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THE HEAD-TAIL EFFECT ENHANCED BY A FAST OSCILLATING OR FAST DECAYING WAKE FIELD

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At the 1973 Particle Accelerator Conference held in San Francisco, the following problem was raised.

The head-tail effect was discovered in the following machines: ADONE, ACO, CEA, SPEAR and NAL Booster. In all these machines only the zero-th mode of the instability, which involves the oscillations of the centre of mass of the beam, was noticed. The instability is usually compensated by cancelling the chromaticity (variation of the tune with momentum) of the machine. In some cases the sign of the chromaticity was also changed because by so doing it was thought the higher modes (which do not involve the motion of the center of mass of the beam) would become unstable. Nevertheless, the higher modes have never been observed. The question was: why?

It was commented that, according to Pellegrini-Sands theory[1,2], <u>if the zeroth mode is stable the higher modes should be unstable and viceversa</u>.

We now believe that this statement is wrong. We believe also that it might be correct for small betatron phase advance across

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the beam-bunch and for smooth, slow decaying wake field. Our concern was mostly about a wake field that rings or decays rather fast within a bunch. Thus, still adopting Pellegrini-Sands theory, we calculated the growth (or damping) rate of the head-tail instability: (1) in the presence of a high-frequency resonator, and (2) for a decaying wake field. Our calculations showed, indeed, that higher modes and zeroth mode can all be be stable (or unstable) at the same time.

Analytic Calculations

1. Let us make use of the notation used in ref. (1). The complex frequency shift due to a wake field $\rho(t)$ for the mode number m is then given by

$$\Delta\omega_m = -\frac{N}{2\pi}\int_{-\pi}^{+\pi} \overline{W}\ (-\psi)\, e^{im\psi}\, d\psi \tag{1}$$

where

$$\overline{W} = \frac{1}{2\omega_o T_s}\int_o^{T_s} e^{-i\omega_o \frac{\xi}{\alpha}(\tau'-\tau)}\ \rho\ (\tau'-\tau)\ dt \tag{2}$$

and

$$\tau'-\tau = 2A\sin\left|\frac{\psi}{2}\right|\cos\omega_s t \tag{3}$$

for a bunch of particles with the same amplitude of synchrotron oscillation.

2. We shall assume that the wake field is originated by a cavity which resonates on a single mode. By making use of the notation used in Appendix B of ref. (2) we have,

$$\rho(t) = \frac{e^2 v Z_o}{d^2 L m_o \gamma} \frac{\omega_r^{\ 2}}{\omega_r^{\ 2} + \Gamma^2} e^{-\Gamma t} \left(\sin \omega_r t + - \frac{1}{4Q^2} \cos \omega_r t \right) H(t)$$

where m_o is the mass of the particle at rest and $m_o\gamma$ the mass of the particle in motion. H (t) is the stepwise function.

Let us write $\rho(t)$ in a more condensed form

$$\rho(t) = CBe^{-\Gamma t} \cos (\omega_r t + g) \; H(t) \tag{4}$$

where

$$C = \frac{e^2 v Z_o}{d^2 L m_o \gamma}$$

$$B = \frac{\omega_r^{\ 2}}{\omega_r^{\ 2} + \Gamma^2} \sqrt{1 + \frac{1}{16Q^4}}$$

$$g = \text{arctg}\,(4Q^2)$$

3. Observing that formally it is

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}$$

and inserting (3) and (4) in (2), and (2) in (1), we obtain

$$\Delta\omega_m = -\frac{NC}{16\pi^2\omega_o} B \left\{ e^{ig} F_m (x - \varepsilon - i\eta) + \right.$$

$$\left. + e^{-ig} F_m (x + \varepsilon - i\eta) \right\}$$

$$= -\frac{NC}{16\pi^2\omega_o} P_m \tag{5}$$

where

$$F_m (z) = \int_{-\pi}^{+\pi} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} e^{-iz\sin\left|\frac{\psi}{2}\right|\cos\phi} e^{im\psi} d\phi d\psi \tag{6}$$

and

$$x = 2A\omega_o \frac{\xi}{\alpha}, \quad \varepsilon = 2A\omega_r, \quad \eta = 2A\Gamma .$$

4. In the Appendix of this paper it is proved that

$$F_m (z) = R_m (z) - i J_m (z) \tag{7}$$

where

$$R_m (z) = 2 \left[\pi J_m \left(\frac{z}{2}\right)\right]^2 \tag{8}$$

and

$$J_m (z) = 4\pi (-1)^m \int_0^{\pi/2} H_o (z\cos\theta) \cos 2m\theta d\theta \tag{9}$$

where J_m is the Bessel function of first kind and m-th order, and H_o is the Struve function of zeroth order. It is easily seen that

$$R_m(z) = R_m(-z) , \qquad J_m(z) = -J_m(-z)$$

and that

$$F_m(z) = o \quad \text{for } z = o \text{ and } |z| \to \infty$$

except for m = o and z = o, in which case

$$F_o(z=o) = 2\pi^2 .$$

5. In the case of small $|z|$ we can expand (8) and (9) up to the first order term in z and use the following approximation

$$F_m(z) \simeq 2\pi^2 \delta_{mo} + i \frac{8z}{4m^2-1}, \quad |z|<<1 \tag{10}$$

where

$$\delta_{mo} = \begin{cases} o & \text{for } m \neq o \\ 1 & \text{for } m = o \end{cases} .$$

By inserting (10) in (5) we obtain for the growth rate β_m, which is the imaginary part of $-\Delta\omega_m$,

$$\beta_m = \frac{ANC\xi}{2\pi^2 Q^2 \alpha (4m^2-1)} \tag{11}$$

which is not in agreement with ref. (2). The difference is not only in a numerical factor but also in the sign.

Numerical Calculations

In the previous section we have seen that for $(x \pm \varepsilon - i\eta)$ small (and not necessarily for x small) the zeroth mode and the higher order modes have different stability cryterion. For large arguments, one has to calculate the frequency shift (5) with a computer. This is what we did by letting the computer calculate (8) and (9) by means of the following series expansions[3]

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{\kappa=o}^{\infty} \frac{(-z^2/4)^\kappa}{k!\ (n+\kappa)\ !}$$

and

$$H_o(z) = \frac{2}{\pi} \sum_{\kappa=1}^{\infty} \frac{(-1)^{\kappa-1}\ z^{2\kappa-1}}{[(2\kappa-1)!!]^2}$$

and by calculating the intergrals of the form

$$\int_o^{\pi/2} \cos^{2\kappa-1}\theta\ \cos 2m\theta\ d\theta$$

as shown in ref. (4). We had to limit our calculation to the first four modes (m = o, 1, 2, 3) and for $x \leq 10$, $\varepsilon \leq 12$. For too large arguments and for too large modes the numerical calculation gets rather inaccurate and unreliable.

Also, it is rather easy to prove that the growth rate change sign if x does too, thus we limited our calculation only to positive values of x.

Finally, we considered only the two extreme cases of very low Q (Q=1) and very high Q ($Q = \infty$). The results of the calculation for Q = 1 are shown in Figures 1 to 4, and for $Q = \infty$ in Figures 5 to 8. The imaginary part of P_m is plotted versus x for various resonating frequency (ε) and modes.

We show in Figures 9 and 10 the stability diagrams. Comparing all the cases with the same Q (which forms a diagram) and the same x (a line of the diagram) we can have, at the glance, the idea of how all the modes under consideration behave all together. For each frequency we have four letters, the first refers to m = o, the second to m = 1 and so on. The letter "s" stays for stable, the letter "u" for unstable.

It is easily seen that, especially for Q = 1 and high frequency, all the modes **own** the same stability cryterion no matter how small x is.

