national accelerator laboratory

FN-249
0402

STOCHASTICITY LIMIT IN ONE-DIMENSION

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October, 1972

Introduction

The purpose of this paper is to calculate the "stochasticity limit" of the NAL main-ring accelerator. We shall follow the same approach E. Keil[1] used to calculate the effect of the nonlinear space-charge forces of one beam on the motion of the other beam in a system of intersecting and storage rings.

In our case, we will not deal with space-charge forces, but we shall assume that the nonlinearities of the guiding (magnetic) field are large enough to have considerable weight on the motion of a charged particle (proton).

The problem is the following: We are able to solve linear differential equations with periodic coefficients. Usually we use the computer to find the solution with whatever accuracy is required. This solution will not have any time limitation. In this case, the pattern of the motion is rather regular.

Unfortunately, we cannot say the same when some nonlinearities are introduced in the equation for motion.

We can state the existence and the uniqueness of the solution of a nonlinear differential equation, but it is practically impossible to find this solution, even with the computer, even if all the coefficients of the equation are periodic, in a form which has no limitation of validity in time. This kind of solution was never required until recently when one wanted charged beams circulating in storage rings for hours or days.

Using different techniques, people could draw trajectories with different amplitude of motion, and we learned that the pattern of the motion gets more and more complicated when the amplitude of the motion is increased. Above some value of the amplitude the pattern is so complicated that the motion looks (but is not) to an observer erratic or stochastic, so much that a system of identical particles can be very well described by kinetic (statistic) equations. Our goal is to calculate this amplitude or boundary of the stochasticity for a particle accelerator. As we shall show, it will be derived by the so-called "stochasticity limit," which is the condition that the nonlinear coefficients have to satisfy at a given amplitude because the motion looks (or is, as we shall say improperly) stochastic.

We wish to emphasize that in this paper we develop concepts that have been already introduced and investigated in many other papers. We believe that a list of references should start with Chirikov's paper (or book?).[2] Almost all the knowledge about nonlinear motion is condensed there, with detailed references of papers about the same subject written before 1970. But if we consider only the applications to the particle accelerator or storage ring, the first ones to think about a stochastic instability of a charged beam were (probably) Goward and Hine.[3]

Recently, other people approached the stochastic stability with various techniques. An incomplete list of their papers can be found in the references (4 to 10 included).

To speed up the formalism of our approach and make clear the preparatory arguments, we shall deal with a one-dimension example. In a later paper we shall discuss the more physical multidimensional problem.

1. The Equation of Motion

Let us consider the following nonlinear differential equation

$$\frac{d^2s}{ds^2} + K_o(s)x = F(s,x). \qquad (1)$$

This might be the equation of motion of a charged particle in a circular accelerator or storage ring, when coupling is absent and first-order derivative quantities are ignored.

$K_o(s)$ and $F(s,x)$ are two periodic functions of s with common period C. $F(s,x)$ includes nonlinear field contributions, field errors, as well as kinematic correction quantities. With a power series expansion we write

$$F(s,x) = a_o(s) + a_1(s)x + \ldots + a_k(s)x^k + \ldots$$

$$= \sum_{k=0}^{\infty} a_k(s)\ x^k,$$

where, also, the $a_k(s)$'s are periodic with period C.

We assume we are able to solve the linear equation

$$\frac{d^2x}{ds^2} + K_o(s)x = 0,$$

the solution of which takes the form

$$x = \sqrt{E\beta(s)}\ \cos\ [\nu_o\phi(s) + \delta]$$

where E and δ are two integration cosntants, $\beta(s)$ is periodic with period C, and

$$\nu_o\phi(s) = \int_o^s \frac{ds}{\beta(s)} .$$

The betatron number ν_o is defined in such a way that $\phi(s)$ adds 2π every cycle of length C.

