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CONCERNING n-DIMENSIONAL COUPLED MOTIONS

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In this report we develop some useful formalisms and theorems for the study of general n-dimensional coupled motions, specifically, general electromagnetic couplings between the motions in three dimensions of charged particles in a beam are studied. Sections I and II are heuristic and are included only as a review. The material contained in these sections can be found in e.g. "Group Theory" by M. Hamermesh (Addison Wesley 1962), "Theory of the Alternating-Gradient Synchrotron" by E. D. Courant and H. S. Snyder (Ann. of Phys. 3, 1-48, 1958) and the references given therein.

The bracketed paragraphs are side-line remarks which help to clarify the mainstream discussion.

I. SYMPLECTIC GROUP

The symplectic group of dimension 2n is the group of linear transformations (M) on 2n objects which obey the condition

$$\tilde{M}GM = G \tag{1}$$

where ~ means transposition and G is an arbitrary antisymmetric 2n x 2n matrix. By properly choosing the basis coordinates one can always reduce G to the canonical form

$$S \equiv \left(\begin{array}{cc:cc:c} 0 & 1 & & & \\ -1 & 0 & & & 0 \\ \hdashline & & 0 & 1 & \\ & & -1 & 0 & \\ \hdashline & 0 & & & \text{etc.} \end{array}\right) \tag{2}$$

A more revealing form of S obtained by rearranging rows and columns is

$$\left(\begin{array}{cccc:cccc} & & & & & & & 1 \\ & 0 & & & & & 1 & \\ & & & & & \cdot & & \\ & & & & \cdot & & & \\ & & & & 1 & & & \\ \hdashline & & & -1 & & & & \\ & & \cdot & & & & & \\ & \cdot & & & & & & \\ -1 & & & & & & 0 & \\ -1 & & & & & & & \end{array}\right)$$

The symplectic condition is, therefore

$$\tilde{M}SM = S. \tag{3}$$

For comparison the condition for the orthogonal group (O) can be written as

$$\tilde{O}FO = F$$

where F is an arbitrary <u>symmetric</u> matrix. By properly choosing the basis coordinates we can reduce F to the canonical form, which is the unit matrix I. The orthogonal condition is, therefore, the familiar

$$\tilde{O}IO = I \qquad \text{or} \qquad \tilde{O}O = I.$$

It can be shown that the dimension of the symplectic group must be even (hence, written as 2n) and that the determinant $|M| = +1$ (taking the determinant of Eq. (3) gives only $|M| = \pm 1$). Therefore, there is no "improper" symplectic group as in the case of the orthogonal group.

For 2n dimension the symplectic condition (3) gives $n(2n-1)$ relations. Thus for real symplectic transformations the number of free parameters is

$$(2n)^2 - n(2n-1) = n(2n+1).$$

Let us write out Eq. (3) explicity for n = 1 and 2.

<u>n = 1</u>

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then Eq. (3) gives

$$\tilde{M}SM = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & ad-bc \\ -(ad-bc) & 0 \end{pmatrix}$$

$$\text{should} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

or

$$|M| = 1.$$

Hence, for n = 1 the symplectic group is identical to the unimodular group.

<u>n = 2</u>

$$\text{Let } M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

where $\sigma \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and M_{11}, M_{12}, M_{21} and M_{22} are all 2x2 matrices. Then Eq. (3) gives

$$\tilde{M}SM = \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$\begin{pmatrix} \tilde{M}_{11}\sigma M_{11} + \tilde{M}_{21}\sigma M_{21} & \tilde{M}_{11}\sigma M_{12} + \tilde{M}_{21}\sigma M_{22} \\ \tilde{M}_{12}\sigma M_{11} + \tilde{M}_{22}\sigma M_{21} & \tilde{M}_{12}\sigma M_{12} + \tilde{M}_{22}\sigma M_{22} \end{pmatrix}$$

$$\text{should} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.$$

The two diagonal equations are simply

$$\begin{cases} |M_{11}| + |M_{21}| = 1 \\ |M_{12}| + |M_{22}| = 1. \end{cases}$$

The two off-diagonal equations are the negative-transpose of each other and we have only the one equation

$$\tilde{M}_{11}\sigma M_{12} + \tilde{M}_{21}\sigma M_{22} = 0$$

which gives four relations. These six relations can be summarized as follows:

Define the "sum-of-determinants" (SOD) of any two columns of M as the sum of the two determinants formed by rows 1 and 2 and rows 3 and 4. For example, the "sum-of-determinants" of columns 1 and 2 of M is just $SOD(1,2) = |M_{11}| + |M_{12}|$. Then the six symplectic relations can be written as

$$SOD(i.j) = \begin{cases} 1 & (i,j) = (1,2) \text{ or } (3,4) \\ 0 & \text{all other } (i,j). \end{cases} \tag{4}$$

The extension of Eq. (4) to arbitrarily high n is obvious.

If the 2n objects on which M operates is written as a column vector Φ then

$$\tilde{\Phi}_2 S \Phi_1 = \text{invariant.}$$

For $n = 1$ and $\Phi = \begin{pmatrix} q \\ p \end{pmatrix}$, this gives

$$q_2 p_1 - q_1 p_2 = \text{invariant}$$

namely, the "cross-product" of Φ_1 and Φ_2, or the "area" formed by Φ_1 and Φ_2 is invariant. (Note that $\Phi S \Phi = 0$.)

[In comparison, for the orthogonal group

$$\tilde{\Phi}_2 I \Phi_1 = \tilde{\Phi}_2 \Phi_1 = \text{invariant}$$

namely, the "dot-product" is invariant.]

It is simple to show that eigenvalues of M must be in reciprocal pairs. Furthermore, for real symplectic transformations their eigenvalues must, of course, also be in complex conjugate pairs.