Fast Decaying Wake Field

We considered, then, a fast, exponentially decaying wake field. The wake form is now

$$\rho(t) = \rho_o \, e^{-\Gamma t} \, H(t) \tag{12}$$

where ρ_o and Γ are two constants.

By inserting (3) and (12) in (2), and (2) in (1), we obtain

$$\Delta\omega_m = - \frac{N\rho_o}{8\pi^2\omega_o} \, F_m \, (x-i\eta). \tag{13}$$

From eq. (10) we derive the growth rate for small argument

$$\beta_m = \frac{2AN\rho_o \xi}{\pi^2 \alpha \ (4m^2-1)} \ . \tag{14}$$

Again observe that this does not necessarily apply for small x, but for small $|x-i\eta|$.

For large argument we calculated $F_m(x-i\eta)$ at the computer. The results of the calculation are shown in Figures 11 to 14. In these figures the imaginary part of F_m is plotted versus x (positive) for several values of η. A stability diagram is also shown in Figure 15. From this we see that, again, no matter how small x is, the zeroth mode and the higher order mode have the same stability cryterion for large η, which is for wake fields which decay very fast compared to the bunch length.

Appendix

From eq. (6) we have

$$F_m(z) = \int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} \cos m\psi \cos\left(z\sin\left|\frac{\psi}{2}\right|\cos\phi\right) d\phi\, d\psi +$$

$$+ \int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} \sin m\psi \sin\left(z\sin\left|\frac{\psi}{2}\right|\cos\phi\right) d\phi\, d\psi +$$

$$+ i\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} \sin m\psi \cos\left(z\sin\left|\frac{\psi}{2}\right|\cos\phi\right) d\phi\, d\psi +$$

$$- i\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} \cos m\psi \sin\left(z\sin\left|\frac{\psi}{2}\right|\cos\phi\right) d\phi\, d\psi .$$

The second and third integrals are identically zero, and we are left with the first and the last one, which we can write, after the change of variable $\phi = \theta - \frac{\pi}{2}$,

$$F_m(z) = \int_{-\pi}^{\pi}\int_{0}^{\pi} \cos m\psi \cos\left(z\sin\left|\frac{\psi}{2}\right|\sin\theta\right) d\theta\, d\psi \quad +$$

$$- i\int_{\pi}^{\pi}\int_{0}^{\pi} \cos m\psi \sin\left(z\sin\left|\frac{\psi}{2}\right|\sin\theta\right) d\theta\, d\psi$$

and, from ref. (5)

$$F_m\ (z) = \pi \int_{-\pi}^{\pi} J_o\ (z\sin|\tfrac{\psi}{2}|)\ \cos m\ \psi\ d\psi\ +$$

$$-i\ \pi \int_{-\pi}^{\pi} H_o\ (z\sin|\tfrac{\psi}{2}|)\ \cos m\ \psi\ d\psi\ .$$

With the change of variable $\phi = \frac{\psi}{2}$, we have

$$F_m\ (z) = 4\pi \int_{o}^{\pi/2} J_o\ (z\sin\phi)\ \cos 2m\ \phi\ d\phi\ +$$

$$-4\pi\ i \int_{o}^{\pi/2} H_o\ (z\sin\phi)\ \cos 2m\ \phi\ d\phi\ .$$

Finally, with the new variable $\phi' = \phi + \frac{\pi}{2}$ we have also

$$F_m\ (z) = 4\pi\ (-1)^m \int_{o}^{\pi/2} J_o\ (z\cos\phi')\ \cos 2m\ \phi'\ d\phi'\ +$$

$$-4\pi i\ (-1)^m \int_{o}^{\pi/2} H_o\ (z\cos\phi')\ \cos 2m\ \phi'\ d\phi'.$$

The first integral, according to ref. (6), is $R_m\ (z)$, and the second integral is $J_m\ (z)$.

References

1. M. Sands, SLAC-TN-69-8, March 1969.

2. C. Pellegrini, Nuovo Cimento, Series X, Vol. 64A, pps. 447-473, Nov. 1969.

3. M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series 55. Issued June 1964. (a) See eq. (9.1.10) on page 360, (b) see eq. (12.1.4) on page 496.