Let us apply to (1) the following transformations

$$\xi = \frac{x}{\sqrt{\beta}} \qquad \text{and} \qquad d\phi = \frac{ds}{\nu_o\beta} .$$

We obtain

$$\begin{aligned} \frac{d^2\xi}{d\phi^2} + \nu_o^2\xi &= \nu_o^2\ \beta^{3/2}\ F(\phi,\xi\ \sqrt{\beta}) \\ &= \nu_o^2 \sum_{k=0}^{\infty} \beta^{\frac{3+k}{2}}\ a_k\xi^k, \end{aligned} \tag{2}$$

where now β and a_k are functions of ϕ, both periodic with period 2π.

We now perform a second transformation, this time into angle-action variables. Let us take

$$\xi = \sqrt{I}\ \cos\ \psi \qquad \text{and} \qquad \xi' = -\nu_o\sqrt{I}\ \sin\ \psi,$$

where prime denotes derivative with respect to ϕ. The inverse transformations are

$$I = \frac{\xi'^2}{\nu_o^2} + \xi^2 \tag{3}$$

$$\psi = -\text{arctg}\,\frac{\xi'}{\nu_o \xi} \tag{4}$$

Differentiating (3) and (4), and by using (2), we obtain finally

$$I' = -2\nu_o\sqrt{I}\,\sin\psi \sum_{k=0}^{\infty} \beta^{\frac{3+k}{2}}\, a_k\, I^{\frac{k}{2}} \cos^k\psi \tag{5}$$

$$\psi' = \nu_o - \nu_o \frac{\cos\psi}{\sqrt{I}} \sum_{k=0}^{\infty} \beta^{\frac{3+k}{2}}\, a_k\, I^{\frac{k}{2}} \cos^k\psi. \tag{6}$$

For the unperturbed motion, that is, when the a_k's are all identically zero, the above equations reduce to

$$I' = 0 \qquad \text{and} \qquad \psi' = \nu_o.$$

In this case, I is a constant of motion; it is the area enclosed by the trajectory of the particle in the (ξ,ξ')-phase plane. This trajectory is an upright ellipse Γ, and looks like that shown in Fig. 1. Also, ψ' is a constant of motion, and is just the betatron number ν_o, i.e. the number of oscillations performed during one period of length C.

The point P, which represents the phase of the particle, goes around the trajectory Γ, ν_o times during one period. If ν_o is not an integer, the particle will end up to occupy another position Q after one period. In the case that ν_o is expressed as the ratio of two integer numbers, namely satisfying the condition

$$n\nu_o + m = 0 \tag{7}$$

with n and m integers, the point P, moving along the trajectory Γ, will go back to the same identical position from where it started, after n periods. In the meantime, it has been going around Γ, m times.

But, if we make some, or all, of the a_k's different from zero, the particle will not end up, in general, to occupy the same position P after n periods, but another point P', which also does not lie on the trajectory Γ. Nevertheless, if the coefficients a_k are small enough, the point P' is relatively close to P, and after several cycles of n periods, will cover a region of the phase space around P. The area of this region will eventually blow up during a very large number of periods.

It is not difficult to see that there are n of such regions, all identical, and they can take the shape shown in Fig. 2 under special circumstances.

In the next two sections, we shall speculate about the nature of the motion around the n fixed points P.

2. The Width of a Nonlinear Resonance

Let us define

$$I = y^2$$

and rewrite (5) and (6) as follows

$$y' = -\nu_o \sin\psi \sum_{k=0}^{\infty} \beta^{\frac{3+k}{2}} a_k y^k \cos^k\psi \tag{8}$$

$$\psi' = \nu_o - \nu_o \cos\psi \sum_{k=0}^{\infty} \beta^{\frac{3+k}{2}} a_k\, y^{k-1} \cos^k\psi. \qquad (9)$$

We can try to solve this system with good accuracy if the a_k's are small. In this case, presumably, y changes very slowly over several periods, so that, during some number of periods we can keep it constant. With this adiabatic approximation, the right-hand sides of (8) and (9) are periodic functions of ψ and ϕ, in both cases with period 2π.