For a given M direct substitution shows that

$$E_k \equiv \tilde{\Phi}(SM^k - \tilde{M}^k S)\Phi = \text{invariant}, \qquad k = 1,2,3.... \tag{6}$$

It can be shown that all bilinear invariants are linear combinations of E_k.

II. JACOBIAN MATRIX AND POINCARÉ INTEGRAL INVARIANTS

"The Jacobian matrix of a canonical transformation is symplectic." We shall indicate the proof only for n = 1.

Let q,p be transformed to Q,P by the generating function G(q,P), so that

$$p = \left[\frac{\partial G}{\partial q}\right]_P \qquad Q = \left[\frac{\partial G}{\partial P}\right]_q$$

where the subscript indicates the variable which is held constant. The Jacobian matrix is

$$J\left(\frac{Q,P}{q,p}\right) \equiv \begin{pmatrix} \left[\frac{\partial Q}{\partial q}\right]_p & \left[\frac{\partial Q}{\partial p}\right]_q \\ \left[\frac{\partial P}{\partial q}\right]_p & \left[\frac{\partial P}{\partial p}\right]_q \end{pmatrix}$$

and

$$\tilde{J}SJ = \begin{pmatrix} 0 & \left[\frac{\partial Q}{\partial q}\right]_p \left[\frac{\partial P}{\partial p}\right]_q - \left[\frac{\partial Q}{\partial p}\right]_q \left[\frac{\partial P}{\partial q}\right]_p \\ \left[\frac{\partial Q}{\partial p}\right]_q \left[\frac{\partial P}{\partial q}\right]_p - \left[\frac{\partial Q}{\partial q}\right]_p \left[\frac{\partial P}{\partial p}\right]_q & 0 \end{pmatrix}.$$

But

$$\begin{aligned}
&\left[\frac{\partial Q}{\partial q}\right]_p \left[\frac{\partial P}{\partial p}\right]_q - \left[\frac{\partial Q}{\partial p}\right]_q \left[\frac{\partial P}{\partial q}\right]_p \\
&\quad = \left[\frac{\partial}{\partial q}\left[\frac{\partial G}{\partial P}\right]_q\right]_p \left[\frac{\partial P}{\partial p}\right]_q - \left[\frac{\partial}{\partial p}\left[\frac{\partial G}{\partial P}\right]_q\right]_q \left[\frac{\partial P}{\partial q}\right]_p \\
&\quad = \left(\left[\frac{\partial}{\partial q}\left[\frac{\partial G}{\partial P}\right]_q\right]_p + \left[\frac{\partial}{\partial P}\left[\frac{\partial G}{\partial P}\right]_q\right]_q \left[\frac{\partial P}{\partial q}\right]_p\right) \left[\frac{\partial P}{\partial p}\right]_q \\
&\qquad\qquad - \left[\frac{\partial}{\partial P}\left[\frac{\partial G}{\partial P}\right]_q\right]_q \left[\frac{\partial P}{\partial p}\right]_q \left[\frac{\partial P}{\partial q}\right]_p \\
&\quad = \left[\frac{\partial}{\partial q}\left[\frac{\partial G}{\partial P}\right]_q\right]_p \left[\frac{\partial P}{\partial p}\right]_q = \left[\frac{\partial}{\partial P}\left[\frac{\partial G}{\partial q}\right]_P\right]_q \left[\frac{\partial P}{\partial p}\right]_q \\
&\quad = \left[\frac{\partial p}{\partial P}\right]_q \left[\frac{\partial P}{\partial p}\right]_q = 1.
\end{aligned}$$

Therefore,

$$\tilde{J}SJ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = S \qquad \text{Q.E.D.}$$

The proof for arbitrary n is similar but more complicated.

Since a Hamiltonian motion is a succession of canonical transformations we have "The Jacobian matrix between two arbitrary times of a Hamiltonian motion is symplectic." This leads to the Poincaré integral invariants.

For an n-dimensional Hamiltonian motion, in the 2n-dimensional phase space of $q_1, p_1, q_2, p_2, \ldots q_n, p_n$, the integrals

$$\left\{\begin{array}{l} P_2 \equiv \displaystyle\iint_{\text{2-D surface}} \sum dq_i dp_i = \iint (dq_1 dp_1 + dq_2 dp_2 + \ldots + dq_n dp_n) \\[2ex] P_4 \equiv \displaystyle\iiiint_{\text{4-D surface}} \sum dq_i dp_i dq_k dp_k \\[2ex] \cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \\[2ex] P_{2n} \equiv \displaystyle\iint_{\text{2n-D volume}} \ldots \int dq_1 dp_1 dq_2 dp_2 \ldots\ldots dq_n dp_n \end{array}\right. \tag{7}$$

are all invariants of motion. Stated in words P_2 = invariant is "The sum of the projections on the n coordinate planes (q_1,p_1), (q_2,p_2), ..., (q_n,p_n) of an arbitrary 2-D surface in the 2n-D phase space is an invariant of motion," and P_{2n} = invariant (the Liouville theorem) is "The volume of an arbitrary 2n-D volume in the 2n-D phase space is an invariant of motion." All other invariants can be similarly stated in words.