4. I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals Series and Products (1965). See eq. (3.631.17) on page 374.

5. As reference (3), (a) see eq. (9.1.18) on page 360, (b) see eq. (12.3.3) on page 498, and eq. (12.3.8) on page 499.

6. As reference (4), see eq. (6.681.1) on page 738.

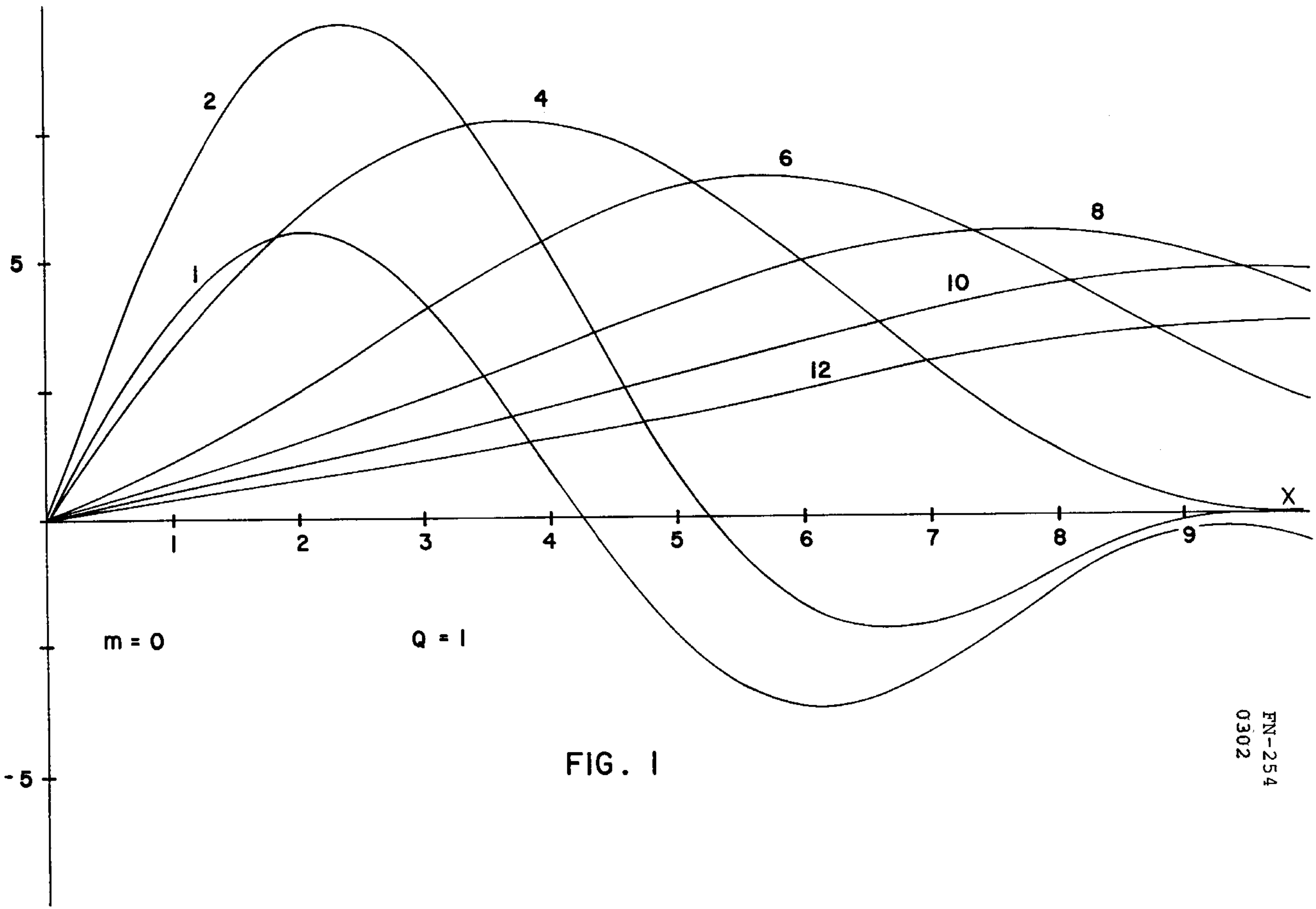

FIG. 1

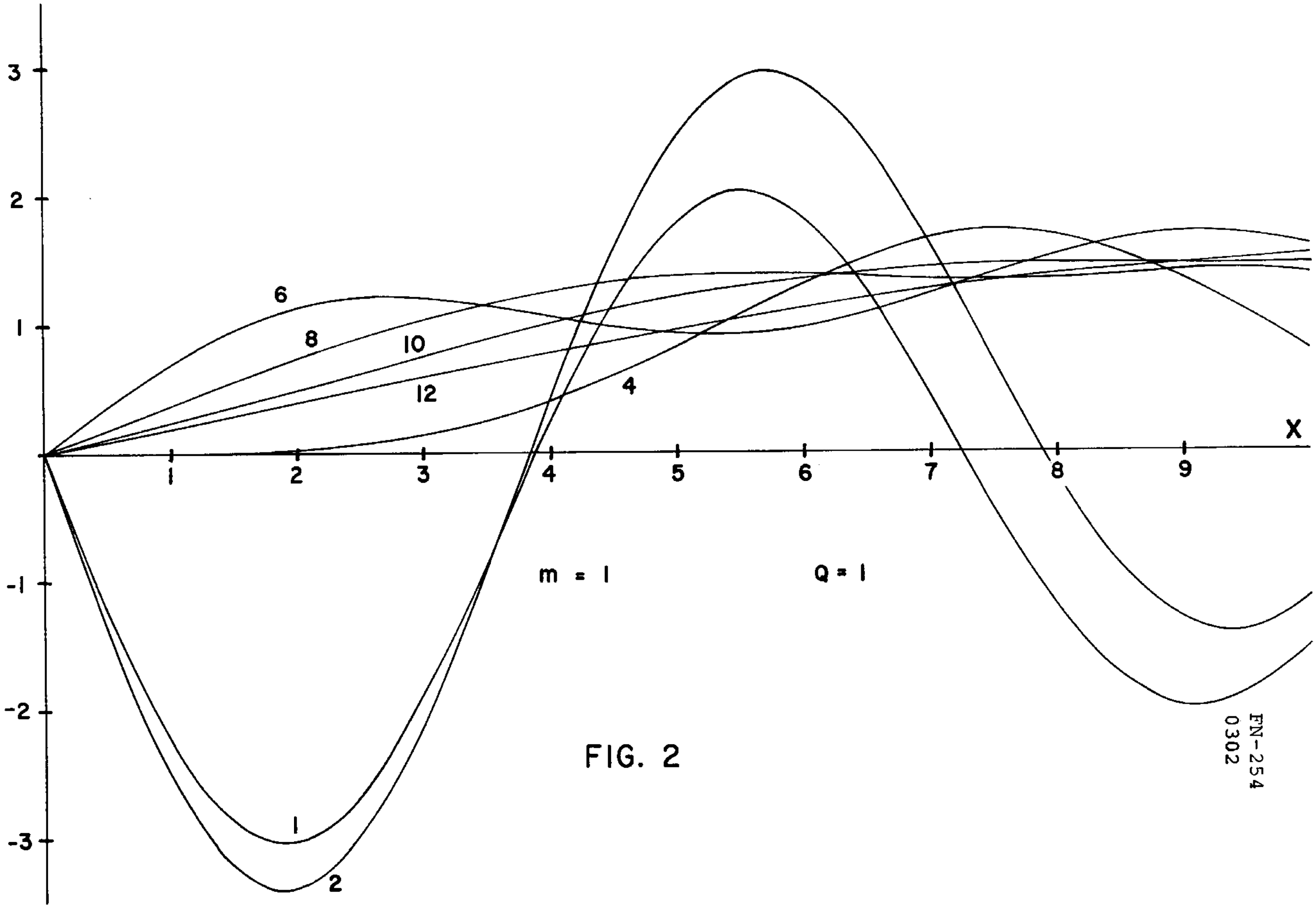

FIG. 2

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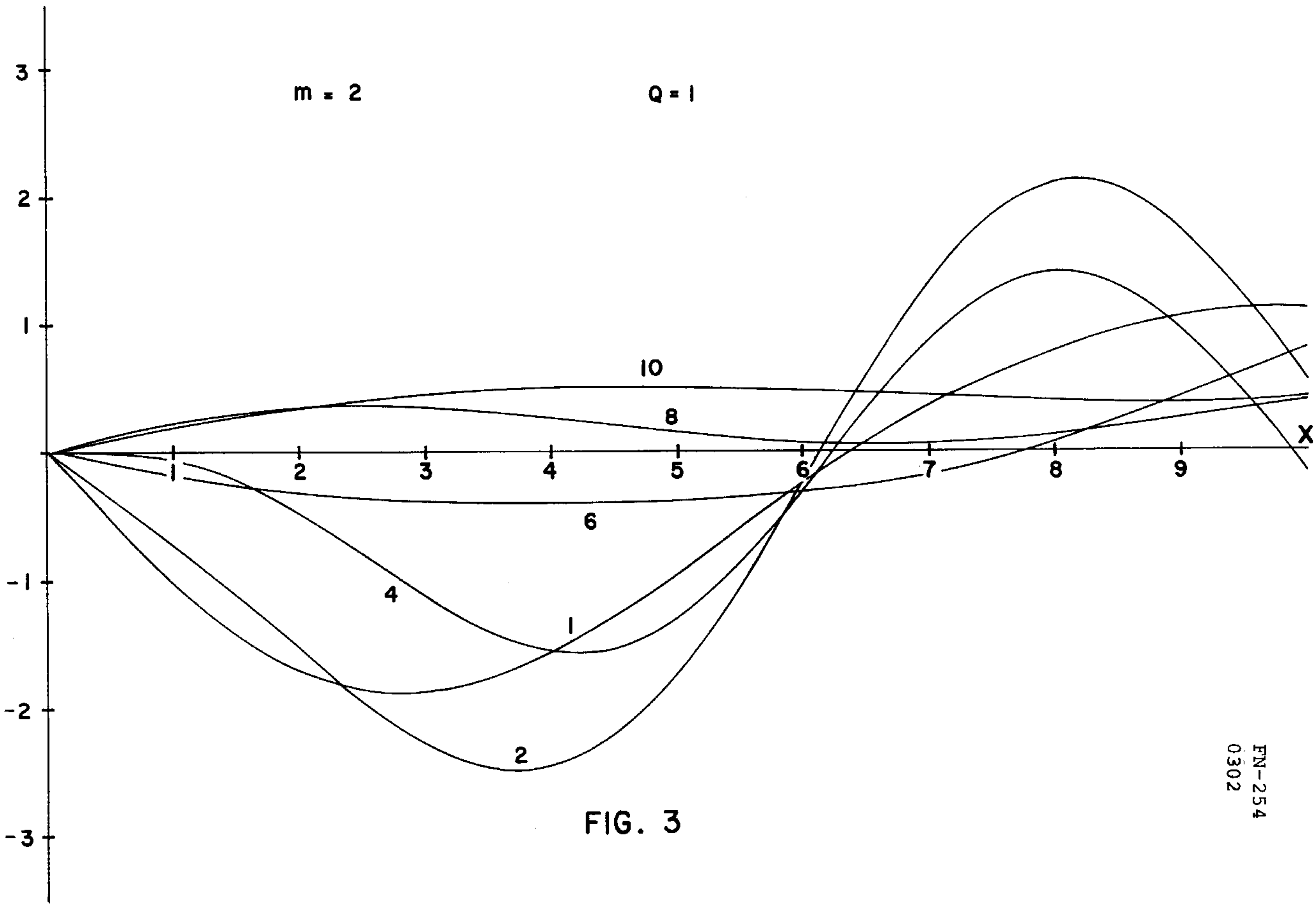