With a double Fourier expansion, we obtain

$$y' = -\nu_o \sum_n \sum_m \sum_{k=0}^{\infty} y^k\, b_{km}\, f_{kn}\, e^{i(n\psi+m\phi)} \qquad (10)$$

$$\psi' = \nu_o - \nu_o \sum_n \sum_m \sum_{k=0}^{\infty} y^{k-1}\, b_{km}\, g_{kn}\, e^{i(n\psi+m\phi)} \qquad (11)$$

where

$$b_{km} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \beta^{\frac{3+k}{2}} a_k\, e^{-im\phi} d\phi \qquad (12)$$

$$f_{kn} = -\frac{n}{k+1}\, i\, g_{kn}$$

$$g_{kn} = \frac{1}{\pi} \int_0^{\pi} \cos^{k+1}\psi \cos n\psi\, d\psi.$$

It is also easy to verify that

$$g_{kn} = g_{k,-n} \qquad \text{and} \qquad f_{kn} = -f_{k,-n}$$

In the following, we shall assume the average over ϕ of all the quantities $\beta^{\frac{3+k}{2}} a_k$ are practically zero, namely

$$b_{ko} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \beta^{\frac{3+k}{2}} a_k \, d\phi = 0$$

for $k = 0,1,2,\ldots$. This is usually satisfied in a particle accelerator or storage ring.

We define a nonlinear resonance when the following quantity

$$X_{nm} = n\psi + m\phi$$

changes very slowly, or, at the limit, is a constant, namely when

$$X'_{nm} = 0. \tag{13}$$

This reduces to the well-known resonance condition (7) for a linear motion. In this case, there are n values of ν_o, in a unit range, that satisfy simultaneously (7). These values are isolated from each other.

As we shall show now, if the motion is not linear, other values of ν_o result to be locked to the resonance (13), in such a way that around each of the previous n values there is a range $\Delta\nu_{nm}$ of values ν_o all locked to the same resonance. We shall refer to $\Delta\nu_{nm}$ as the <u>width</u> of the resonance (13).

We can calculate $\Delta\nu_{nm}$ assuming that (13) is the only predominant resonance. In this case, ignoring the fast oscillatory terms, we obtain from (10) and (11)

$$y' = -2n\nu_o \left| \sum_{k=0}^{\infty} y^k \frac{b_{km} g_{kn}}{k+1} \right| \sin\left(X_{nm} + S_{nm}\right) \tag{14}$$

and

$$X'_{nm} = -2n\nu_o \left| \sum_{k=0}^{\infty} y^{k-1} b_{km} g_{kn} \right| \cos\left(X_{nm} + C_{nm}\right) + (n\nu_o + m) \tag{15}$$

where S_{nm} and C_{nm} are appropriate phase angles, and we remind that b_{km} is generally a complex number. From (15) we quickly see that

$$\Delta\nu_{nm} = 4\nu_o \left| \sum_{k=0}^{\infty} y^{k-1} b_{km} g_{kn} \right|. \tag{16}$$

This quantity depends on the amplitude of motion y, and, in order to make sense, applies only when the a_k's are so small that there is only a little change of y during any n periods. For the same reason, (16) does not apply for too small values of y. But, all this was already assumed in the adiabatic approximation we stated at the beginning of this section.

There is also another meaning of $\Delta\nu_{nm}$. In the linear motion ν_o is exactly the number of betatron oscillations performed per period by any particle, whatever are the initial conditions. But in the nonlinear machine, ν_o is nothing else than a measure of the accelerator setting, the number of oscillations per period being now given by ψ'. Thus, in the case of a single resonance, $\Delta\nu_{nm}$ is also the interval of the variation of the actual betatron number. Besides, in this case, the equations (14) and (15) describe the motion around the n fixed points we have been talking about at the end of the previous section. The trajectories look like that shown in Fig. 2. They are regular curves, and will be still regular (but a little more complicated), also, when we retain more than one equally predominant resonating term in the equations (10) and (11), provided that the corresponding resonances are far enough away from each other.