To prove the invariance of P_2 let the motion transform q_i,p_i at time t_1 to Q_k,P_k at time t_2. Then denoting by $\frac{\partial(Q,P)}{\partial(q,p)}$ the Jacobian determinant between Q,P and q,p; we have

$$
\begin{aligned}
P_2(t_2) &= \iint (dQ_1 dP_1 + dQ_2 dP_2 + \dots + dQ_n dP_n) \\
&= \iint \left\{ \left|\frac{\partial(Q_1,P_1)}{\partial(q_1,p_1)}\right| dq_1 dp_1 + \dots + \left|\frac{\partial(Q_1,P_1)}{\partial(q_n,p_n)}\right| dq_n dp_n \right. \\
&\quad + \left|\frac{\partial(Q_2,P_2)}{\partial(q_1,p_1)}\right| dq_1 dp_1 + \dots + \left|\frac{\partial(Q_2,P_2)}{\partial(q_n,p_n)}\right| dq_n dp_n \\
&\quad + \dots \\
&\quad \left. + \left|\frac{\partial(Q_n,P_n)}{\partial(q_1,p_1)}\right| dq_1 dp_1 + \dots + \left|\frac{\partial(Q_n,P_n)}{\partial(q_n,p_n)}\right| dq_n dp_n \right\} \\
&= \iint \Big\{ [\mathrm{SOD}(q_1,p_1) \text{ of } J] dq_1 dp_1 + [\mathrm{SOD}(q_2,p_2) \text{ of } J] dq_2 dp_2 \\
&\quad + \dots + [\mathrm{SOD}(q_n,p_n) \text{ of } J] dq_n dp_n \Big\} \\
&= \iint (dq_1 dp_1 + dq_2 dp_2 + \dots + dq_n dp_n) = P_2(t_1)
\end{aligned}
$$

since J is symplectic. The proofs for the higher invariants are similar but more complex. The proof for P_{2n} (the Liouville invariant) is, however, very simple. It is a direct consequence of Jacobian determinant = $|J| = 1$.

As an example of application consider the coupled transverse motions of an unaccelerated particle beam (phase space x, x', y, y'). If, initially, the particles have no y-motion (y = y' = 0) and populate an area A in the x,x' plane, the invariance of P_2 states that at any later time the sum of the emittances (projection areas) of the beam in x,x' and y,y' is always equal to A. Note here, however, that the projection areas (emittances) must be taken algebraically with proper signs. This is a rather severe limitation on the usefulness of all the invariants except the widely applied Liouville invariant.

It is interesting to point out that for linear motion the above theorems become very transparent. A general quadratic 2n-dimensional Hamiltonian can be written as

$$\mathcal{H} = \frac{1}{2}\,\tilde{\Phi}H\Phi \qquad (8)$$

where $\tilde{\Phi} \equiv (q_1\ p_1\ q_2\ p_2\ \cdots\ q_n\ p_n)$ and $H = H(t)$ is a general <u>symmetric</u> 2n x 2n matrix. The canonical equations of motion can be written as (dot $= \frac{d}{dt}$)

$$\dot{\Phi} = SH\Phi. \qquad (9)$$

If Φ_1 and Φ_2 are two solutions of Eq. (9)

$$\tilde{\Phi}_2 S\Phi_1 = \text{invariant}$$

because

$$\begin{aligned}\frac{d}{dt}\left(\tilde{\Phi}_2 S\Phi_1\right) &= \dot{\tilde{\Phi}}_2 S\Phi_1 + \tilde{\Phi}_2 S\dot{\Phi}_1\\ &= \tilde{\Phi}_2\tilde{H}\tilde{S}S\Phi_1 + \tilde{\Phi}_2 SSH\Phi_1\\ &= \tilde{\Phi}_2\tilde{H}\Phi_1 - \tilde{\Phi}_2 H\Phi_1 = 0.\end{aligned}$$

The transfer matrix M is symplectic because it leaves $\tilde{\Phi}_2 S\Phi_1$ invariant. For this case of linear motion the transfer matrix M is identical to the Jacobian matrix J.

III. PARAMETRIZATION OF SYMPLECTIC MATRICES

A. For $n = 1$ the 2×2 real symplectic matrix is unimodular and has three parameters. The parametrization was given by Courant and Snyder

$$M_2 = \cos\mu + \begin{pmatrix} \alpha & \beta \\ -\frac{1+\alpha^2}{\beta} & -\alpha \end{pmatrix} \sin\mu . \qquad (10)$$

B. For $n = 2$ the 4×4 real symplectic matrix has ten parameters. When the two spatial dimensions are uncoupled the matrix is

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$$

where X and Y are 2×2 symplectic (unimodular) matrices. This contains six parameters. For coupling we can use the "symplectic rotation" matrix

$$\begin{pmatrix} \cos\theta & K^{-1}\sin\theta \\ -K\sin\theta & \cos\theta \end{pmatrix}$$

where K is again a 2×2 symplectic matrix. This matrix contains four parameters and is symplectic as can be shown by direct substitution into Eq. (3). The parametrized form of a general 4×4 symplectic matrix can, therefore, be written as

$$M_4 = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} \cos\theta & K^{-1}\sin\theta \\ -K\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} X\cos\theta & XK^{-1}\sin\theta \\ -YK\sin\theta & Y\cos\theta \end{pmatrix} \qquad (11)$$

when θ measures the "strength" and K gives the "structure" of the coupling.

