{dn} = 0.00119$ ($\Delta\nu = 0.04$ in 20 μsec or 33.7 revolutions). This gives

$$K = 0.0633 \quad \text{and} \quad \cos\Delta\theta = 0.987$$

showing essentially no "depolarization". The required range of the ν-jump is given by Eq. (22) to be

$$\Delta\nu = 0.055.$$

The applied range of $\Delta\nu = 0.04$ is a little too small but not unreasonably so.

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*Two errors in the computation of G_8 in the reference were corrected.

(1) The Thomas precession term $-\frac{2}{g}(\gamma-1)\vec{B}_\perp$ where $\vec{B}_\perp$ is the transverse field component in the laboratory frame, must be added to give $\vec{B}$ in the rotating rest-frame.

(2) Across a straight section $B_v=0$ and there is no vertical precession. The contribution to B_h from the horizontal field components at the ends of a straight section must be treated specially.

When the ν-jump is timed early so that the resonance is crossed on the negative $\frac{d\nu}{dn}$ side, the crossing is slower and the "depolarization" is larger than without ν-jump, resulting in a maximum negative polarization of $P_f = -25\%$. This gives

$$\cos\Delta\theta = -\frac{25}{65} = -0.385,$$

$$K = 0.866,$$

and

$$\frac{d\nu}{dn} = -1.14\text{x}10^{-5}.$$

This is the maximum negative slope compared to an average slope of $\frac{d\nu}{dn} = -0.8\text{x}10^{-5}$ ($\Delta\nu = -0.04$ in ~ 3 msec or ~ 5000 revolutions). If the fall-time of the ν-jump quadrupoles can be shortened to give $\frac{d\nu}{dn}$ close to $-1.79\frac{d\gamma}{dn} = -1.79\text{x}10^{-5}$ one can obtain a total flip to give $P_f \cong -P_o = -65\%$.

Here, we studied only the "intrinsic" resonances due to vertical oscillations. The "error" resonances due to closed-orbit distortions are generally unimportant for weak focusing machines, but can be rather destructive for strong focusing machines. They must be eliminated (making $B_h = 0$, hence $r = K = 0$) by using correction dipoles.

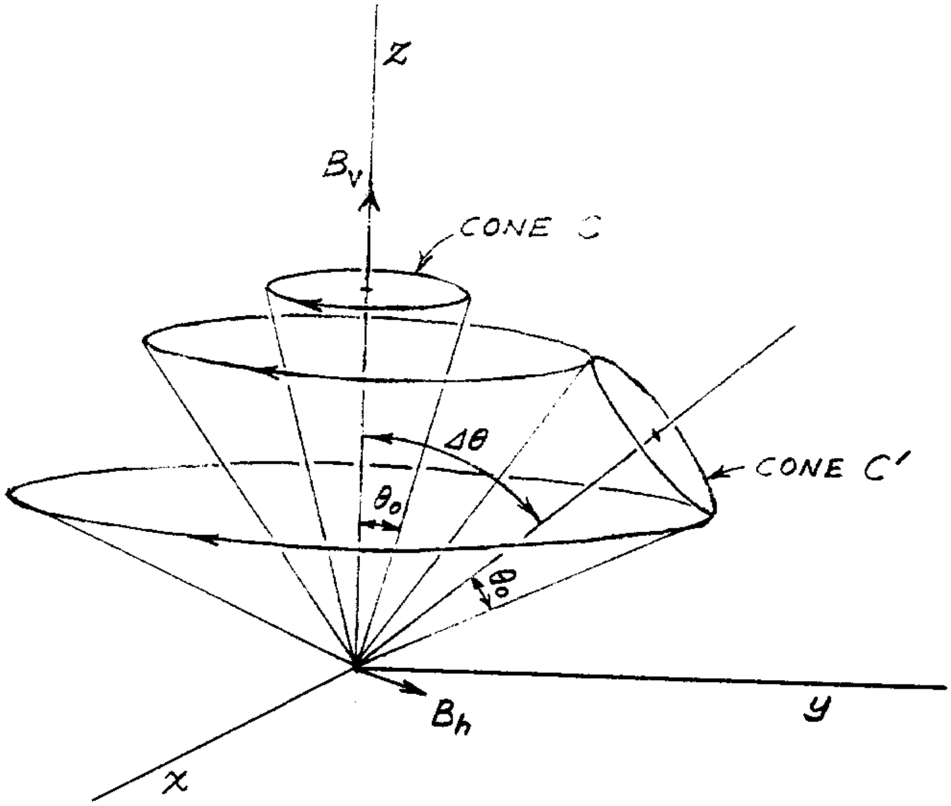

FIGURE 1 "DEPOLARIZATION" PROCESS

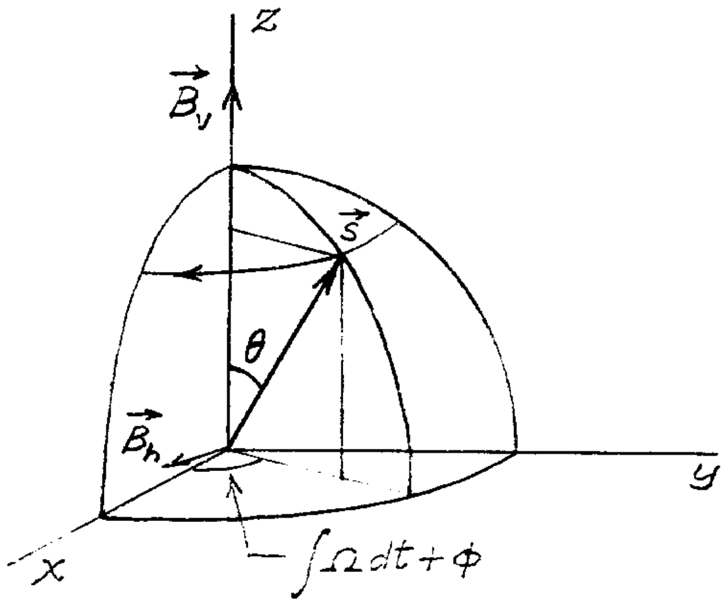

FIGURE 2 COORDINATES ADOPTED