Accurate numerical calculation of the resonance width $\Delta\nu_{nm}$ is made possible only by a good knowledge of the Fourier coefficients b_{km}. In practice, at most, we know the root mean square average $\langle a_k \rangle$ of the function $a_k(\phi)$. In this case, if we have M magnets all with the same short length L, all of them, and only them, holding lumped field errors, each other uncorrelated, the rms average of the expectation value of b_{km} is, from (12)

$$\langle b_{km} \rangle = \frac{\sqrt{M}\ L}{2\pi\nu_o}\, \bar{\beta}^{\frac{k+1}{2}} \langle a_k \rangle \tag{17}$$

for $|m| \lesssim M$ and

$$\langle b_{km} \rangle = 0 \qquad \text{for} \qquad |m| \gtrsim M.$$

$\bar{\beta}$ is the average value of β over one period.

Then, we can calculate the rms average of the expectation value of the width $\Delta\nu_{nm}$. This is, since the rms averages sum quadratically,

$$\langle \Delta\nu_{nm} \rangle = \sqrt{\sum_{k=0}^{\infty} \left(4\nu_o Y^{k-1}\ g_{kn} \langle b_{km} \rangle\right)^2}$$

which is zero for $m \gtrsim |M|$ and, from (17)

$$\langle \Delta\nu_{nm} \rangle = \frac{2}{\pi} \sqrt{M}\; L\overline{\beta} \sqrt{\sum_{k=0}^{\infty} \left(\langle a_k \rangle\; g_{kn}\; \hat{x}^{k-1} \right)^2} \tag{18}$$

for $|m| \lesssim M$.

In this expression $x = \sqrt{I\overline{\overline{\beta}}}$ is some sort of average (over one period) of the amplitude of the motion, essentially the same x appearing in (1).

3. The Stochastic Limit

From (11) and (16), we have formerly

$$\psi' = \nu_o - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Delta\nu_{nm} \cos\,(n\psi + \overline{\psi}_n)\; \cos(m\phi + \overline{\phi}_m) \tag{19}$$

where $\overline{\psi}_n$ and $\overline{\phi}_m$ are appropriate phase angles.

Let us assume we are able to integrate the equation (19), and we know the function

$$\psi = \psi(\phi)$$

which satisfies (19) with given initial conditions. Thus, if we know ψ at some $\phi = \phi_1$, say

$$\psi(\phi = \phi_1) = \psi_1,$$

we can calculate also ψ at $\phi = \phi_1 + 2\pi$, namely

$$\psi(\phi = \phi_1 + 2\pi) = \psi_2,$$

and we can consider ψ_2 as a function of ψ_1.

In order to judge how much regular is the pattern of the motion, we calculate the following correlation factor[11]

$$P = \frac{\left| \int_{-\pi}^{+\pi} \psi_2 \psi_1 d\psi_1 \right|}{\int_{-\pi}^{+\pi} \psi_1^2 d\psi_1} .$$

This factor, through the $\Delta\nu_{nm}$'s, is a function of the amplitude of the motion $\hat{x}$. In particular, if all the $\Delta\nu_{nm}$'s are zero, we have $P = 1$ for whatever $\hat{x}$, and the pattern of the motion has the maximum of regularity, since there is also the maximum of correlation between ψ_2 and ψ_1 for whatever ψ_1.

When $P = 0$, ψ_2 and ψ_1 are completely uncorrelated for any ψ_1. In this case, the motion is stochastic. Thus the correct stochasticity limit is

$$P \sim 0 \tag{20}$$

which conversely can be used to calculate those amplitudes $\hat{x}$ by which the motion is erratic.