The parametrized form can equally well be written as

$$M_4 = \begin{pmatrix} \cos\theta & L^{-1}\sin\theta \\ -L\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = \begin{pmatrix} X\cos\theta & L^{-1}Y\sin\theta \\ -LX\sin\theta & Y\sin\theta \end{pmatrix} \quad (12)$$

where L is related to K by

$$L = YKX^{-1}. \quad (13)$$

We can also parametrize the matrix in the "symplectic rotation" form, namely,

$$M_4 = \begin{pmatrix} \cos\alpha & -R^{-1}\sin\alpha \\ R\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \cos\alpha & R^{-1}\sin\alpha \\ -R\sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} A\cos^2\alpha + R^{-1}BR\sin^2\alpha & (AR^{-1} - R^{-1}B)\sin\alpha\cos\alpha \\ (RA - BR)\sin\alpha\cos\alpha & B\cos^2\alpha + RAR^{-1}\sin^2\alpha \end{pmatrix}. \quad (14)$$

The parameters θ, X, Y and K are related to α, A, B and R by

$$\begin{cases} \sin^2\theta = [2 - \mathrm{Tr}D]\sin^2\alpha\cos^2\alpha \\ X\cos\theta = (AR^{-1}\cos^2\alpha + R^{-1}B\sin^2\alpha)R \\ Y\cos\theta = R(R^{-1}B\cos^2\alpha + AR^{-1}\sin^2\alpha) \\ K\cos\theta = \dfrac{1}{(2 - \mathrm{Tr}D)^{1/2}}[(1-D)\cos^2\alpha - (1-D^{-1})\sin^2\alpha]R \end{cases} \quad (15)$$

where

$$D \equiv B^{-1}RAR^{-1} \qquad \mathrm{Tr}D \equiv \text{Trace of } D.$$

C. For n = 3 the 6 x 6 real symplectic matrix has 21 parameters. The general parametrized form corresponding to Eq. (11) is obviously

$$M_6 = \begin{pmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix} \begin{pmatrix} \cos\theta_1 & K_1^{-1}\sin\theta_1 & 0 \\ -K_1\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & K_2^{-1}\sin\theta_2 \\ 0 & -K_2\sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_3 & 0 & -K_3\sin\theta_3 \\ 0 & 1 & 0 \\ K_3^{-1}\sin\theta_3 & 0 & \cos\theta_3 \end{pmatrix} \tag{16}$$

where X, Y, Z, K_1, K_2 and K_3 are all 2 x 2 symplectic matrices. The significances of these matrices and θ_1, θ_2, and θ_3 are clear. The extension to matrices with n > 3 is obvious.

The parametrized forms exhibit the effect of coupling explicity, thereby facilitating interpretation. For example, for a 2-dimensional motion (n = 2) if the 4 x 4 transfer matrix is parametrized in the form of Eq. (11) it becomes immediately evident that to decouple the motions one should multiply the transfer matrix from the right by

$$\begin{pmatrix} \cos\theta & K^{-1}\sin\theta \\ -K\sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & -K^{-1}\sin\theta \\ K\sin\theta & \cos\theta \end{pmatrix} . \tag{17}$$

The "symplectic rotation" form is useful in many applications. For example, if the transverse motions of a beam through an uncoupled transport section is given by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. When this transport section is rotated about the beam as axis through an angle α the transfer matrix becomes

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{18}$$

which is in the form of Eq. (14). Eq. (15) can be used to put Eq. (18) into the forms of Eq. (11). Combined with Eq. (17) this gives a simple recipe for designing a decoupler which is obtained by rotating an uncoupled transport section.

IV. ELECTROMAGNETIC COUPLER (OR DECOUPLER) FOR MOTIONS OF CHARGED PARTICLES IN A BEAM

We assume that the centroid of the bunch of charged particles (charge e, rest mass m) to be moving in the z direction with velocity βc (c = speed of light). The quantities in the rest-frame (subscript o) are related to those in the lab-frame (no subscript) by the Lorentz transformation.

Field Intensities

$$\begin{cases} E_{xo} = \gamma(E_x - \beta B_y) \\ E_{yo} = \gamma(E_y + \beta B_x) \\ E_{zo} = E_z \end{cases} \qquad \begin{cases} B_{xo} = \gamma(B_x + \beta E_y) \\ B_{yo} = \gamma(B_y - \beta E_x) \\ B_{zo} = B_z \end{cases} \tag{19}$$

Potentials

$$\begin{cases} A_{xo} = A_x \\ A_{yo} = A_y \\ A_{zo} = \gamma(A_z - \beta\phi) \end{cases} \qquad \phi_o = \gamma(\phi - \beta A_z) \tag{20}$$

Coordinates

$$\begin{cases} x_o = x \\ t_o = \gamma(t - \beta\frac{z}{c}) \end{cases} \qquad y_o = y \qquad z_o = \gamma(z - \beta ct) \tag{21}$$

The inverse transformations are obtained simply by reversing the sign of β $\left(\gamma \equiv (1-\beta^2)^{-1/2}\right)$.

These relations can also be put in the more general vectorial forms

$$\begin{cases} \vec{E}_o = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - (\gamma-1)\hat{\beta}\ \hat{\beta}\cdot\vec{E} \\ \vec{B}_o = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - (\gamma-1)\hat{\beta}\ \hat{\beta}\cdot\vec{B} \end{cases}$$

$$\begin{cases} \vec{A}_o = \gamma(\vec{A} - \vec{\beta}\phi) + (\gamma-1)\hat{\beta}\times(\hat{\beta}\times\vec{A}) \\ \phi_o = \gamma(\phi - \vec{\beta}\cdot\vec{A}) \end{cases}$$

$$\begin{cases} \vec{r}_o = \gamma(\vec{r} - \vec{\beta}ct) + (\gamma-1)\hat{\beta}\times(\hat{\beta}\times\vec{r}) \\ t_o = \gamma\left(t - \frac{\vec{\beta}\cdot\vec{r}}{c}\right) \end{cases}$$

To exhibit the general features it is adequate to consider only motions linear in the coordinates x_o, y_o, z_o and their conjugate variables p_{xo}, p_{yo}, p_{zo}. The Hamiltonian in the rest-frame is, then,

$$\begin{aligned} H_o &= \sqrt{m^2c^4 + \left(\vec{p}_o - \frac{e}{c}\vec{A}_o\right)^2 c^2} + e\phi_o \\ &\cong \frac{\left(\vec{p}_o - \frac{e}{c}\vec{A}_o\right)^2}{2m} + e\phi_o \end{aligned} \tag{22}$$

where in ϕ_o only terms up to the second degree in x_o, y_o, z_o are kept and in $\vec{A}_o$ only terms up to the first degree are retained. There are two general types of coupling (electric and magnetic). We shall consider these two types for the cases of transverse-transverse (x,y) coupling and transverse-longitudinal (y,z) coupling.