FIG. 3

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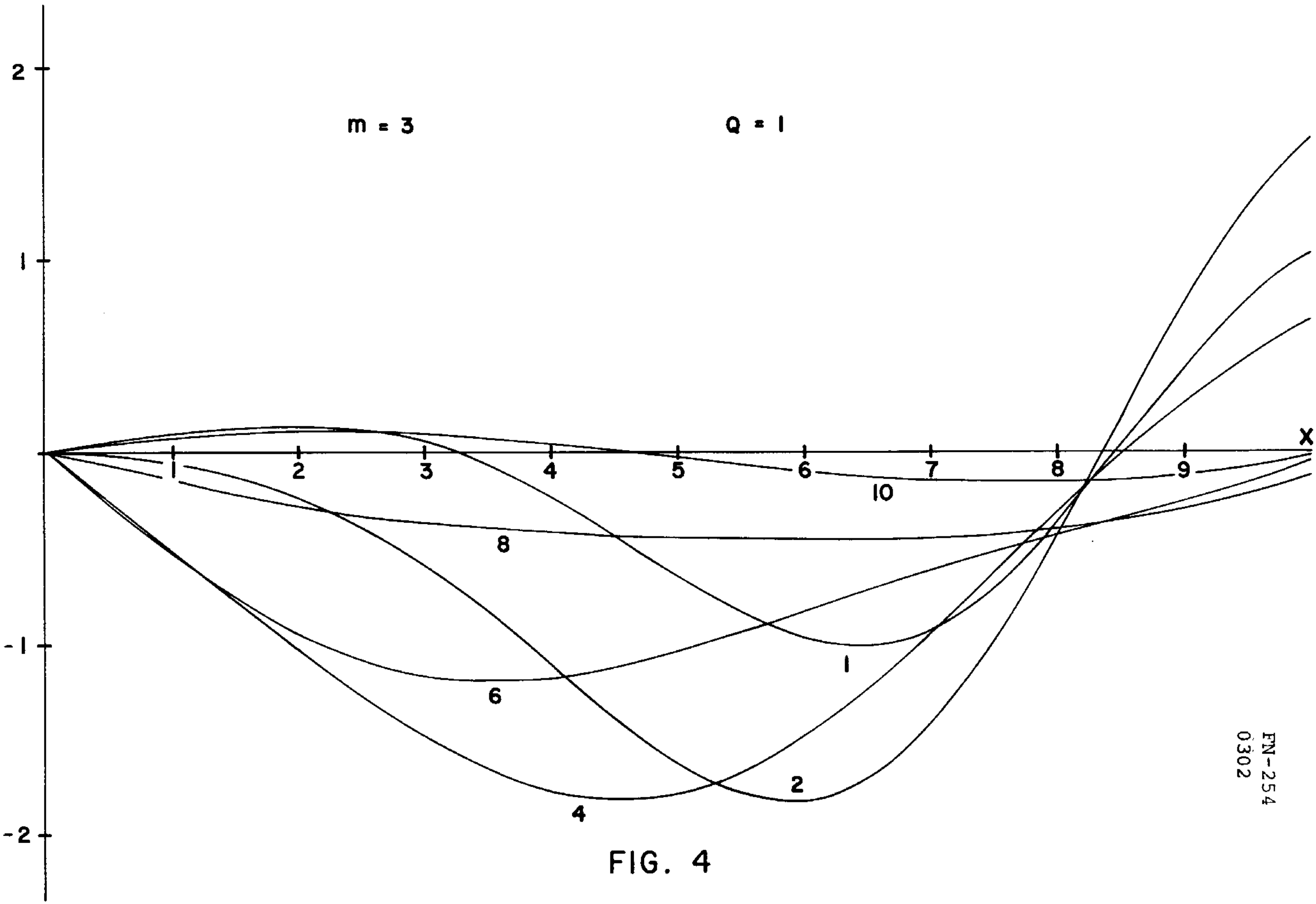