In practice, (20) is useless, since to calculate P we must know ψ_2 as a function of ψ_1. Several authors replace (20) with the more suitable condition that more resonances overlap each other in order to cause the motion to be stochastic. In a unit interval of ν_o, there are n resonances (7) for every value of n, thus the condition of overlapping is

$$\sum_n n\ \Delta\nu_{nm} \gtrsim 1, \tag{21}$$

where the summation is extended only to those resonances contained in a unit interval of ν_o.

We are not able to give any straight justification of the relation (21) as a consequence of (20), but we shall adopt it as it is commonly accepted.[1,2,6]

Thus, let us calculate the boundary of <u>stochasticity</u> $\hat{x}$, which satisfies the equation

$$\sum_n n \, \Delta\nu_{nm} = 1 \tag{22}$$

Again, we can calculate only the rms average of the expectation value of the function $\sum\limits_n n\Delta\nu_{nm}$. Reminding once more that rms averages sum quadratically, we replace equation (22) by the following

$$\sum_n \; n \; \langle\Delta\nu_{nm}\rangle^2 = 1.$$

From (18), this becomes, finally

$$\frac{4}{\pi^2} \, ML^2 \; \overline{\beta}^2 \; \sum_{k=0}^{\infty} \langle a_k\rangle \; q_k^2 \; \hat{x}^{2(k-1)} = 1 \tag{23}$$

where

$$q_k^2 = \sum_{n=0}^{M/\nu_o} n \; g_{kn}^2 .$$

With a little work,[12] we have

$$q_k^2 = \sum_{n=0}^{\Omega} \frac{n}{4^{k+1}} \left[\begin{pmatrix} k+1 \\ \frac{k+1-n}{2} \end{pmatrix} \right]^2$$

where Ω is the minor of k+1 and M/ν_o, and the summation is extended only to those values of n such that (k+1-n) is an even number. Values of q_k^2 are listed in Table 1 for $M/\nu_o > 20$.

Let us introduce the following polynomial equation of order 2p

$$A^2 \sum_{k=0}^{p} q_k^2 <a_k>^2 \hat{x}^{2k} = \hat{x}^2, \ p \geq 2$$

where

$$A^2 = \frac{4}{\pi^2} M L^2 \bar{\beta}^2.$$

Using the rule of the variation of the signs, we see this equation has only two positive roots if, as we assume,

$$A^2 q_1^2 <a_1>^2 <1.$$

The smaller root is too close to zero. Thus, because of the <u>adiabatic approximation</u> we used in our calculation of section 2, we shall disregard it. Let us denote the larger root by $\hat{x}_p$. This is, finally, the amplitude above which the motion is stochastic if we ignore all the terms with k>p in the summation at the r.h.s. of (23). Hence, we have a series of values

$$\hat{x}_2, \ \hat{x}_3, \ \hat{x}_4, \ldots, \hat{x}_p, \ \hat{x}_{p+1}, \ldots, \hat{x}_\infty$$

where $\hat{x}_\infty$ is the large positive root of the equation (23).

For example, in the particular case $<a_o>$ and $<a_1>$ are neglected, we have

$$\hat{x}_2 = 1/A \ q_2 <a_2> \tag{24}$$

and

$$\hat{x}_3 = \frac{\left\{-(A\ q_2\langle a_2\rangle)^2 + \sqrt{(A\ q_2\langle a_2\rangle)^4 + 4(A\ q_3\langle a_3\rangle)^2}\right\}^{1/2}}{\sqrt{2}\ A\ q_3\langle a_3\rangle} \qquad (25)$$

It can be easily shown that

$$0 \le \hat{x}_\infty < \ldots < \hat{x}_{p+1} < \hat{x}_p < \hat{x}_{p-1} < \ldots < \hat{x}_3 < \hat{x}_2 .$$

Thus every higher order term taken into account in the summation (23) has the effect to push the boundary slightly more toward the center $x = 0$.