A. TRANSVERSE-TRANSVERSE x,y COUPLING

1. Electric Coupling

This is the coupling where

$$\begin{cases} \phi_o = a\, x_o\, y_o \\ \vec{A}_o = 0. \end{cases} \tag{23}$$

When only linear motions are considered, this is provided by a simple 45° skew quadrupole magnet in the lab-frame. In the lab-frame write

$$\begin{cases} B_x = -\frac{a}{\beta\gamma}\, x \\ B_y = \frac{a}{\beta\gamma}\, y \\ B_z = 0. \end{cases} \qquad \vec{E} = 0 \tag{24}$$

Transformed to the rest frame this gives

$$\begin{cases} E_{xo} = -a\, y_o \\ E_{yo} = -a\, x_o \\ E_{zo} = 0 \end{cases} \qquad \begin{cases} B_{xo} = -\frac{a}{\beta}\, x_o \\ B_{yo} = \frac{a}{\beta}\, y_o \\ B_{zo} = 0 \end{cases}$$

or

$$\begin{cases} \phi_o = a\, x_o\, y_o \\ A_{xo} = A_{yo} = 0, \quad A_{zo} = -\frac{a}{\beta}\, x_o\, y_o \simeq 0 \text{ (higher order)} \end{cases}$$

which is the same as Eq. (23).

2. Magnetic Coupling

This is the coupling where

$$\begin{cases} A_{xo} = -\frac{b}{2}\, y_o \\ A_{yo} = \frac{b}{2}\, x_o \\ A_{zo} = 0 \end{cases} \qquad \phi_o = 0 \tag{25}$$

and corresponds to <u>a longitudinal magnetic field in the lab-frame</u>, namely,

$$\begin{cases} B_z = B_{zo} = b \\ B_x = B_y = B_{xo} = B_{yo} = 0 \\ \vec{E} = \vec{E}_o = 0. \end{cases} \tag{26}$$

B. <u>TRANSVERSE-LONGITUDINAL y,z COUPLING</u>

1. <u>Electric Coupling</u>

We want

$$\begin{cases} \phi_o = a\, y_o\, z_o \\ \vec{A}_o = 0. \end{cases} \tag{27}$$

In the lab-frame this corresponds to the travelling fields

$$\begin{cases} E_x = 0 \\ E_y = -a\gamma^2 (z-\beta ct) \\ E_y = -ay \end{cases} \qquad \begin{cases} B_x = a\beta\gamma^2 (z-\beta ct) \\ B_y = 0 \\ B_z = 0 \end{cases} \tag{28}$$

which has to be produced by an RF cavity. For small $y_o = y$ and $z_o = \gamma(z - \beta ct)$ (More precisely, $ky_o << 2\pi$, $kz_o << 2\pi$. These conditions will determine the choice of the wave number k.) we can approximate this field by

$$\begin{cases} E_x = 0 \\ E_y = -\frac{a}{k}\,\gamma \cosh ky \sin k\gamma(z-\beta ct) \\ E_z = -\frac{a}{k} \sinh ky \cos k\gamma(z-\beta ct) \end{cases}$$

$$\begin{cases} B_x = \frac{a}{k}\,\beta\gamma \cosh ky \sin k\gamma(z-\beta ct) \\ B_y = B_z = 0. \end{cases}$$

Furthermore, this field is just the +z travelling wave component of the standing wave

$$\begin{cases} E_x = 0 \\ E_y = 2\frac{a}{k}\,\gamma \cosh ky \cos k\gamma z \sin \omega t \\ E_z = -2\frac{a}{k} \sinh ky \sin k\gamma z \sin \omega t \end{cases} \tag{29}$$

$$\begin{cases} B_x = 2\frac{a}{k}\,\beta\gamma \cosh ky \sin k\gamma z \cos \omega t \\ B_y = B_z = 0 \end{cases}$$