FIG. 4

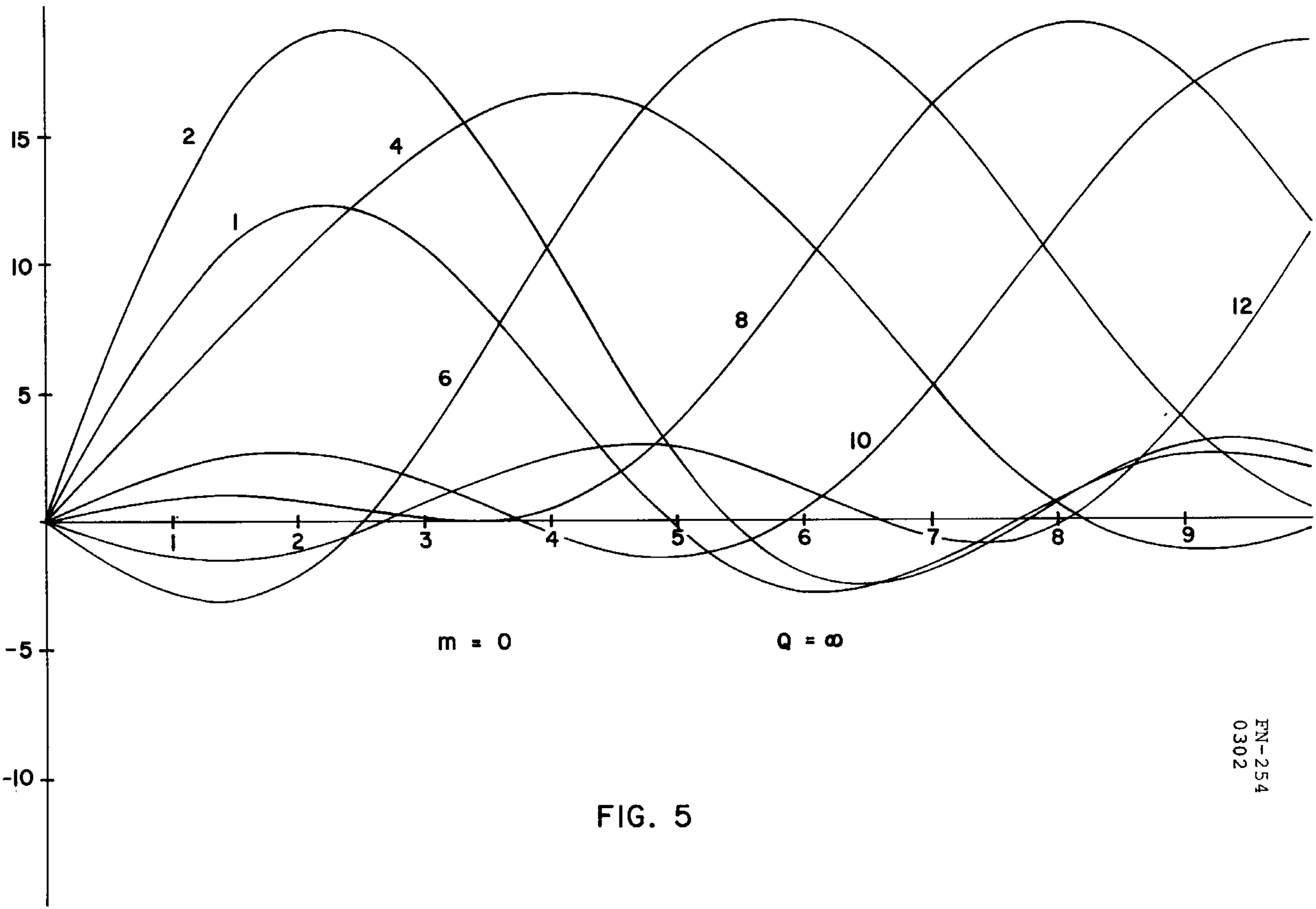

FIG. 5

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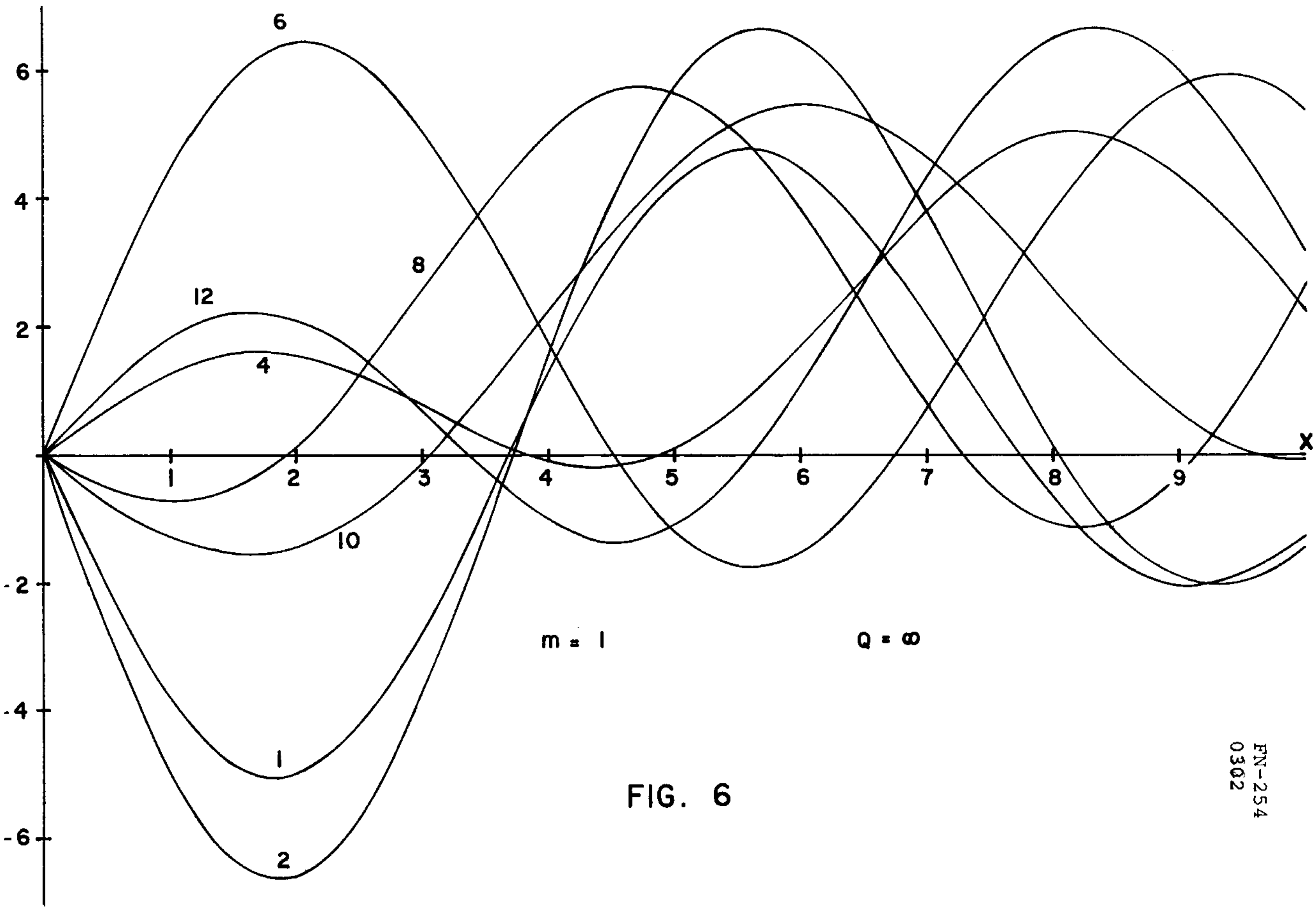