4. Application to the NAL Main Ring

We have some information about the field distribution in the main ring magnets up to the third order. Ignoring dipole and quadrupole contribution, we are able to calculate the boundary of stochasticity $\hat{x}_2$ and $\hat{x}_3$, taking into account in the first case (a) only the sextupole field, and in the latter (b) the sextupole and octupole fields together.

The following numbers apply to the injection energy of 7.2 GeV.

(a) Sextupole field. This is essentially due to the remanent field in the dipoles.

It is

$$M = 774, \qquad L = 6\text{m}, \qquad \overline{\beta} = 50\text{m}$$

from which

$$A = 5.3 \times 10^3\ \text{m}^2.$$

The rms value of the sextupole strength is about 0.3 kG/m^2, and the magnet rigidity is $B_o\rho$ = 262 kG·m, which gives

$$\langle a_2 \rangle = 0.0011\ \mathrm{m}^{-3}.$$

We obtain from (24)

$$\hat{x}_2 = 40\ \mathrm{cm}.$$

(b) Octupole field. This is essentially due to the remanent field in the quadrupoles. It is

$$M = 240, \qquad L = 2\mathrm{m}, \quad \text{and} \quad A = 987\ \mathrm{m}^2.$$

The rms value of the octupole field is about 3 kG/m^3, which gives

$$\langle a_3 \rangle = 0.011\ \mathrm{m}^{-4} \quad \text{and} \quad A\ q_3 \langle a_3 \rangle = 5.4\ \mathrm{m}^{-2}.$$

We derive from (25) the new limit

$$\hat{x}_3 = 32\ \mathrm{cm}.$$

We are now interested to calculate the rate by which the boundary of stochasticity reduces when higher-order terms are included in the summation (23). Unfortunately, our information about higher-order field distribution is limited only to detailed measurement made on the field profile of one magnet of type B1 and one magnet of type B2. We shall assume these two magnets are typical and represent well enough in average all the other magnets.

By using the least squares method and the Gauss criterion, we found that the two sets of measurements can be fit by polynomials of fifth order. The coefficients of the polynomials are given in Table 2, together with their quadratic averages which we use in the next calculation. The generic $\langle a_k \rangle$ is given in $(\text{meter})^{-(k+1)}$ unit. We have replaced $\langle a_2 \rangle_{q.a.}$ with the contribution from the remanent field in the dipoles, and we added to $\langle a_3 \rangle_{a.a.}$ the contribution from the remanent field in the quadrupoles. Contributions of the quadrupoles to the higher-order term are ignored. Observe, also, that the kinematic corrections have been ignored in our calculation.

The stochastic limits are listed in Table 3 for $p = 2$ to 5. We infer from these numbers that, likely, in the main ring $\hat{x}_\infty \sim 10$ cm, which is a large number compared to the beam size. But what would the effect on $\hat{x}_\infty$ be if the coupling motion were taken into account?

REFERENCES

1. E. Keil, CERN/ISR-TH/72-7 and CERN/ISR-TH/72-25.

2. B. Chirikov, Research concerning the theory of nonlinear resonance and stochasticity. Novosibirsk 1969 (CERN Trans. 71-40).