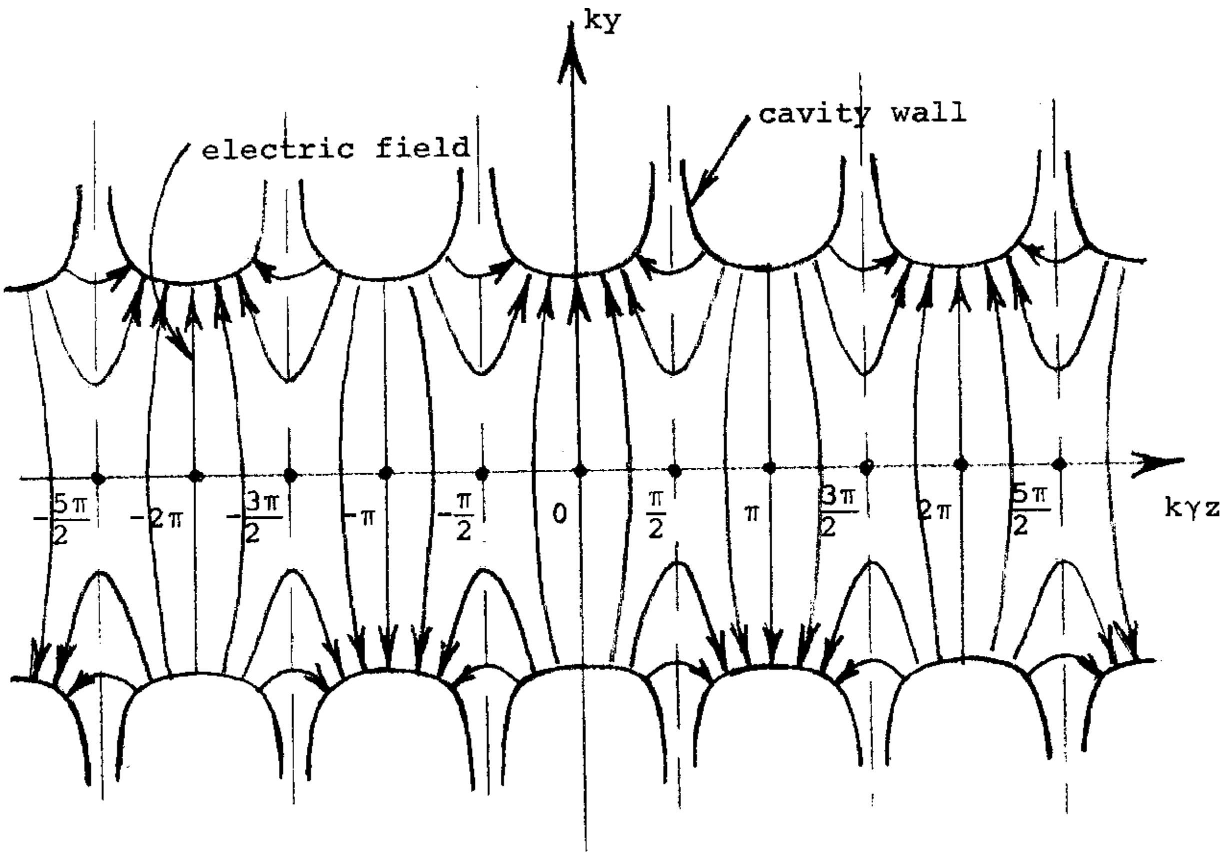
where $\omega \equiv \beta\gamma kc$ is the angular frequency. The cavity (assumed to be perfectly conducting) producing this standing wave is uniform along x and has a y,z shape given by

$$\frac{dy}{dz} = -\frac{E_z}{E_y} = \frac{\sinh ky \sin k\gamma z}{\gamma \cosh ky \cos k\gamma z}$$

or

$$(\sinh ky)^{\gamma^2} = \frac{\text{constant}}{\cos k\gamma z}$$

which looks like



This cavity is a little unrealistic, especially the uniformity along x. But there is presumably no difficulty in making it more realistic without spoiling the approximate field for small x_o, y_o and z_o. By analogy we may call this a "travelling y,z electric quadrupole."

2. Magnetic Coupling

We want

$$\begin{cases} A_{xo} = 0 \\ A_{yo} = -\frac{b}{2} z_o \\ A_{zo} = \frac{b}{2} y_o . \end{cases} \qquad \phi_o = 0 \tag{30}$$

In the rest-frame this is just a $B_{xo} = b$ field. In the lab-frame this corresponds to the uniform and static fields

$$\begin{cases} E_x = 0 \\ E_y = -b\beta\gamma \\ E_z = 0 \end{cases} \qquad \begin{cases} B_x = b\gamma \\ B_y = 0 \\ B_z = 0 \end{cases} \tag{31}$$

which is just a simple transverse cross electric and magnetic field and is produced in an ordinary cross-field electrostatic particle separator.

In contrast to transverse-transverse coupling for transverse-longitudinal coupling electric fields either dc or rf are needed in the lab-frame. Since electric field intensities are rather severely limited by breakdowns it is difficult in practice to provide strong transverse-longitudinal coupling.

To see the effects due to these two types of coupling it suffices to write down the 4 x 4 transfer matrices for δ-function fields (thin coupling lenses). In the rest-frame take the two coupled coordinates to be x and y (we shall drop the subscript o from now on). The second order Hamiltonian from Eq. (22) is

$$H = \frac{1}{2m}\left[\left(p_x - \frac{e}{c}A_x\right)^2 + \left(p_y - \frac{e}{c}A_y\right)\right] + e\phi. \tag{32}$$

Electric Coupling Thin Lens

$$H = \frac{1}{2m}\left(p_x^{\,2} + p_y^{\,2}\right) + eaxy \tag{33}$$

and the equations of motion are

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{p_x}{m} \\[2ex] \dfrac{dp_x}{dt} = -eay \end{cases} \qquad \begin{cases} \dfrac{dy}{dt} = \dfrac{p_y}{m} \\[2ex] \dfrac{dp_y}{dt} = -eax . \end{cases} \tag{34}$$

For a thin lens the transfer matrix is

$$M_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \bar{a} & 0 \\ 0 & 0 & 1 & 0 \\ \bar{a} & 0 & 0 & 1 \end{pmatrix} \quad \text{where} \quad \bar{a} \equiv -e \int a dt . \tag{35}$$

Magnetic Coupling Thin Lens

$$H = \frac{1}{2m} \left[\left(p_x + \frac{eb}{2c} y \right)^2 + \left(p_y - \frac{eb}{2c} x \right)^2 \right] \tag{36}$$

and the equations of motion are

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{m} \left(p_x + \dfrac{eb}{2c} y \right) \\[2ex] \dfrac{dp_x}{dt} = \dfrac{eb}{2mc} \left(p_y - \dfrac{eb}{2c} x \right) \end{cases} \qquad \begin{cases} \dfrac{dy}{dt} = \dfrac{1}{m} \left(p_y - \dfrac{eb}{2c} x \right) \\[2ex] \dfrac{dp_y}{dt} = -\dfrac{eb}{2mc} \left(p_x + \dfrac{eb}{2c} y \right) . \end{cases} \tag{37}$$