FIG. 6

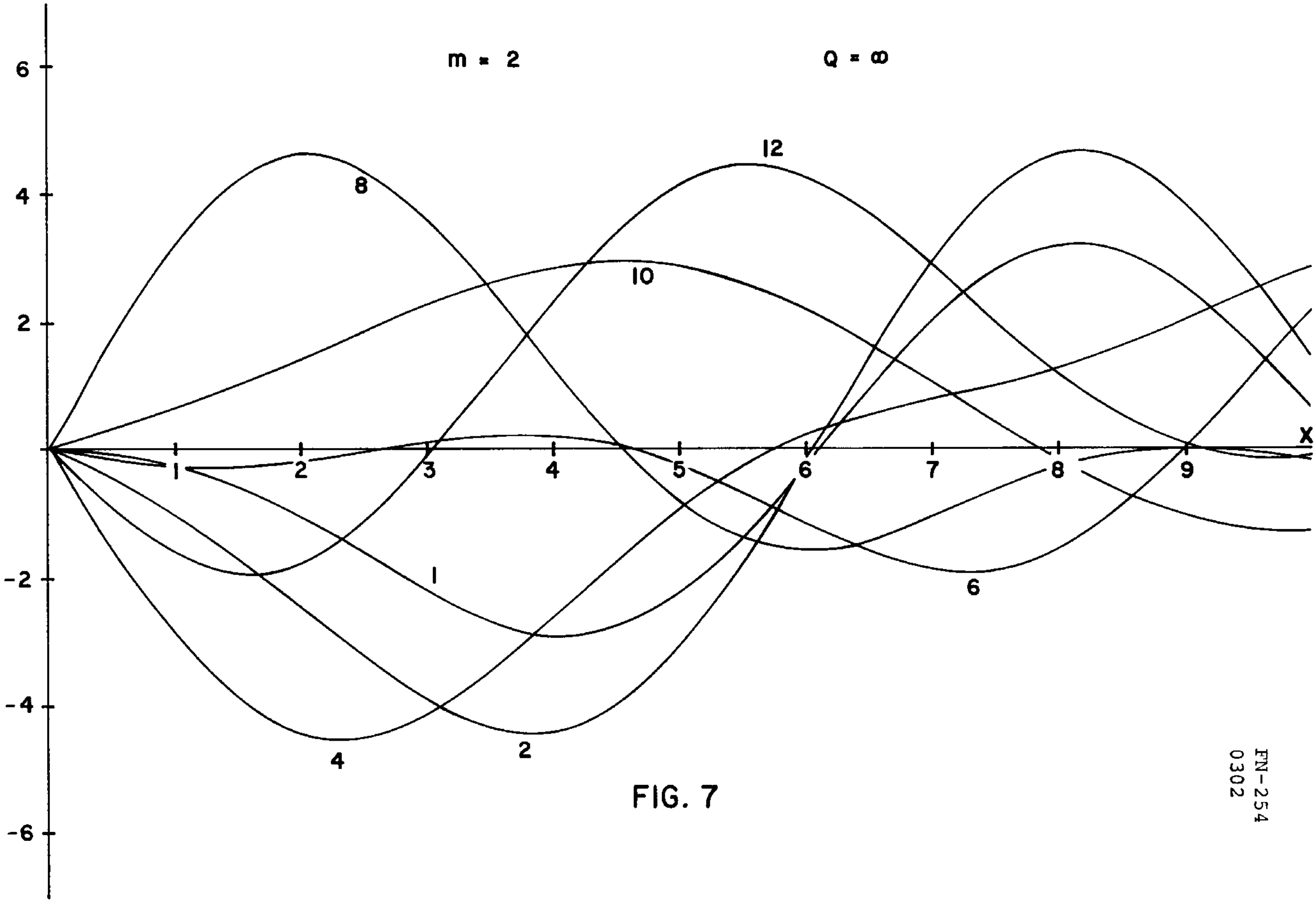

FIG. 7

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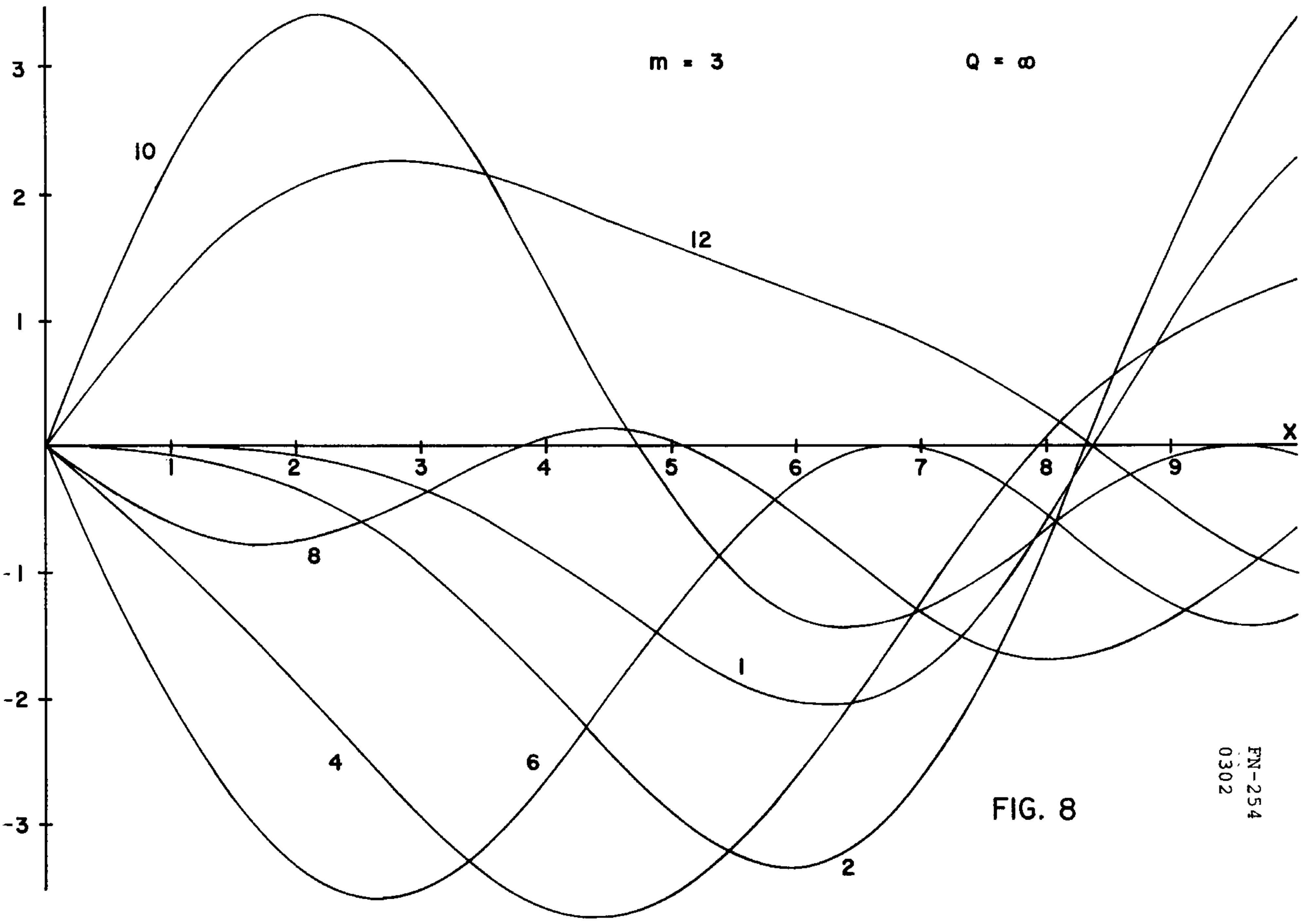

FIG. 8

X \ ε	1	2	4	6	8	10	12	
0.2	s u u s	s u u u	s s u u	s s u u	s s s u	s s s s	s s s s	
1	s u u s	s u u u	s s u u	s s u u	s s s u	s s s s	s s s s	
2	s u u s	s u u u	s s u u	s s u u	s s s u	s s s s	s s s s	
3	s u u s	s u u u	s s u u	s s u u	s s s u	s s s s	s s s s	
4	s s u u	s s u u	s s u u	s s u u	s s s u	s s s s	s s s s	
5	u s u u	s s u u	s s u u	s s u u	s s s u	s s s u	s s s s	
6	u s u u	u s u u	s s u u	s s u u	s s s u	s s s u	s s s s	
7	u s s u	u s s u	s s s u	s s u u	s s s u	s s s u	s s s s	
8	u u s u	u u s u	s s s u	s s s u	s s s u	s s s u	s s s s	
9	u u s s	u u s s	s s s s	s s s u	s s s u	s s s u	s s s s	
10	u u s s	u u s s	u s u s	s s s u	s s s u	s s s u	s s s s	

Q = 1

Figure 9

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X \ E	1	2	4	6	8	10	12	
0.2	s u u u	s u u u	s s u u	u s u u	s u s u	s u s s	u s u s	
1	s u u u	s u u u	s s u u	u s u u	s u s u	s u s s	u s u s	
2	s u u u	s u u u	s s u u	u s u u	s s s u	s u s s	u s u s	
3	s u u u	s u u u	s s u u	s s s u	s s s u	s u s s	s s u s	
4	s s u u	s s u u	s u u u	s s s u	s s s s	u s s s	s u s s	
5	u s u u	s s u u	s s u u	s u u u	s s u s	u s s u	s u s s	
6	u s u u	u s u u	s s s u	s u u u	s s u u	s s s u	s s s s	
7	u s s u	u s s u	s s s u	s s u u	s s u u	s s s u	u s s s	
8	s u s u	s s s u	s s s s	s s u u	s u u u	s s u u	u s s s	
9	s u s s	s u s s	u s s s	s s s u	s u u u	s s u u	s s s u	
10	s u s s	s u s s	u s s s	s s s u	s s u u	s u u u	s s u u	