3. M.G.N. Hine, Proc. CERN Conference on the Proton 20-25 BeV Strong Focusing Synchrotron (1953).

4. L. J. Laslett, Soviet Conference on Charged-Particle Accelerators. Moscow Oct. 1968, Vol. II, p. 301.

5. K.R. Symon, A.M. Sessler, CERN Symposium 1956, p. 44.

6. B. Chirikov, E. Keil, A. Sessler, ISR-TH/69-59.

7. I. Gumowski, C. Mira, Proc. of the 8th Int. Conf. on High Energy Accelerators., CERN 1971, p. 374.

8. I. Gumowski, CERN/SI/Int. BR/72-1.

9. E. Keil, Proc. of the 8th Int. Conf. on High Energy Accelerators, CERN 1971, p. 372.

10. H.G. Hereward, CERN/ISR-DI/72-26.

11. G. Zaslavsky, B. Chirikov, Dokl. Akad. Nauk. SSSR 169,306 (1964).

12. I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press, New York (1965), p. 374 (3.631.17).

13. H. Hinterberger, et.al. 1971 Particle Accel. Conf., Chicago, Ill., p. 853.

K	q_k	q_k^2
1	0.3536	0.1250
2	0.4330	0.1875
3	0.3750	0.1406
4	0.4193	0.1758
5	0.3827	0.1465
6	0.4134	0.1709
7	0.3867	0.1495
8	0.4102	0.1682
9	0.3891	0.1514
10	0.4081	0.1665
11	0.3907	0.1527
12	0.4067	0.1654
13	0.3919	0.1536
14	0.4056	0.1645
15	0.3928	0.1543
16	0.4048	0.1639
17	0.3934	0.1548
18	0.4042	0.1634
19	0.3940	0.1552
20	0.4037	0.1630

Table 2

k	$\langle a_k \rangle_{B1}$	$\langle a_k \rangle_{B2}$	$\langle a_k \rangle_{q.a}$
0	-	-	-
1	-	-	-
2	$9.5\ 10^{-4}$	$4.3\ 10^{-4}$	$1.1\ 10^{-3}$
3	$2.4\ 10^{-3}$	$3.0\ 10^{-3}$	$4.7\ 10^{-3}$
4	$5.2\ 10^{-2}$	$3.8\ 10^{-2}$	$4.6\ 10^{-2}$
5	$4.6\ 10^{-1}$	$7.2\ 10^{-1}$	$6.0\ 10^{-1}$