In this case it is simpler to use the vector $(x\ \dot{x}\ y\ \dot{y})$. The equations of motion are then

$$\begin{cases} \dfrac{dx}{dt} = \dot{x} \\[2ex] \dfrac{d\dot{x}}{dt} = \dfrac{eb}{mc} \dot{y} \end{cases} \qquad \begin{cases} \dfrac{dy}{dt} = \dot{y} \\[2ex] \dfrac{d\dot{y}}{dt} = -\dfrac{eb}{mc} \dot{x} \end{cases} \tag{38}$$

and for a thin lens the transfer matrix is

$$M_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \bar{b} \\ 0 & 0 & 1 & 0 \\ 0 & -\bar{b} & 0 & 1 \end{pmatrix} \quad \text{where } \bar{b} \equiv \frac{e}{mc} \int b dt \ . \tag{39}$$

It is interesting and useful to work out the cases of uniform thick lenses where a and b are constant over a given length ℓ along the beam in the lab-frame. The computation is straightforward but tedious and is left to the reader. It suffices to point out here that each such thick coupling lens supplies two adjustable parameters--the length (ℓ) and the strength (a or b). In order to produce an arbitrary desired coupling with four parameters [θ and three parameters in K from Eq. (11)] one needs a minimum of two thick lenses.

V. FURTHER DISCUSSION ON MAGNETIC COUPLING

In this section we discuss a different way of visualizing the magnetic coupling which gets away from the linear approximation and is useful in some applications. In the rest-frame the effect of a magnetic field is simply to couple the motions in the two dimensions transverse to the field through the cyclotron rotation about the field. The exact equation of motion (still in the rest-frame, therefore, $\gamma \approx 1$) is well known.

$$\dot{\vec{v}} = \frac{e}{mc} \vec{v} \times \vec{B} \tag{40}$$

where $\vec{v}$ is the velocity. This equation has the same form as the precession equation for a magnetic moment and shows that $\vec{v}$ simply precesses about $\vec{B}$ with the angular velocity $\omega = \frac{eB}{mc}$ = independent of v. Since the position of the particle is dependent on $\vec{v}$ our intuitive feelings of the precessional motion of a magnetic moment is most directly applicable to that of $\vec{v}$ when $\vec{B}$ is spatially uniform. Several interesting conclusions follow immediately.

A. If in the rest-frame of the beam bunch the velocities of all the particles are along the same direction (the velocity line) one can precess the velocity line to whatever orientation by a proper uniform magnetic field.

B. If the velocities of all the particles are in a given plane with normal $\hat{n}$ one can precess the velocity plane to any orientation by a proper uniform magnetic field. To see this, one has only to show that the normal $\hat{n}$ of the velocity plane obeys the same precession equation.

The equation of the velocity plane is

$$\hat{n}\cdot\vec{v} = 0.$$

Differentiating this equation and substituting Eq. (40) we get

$$0 = \frac{d}{dt}(\hat{n}\cdot\vec{v}) = \dot{\hat{n}}\cdot\vec{v} + \hat{n}\cdot\dot{\vec{v}}$$

$$= \dot{\hat{n}}\cdot\vec{v} + \frac{e}{mc}\,\hat{n}\cdot(\vec{v}\times\vec{B}) = \left[\dot{\hat{n}} - \frac{e}{mc}(\hat{n}\times\vec{B})\right]\cdot\vec{v}.$$

This is true for all $\vec{v}$ in the velocity plane, therefore, the vector in the bracket must be $\perp$ to the velocity plane, namely, along $\hat{n}$. But neither $\dot{\hat{n}}$ (because $\hat{n}$ is a unit vector) nor $\hat{n} \times B$ has a component along $\hat{n}$. Hence, the vector in the bracket must vanish identically giving

$$\dot{\hat{n}} = \frac{e}{mc} \hat{n} \times \vec{B}. \tag{41}$$

C. This same transition from vectors-in-plane to normal-of-plane can be repeated over and over again. Namely, if the normals $\hat{n}$ of many velocity planes themselves lie in a plane ($\hat{n}$-plane) with a normal $\hat{\tau}$. The equation for $\hat{\tau}$ is again the same as that for $\hat{n}$.

One of the main purposes to consider coupling is to change the emittances in individual dimensions. This means manipulating the 2n-dimensional phase space volume to change its projectional areas on individual coordinate planes in conformity with the Poincaré invariants. For simplicity, we consider only $n = 2$ (say, x and y). The magnetic coupling by a uniform B_z is simple enough for visualizing the design of a system which reduces the x-emittance at the expense of increasing the y-emittance.

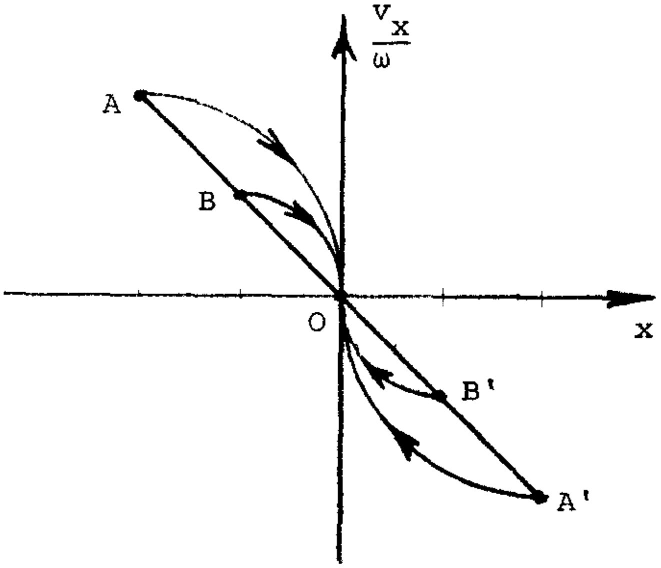
To further simplify the problem we assume the initial emittances to be "first-order zeros," i.e., areas of lines. The motions in a uniform magnetic field B_z are

$$\begin{cases} v_x = v_i \cos \omega t \\ v_y = v_i \sin \omega t \end{cases} \qquad \begin{cases} x = x_i + \dfrac{v_i}{\omega} \sin \omega t \\ y = y_i + \dfrac{v_i}{\omega}(1 - \cos \omega t) \end{cases} \tag{40}$$