$Q = \infty$

Figure 10

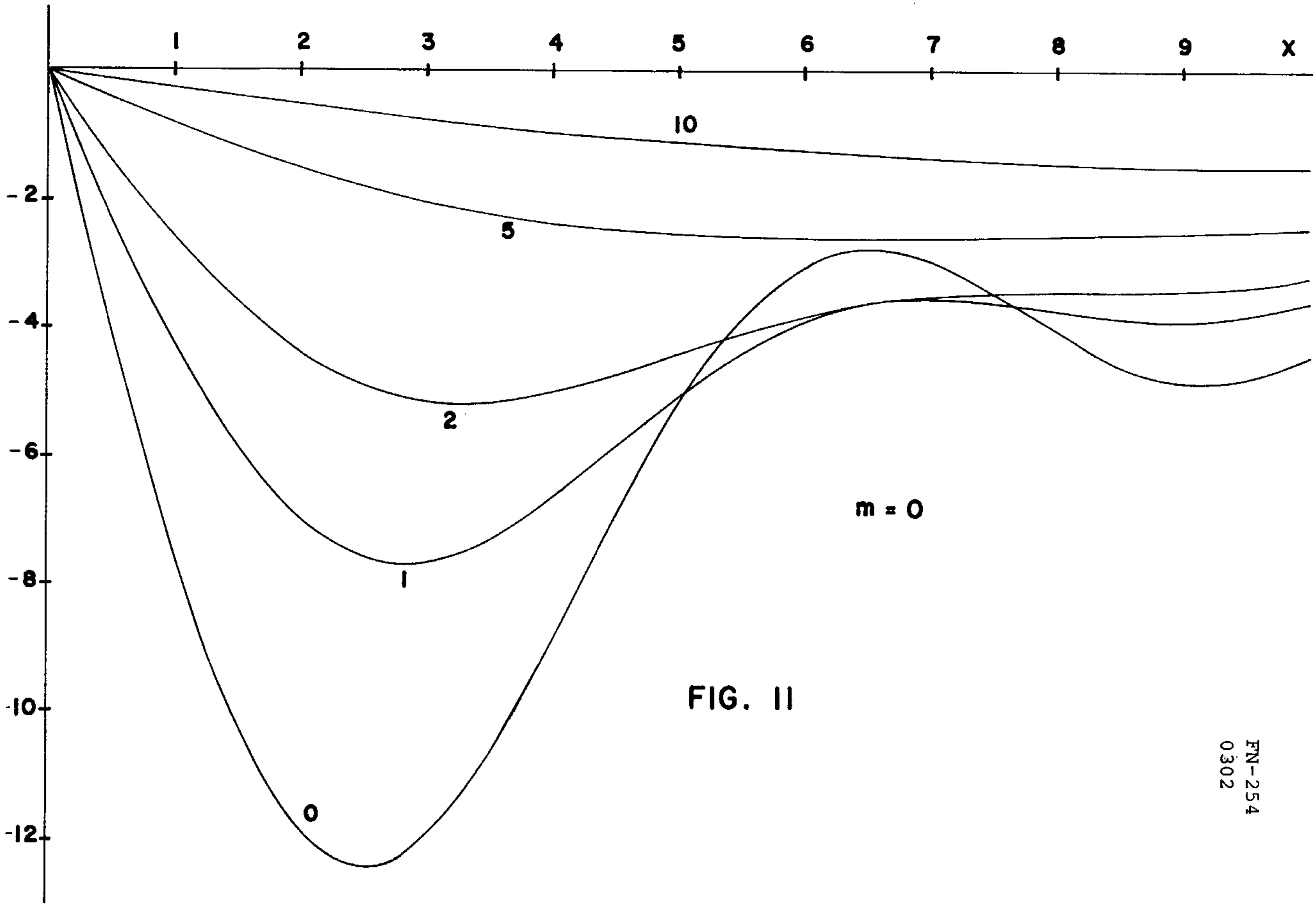

FIG. 11

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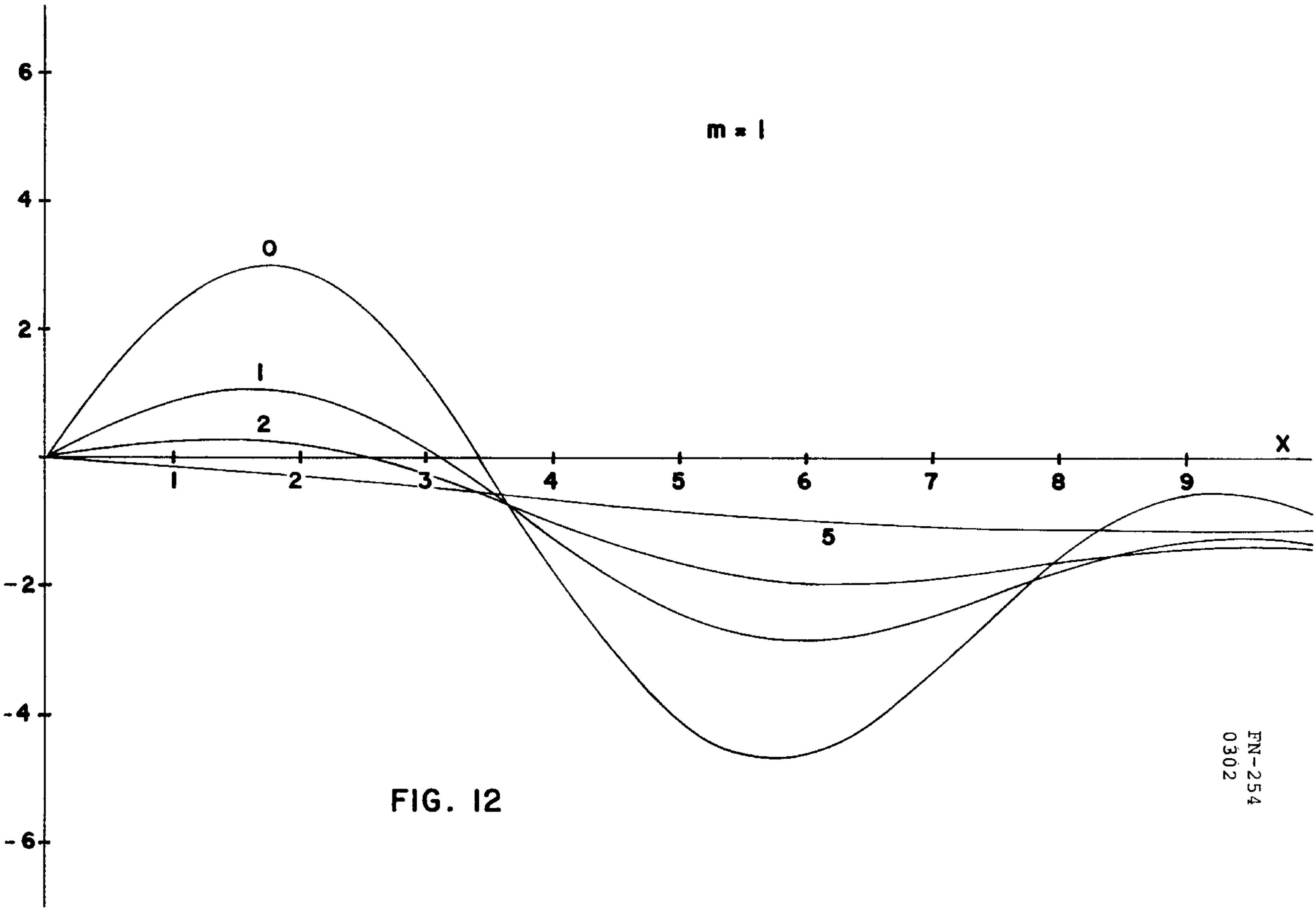