Table 3

p	$\hat{x}_p$
2	40 cm
3	28 cm
4	20 cm
5	16 cm

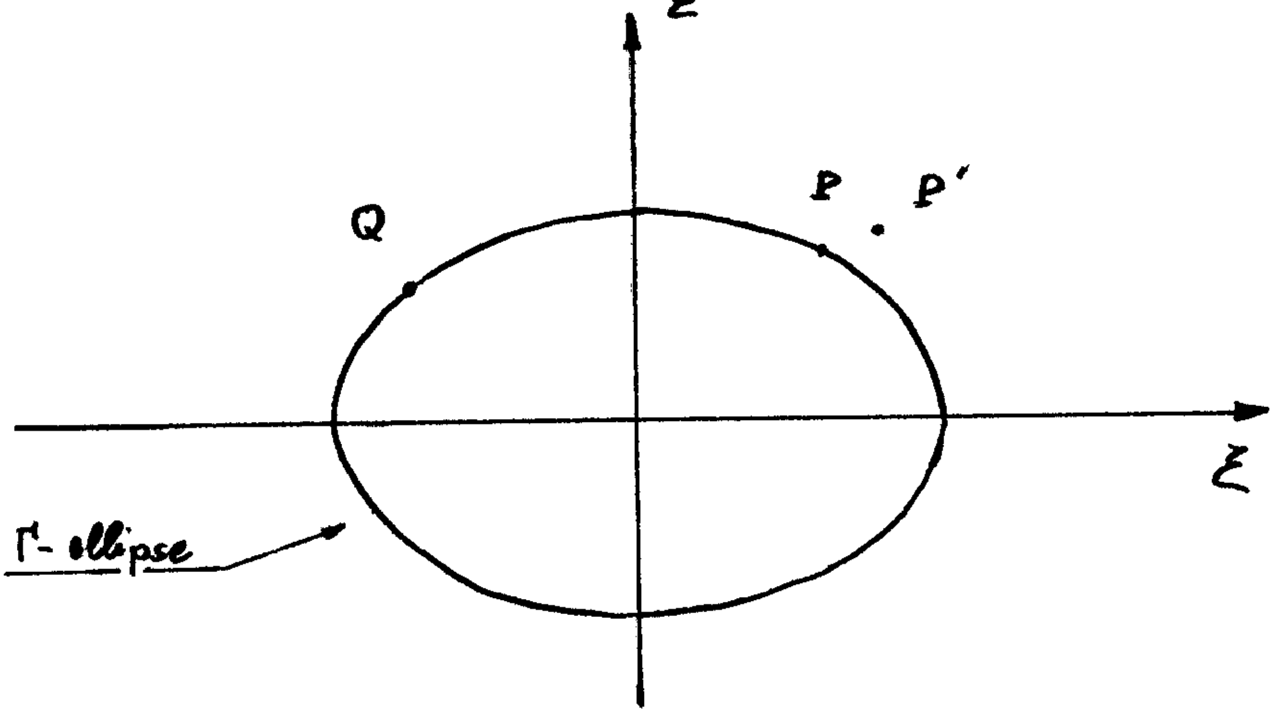

Fig. 1

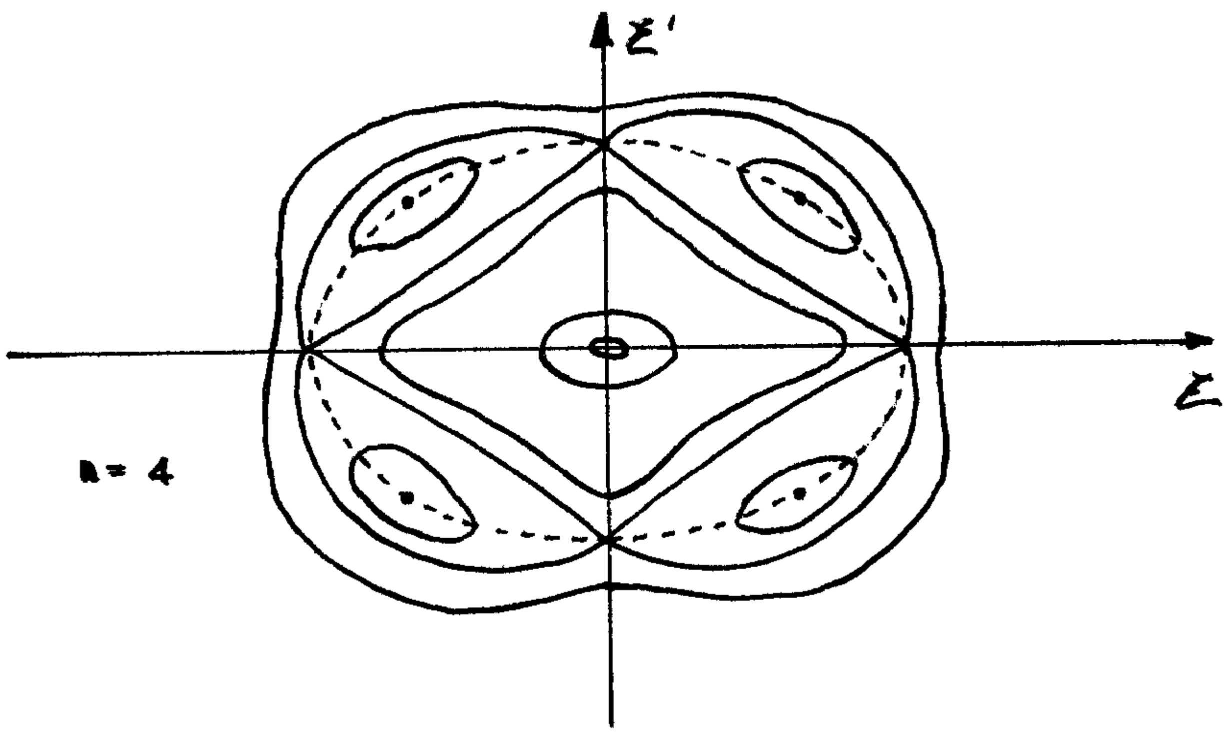

Fig. 2