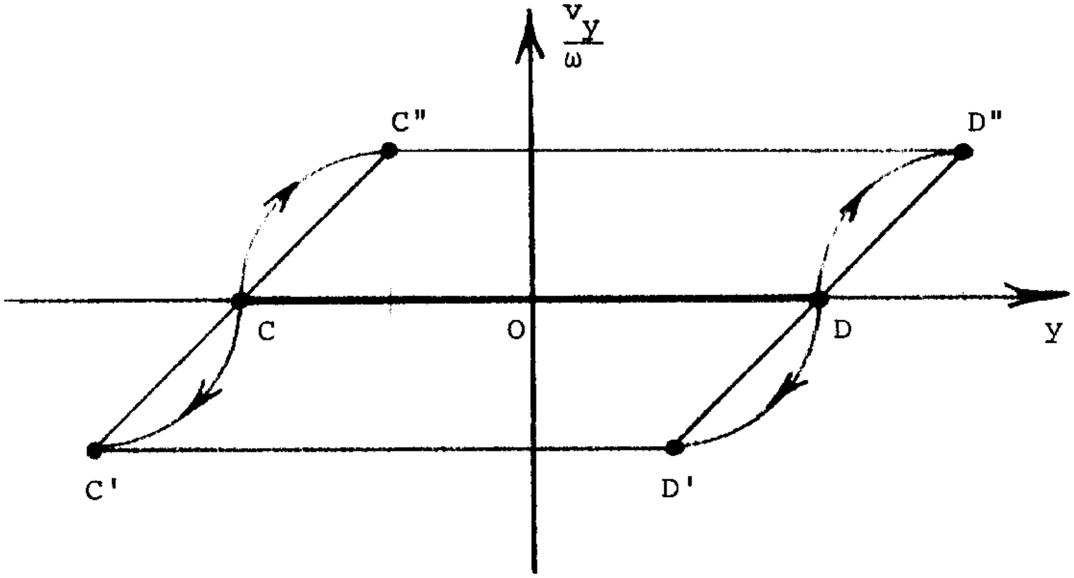
where $\omega = \frac{eB_z}{mc}$ and $x = x_i$, $y = y_i$, $v_x = v_i$, $v_y = 0$ at $t = 0$ when B_z is turned on. At $\omega t = \frac{\pi}{2}$, $v_x = 0$, $x_f = x_i + \frac{v_i}{\omega}$ and B_z is turned off. In order that $x_f = 0$ we must have

$$x_i + \frac{v_i}{\omega} = 0$$

which is shown as line AOA' in the $x, \frac{v_x}{\omega}$ plot below. All phase points starting on line AOA' at $t = 0$ traverse quarter circles and end up at the origin O at $\omega t = \frac{\pi}{2}$. We have thus reduced the x-emittance (from the 1st order zero area of a line to the 2nd order zero area of a point). In the $y, \frac{v_y}{\omega}$ plane the points C ($y = y_i$, $v_y = 0$)



with different values of $v_x = v_i$ will again traverse quarter circles and end up along C'CC" at $\omega t = \frac{\pi}{2}$. The y-emittance is therefore increased by the area C'C"D"D'. This is not, however, in violation of the Poincaré invariant P_2 because when proper signs are taken into account the area CC"D"D above the y-axis is positive and the area C'CDD' below the y-axis is negative. The algebraic total of the y, $\frac{v_y}{\omega}$ area increase is, therefore, zero.



Thus, at $t = 0$, B_z is turned on; and the beam should be large and focusing in x with $\frac{v_x}{x} = \omega \equiv \frac{eB_z}{mc}$, and large and parallel ($v_y = 0$) in y. The x-emittance will be a minimum at $\omega t = \frac{\pi}{2}$ when B_z should be turned off.

The most interesting application is in coupling the longitudinal motion to a transverse motion as described in B.2 of Section IV. By applying the scheme twice we can, in principle, reduce the beam emittances in both transverse dimensions to arbitrarily small values at the expense of increasing the longitudinal emittance, namely, increasing either the bunch length or the momentum spread.

For transverse-longitudinal coupling (say, between y_o and z_o following the notation of B.2 of Section IV and restoring the subscript o for the rest-frame) the coupling strength is limited by the maximum value of E_y attainable in the lab-frame. Taking an optimistic value of E_y = 300 kV/cm = 1000 esu and a proton beam of, say, 200 MeV (β = 0.566, γ = 1.213) we get b = 1.456 kG. If the crossed E_y, B_x field extends in the lab-frame over a length z along the beam we get

$$\omega_o t_o = \frac{\omega_o}{\gamma} t = \frac{\omega_o}{c\beta\gamma} z = \frac{eb}{mc^2\beta\gamma} z = \frac{b}{(B\rho)} z$$

where $(B\rho)$ is the usual magnetic rigidity of the particle. For 200 MeV protons $(B\rho)$ = 21.5 kGm and to get $\omega_o t_o = \frac{\pi}{2}$ we must have z = 23.2 m which is rather long.

In view of the rather weak maximum available transverse-longitudinal coupling, its application to a circular accelerator or a storage ring where the beam passes repeatedly through the coupler would be more practical.