FIG. 12

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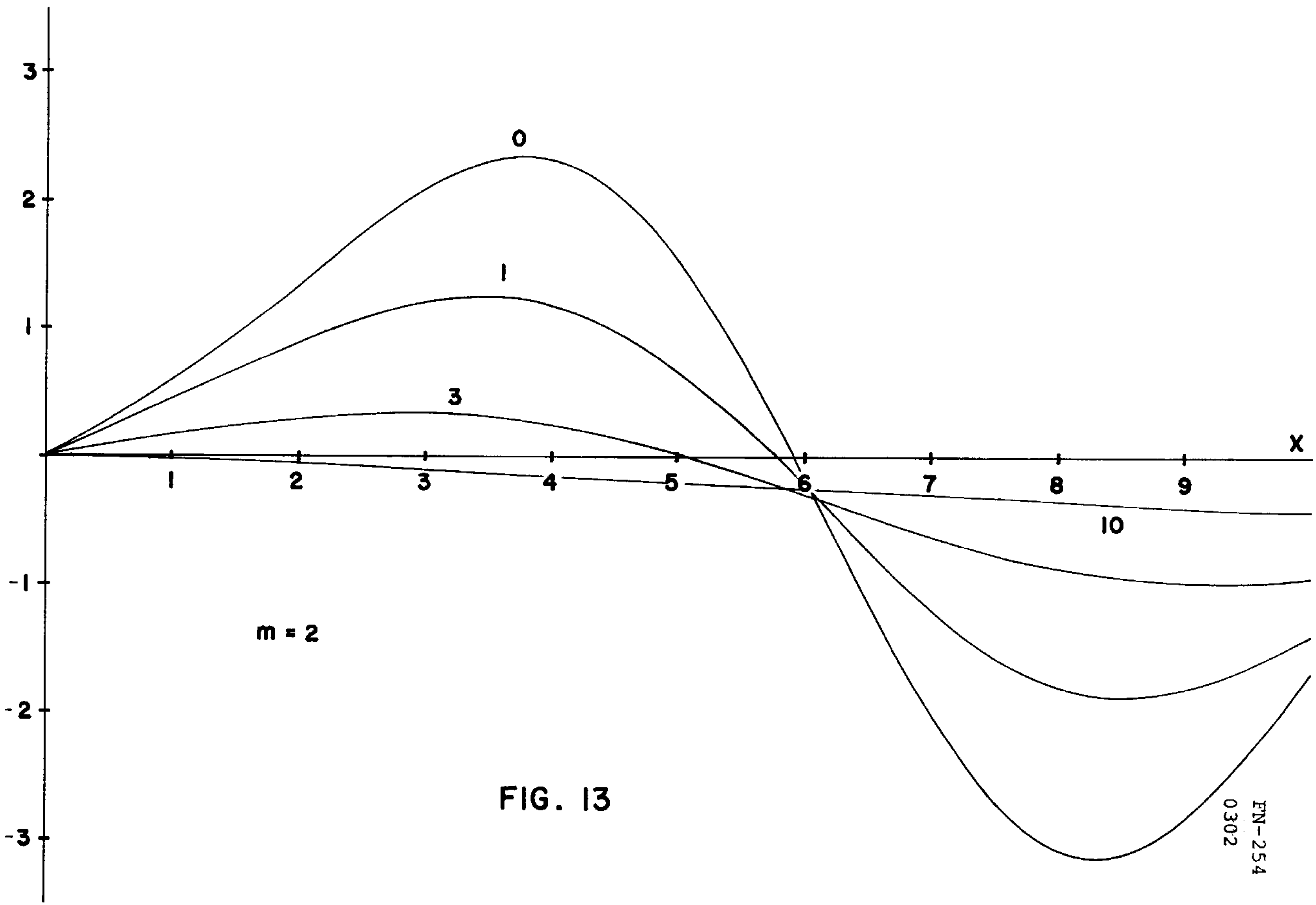

FIG. 13

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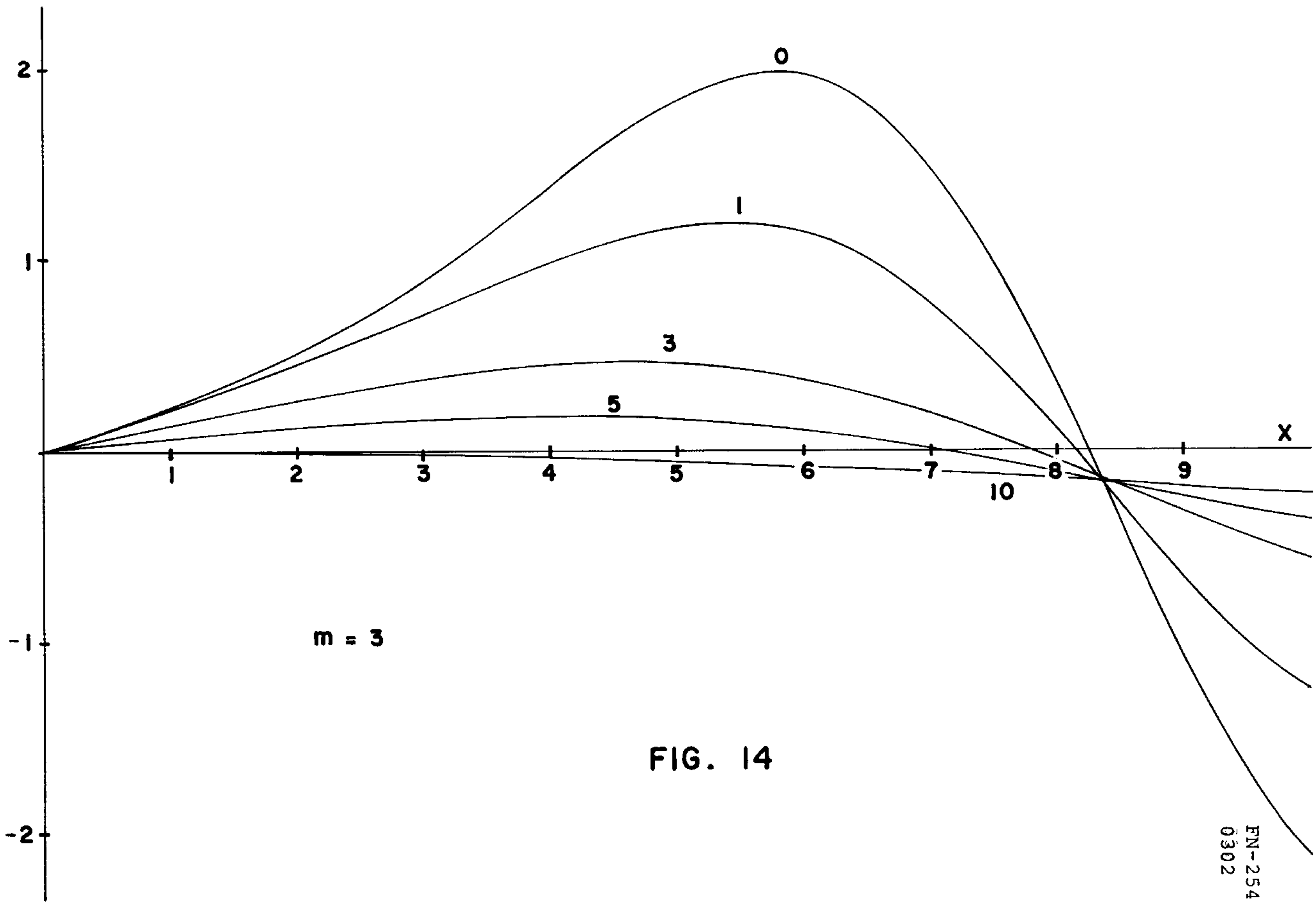

FIG. 14

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X \ η	0	1	2	3	5	10	
0.2	u s s s	u s s s	u s s s	u s s s	u u s s	u u u u	
1	u s s s	u s s s	u s s s	u s s s	u u s s	u u u u	
2	u s s s	u s s s	u s s s	u u s s	u u s s	u u u u	
3	u s s s	u s s s	u u s s	u u s s	u u s s	u u u u	
4	u u s s	u u s s	u u s s	u u s s	u u u s	u u u u	
5	u u s s	u u s s	u u s s	u u s s	u u u s	u u u u	
6	u u u s	u u u s	u u u s	u u u s	u u u s	u u u u	
7	u u u s	u u u s	u u u s	u u u s	u u u s	u u u u	
8	u u u s	u u u s	u u u s	u u u u	u u u u	u u u u	
9	u u u u	u u u u	u u u u	u u u u	u u u u	u u u u	
10	u u u u	u u u u	u u u u	u u u u	u u u u	u u u u	

Figure 15