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NUMERICAL STUDY OF BEAM BUNCH LENGTH MATCHING AT TRANSITION USING THE γ_t-JUMP METHOD

W. Lee and L. C. Teng

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The bunch length equation (envelope equation) near transition for phase oscillation with space charge was derived by A. Sørenssen (CERN MPS/Int. MU/EP 67-2, 1967) and used by E. D. Courant (NAL Report FN-187, 1969) for a preliminary study of the γ_t-jump scheme for bunch length matching at transition. For completeness we shall give a brief outline of the derivation of the envelope equation here.

The equations for small phase oscillations of individual particles are

$$\begin{cases} \dfrac{d\psi}{dt} = aw & a = a(t) = \dfrac{h^2}{mR^2\gamma}\left(\dfrac{1}{\gamma_t^2} - \dfrac{1}{\gamma^2}\right) \\ \\ \dfrac{dw}{dt} = -b\psi & b = b(t,\theta) \end{cases} \qquad (1)$$

where ψ and w are small deviations of the phase and the conjugate "momentum" variable from their synchronous values. The first equation is a kinematic (geometric) relationship and does not depend on the space charge. The second is a dynamic equation where b is the sum of two terms:

RF term: This term is proportional to the time-slope of the cavity voltage at the synchronous phase and, therefore, changes sign at the RF phase jump.

Space charge term: This term is proportional to the second derivative of the linear charge density of the beam bunch. For the lowest moment of the linear charge density this term is proportional to $\frac{1}{\theta^3}$ where θ is the bunch half-length (envelope of ψ).

The envelope equation of Eq. (1) is

$$\frac{d}{dt}\left(\frac{1}{a}\frac{d\theta}{dt}\right) + b\theta - \left(\frac{S}{\pi}\right)^2 \frac{a}{\theta^3} = 0 \qquad (2)$$

where $S(>0)$ is the phase space area occupied by the beam bunch in $w\psi$ unit. At transition $a = 0$, $t = 0$. Near transition a is approximately proportional to t. Since the dependence of a on the transition γ_t is through the factor $\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$, to incorporate γ_t-jump we shall write

$$a = \frac{2h^2\dot{\gamma}}{mR^2\gamma^4}\left(1 - \frac{\dot{\gamma}_t}{\dot{\gamma}}\right) t \equiv a_o A t \qquad a_o = \text{const} > 0.$$

Without γ_t-jump, $A = 1$ and a_o is just $\dot{a}$. With γ_t-jump $A = A(t) > 1$. To simplify the problem we shall assume that the γ_t-jump is turned on and off suddenly and that during the jump $A = \text{const} > 1$.

For b we shall write

$$b = \pm b_o + \frac{B}{\theta^3} = b_o\left(\delta + \frac{B}{b_o\theta^3}\right)$$

where

$$\begin{cases} b_o = \dfrac{ev}{2\pi h} \; |\cos \phi_s| = \text{const} > 0. \\[2ex] \delta = \begin{cases} +1 & \cos \phi_s < 0, \text{ after RF phase jump} \\ -1 & \cos \phi_s > 0, \text{ before RF phase jump} \end{cases} \\[2ex] B = \dfrac{3}{2} \; \dfrac{e^2 Ng}{R\gamma^2} > 0 \;\; (g = \text{geometrical factor} \approx 4.5) \end{cases}$$

The envelope equation (2) becomes

$$\frac{d}{dt} \left(\frac{1}{t} \frac{d\theta}{dt} \right) + a_o b_o A \left(\delta + \frac{B}{b_o \theta^3} \right) \theta - \left(\frac{a_o S}{\pi} \right)^2 A^2 \frac{t}{\theta^3} = 0. \quad (3)$$

Now, we change the scale of t and θ to kill the complicated constants. Let

$$t = Tx \quad \text{and} \quad \theta = Ky$$

and rewrite Eq. (3) as

$$\frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dx} \right) + a_o b_o T^3 A \left(\delta + \frac{B}{b_o K^3 y^3} \right) y - \left(\frac{a_o S}{\pi} \frac{T^2}{K^2} \right)^2 A^2 \frac{x}{y^3} = 0. \quad (4)$$

Setting $a_o b_o T^3 = 1$ and $\dfrac{a_o S}{\pi} \dfrac{T^2}{K^2} = 1$ we get

$$T^3 = \frac{1}{a_o b_o}, \quad K^3 = T^3 \left(\frac{a_o S}{\pi} \right)^{3/2} = \frac{a_o^{1/2}}{b_o} \left(\frac{S}{\pi} \right)^{3/2}$$

and the equation

$$\left(\frac{y'}{x} \right)' + A \left(\delta + \frac{C}{y^3} \right) y - A^2 \frac{x}{y^3} = 0 \quad (5)$$

where prime means $\frac{d}{dx}$, and

$$\left\{\begin{aligned} \delta &= \begin{cases} +1 & \cos\phi_s < 0,\ \text{after RF phase jump} \\ -1 & \cos\phi_s > 0,\ \text{before RF phase jump} \end{cases} \\ A &= 1 - \frac{\dot{\gamma}_t}{\dot{\gamma}} \\ C &= \frac{B}{b_o K^3} = \frac{B}{a_o^{1/2}} \left(\frac{\pi}{S}\right)^{3/2} \end{aligned}\right.$$

When all the parameters are substituted and reduced we get

$$T^3 = \frac{1}{2} \frac{\beta^2 \gamma^4}{\omega^3} \left(\frac{\omega}{\dot{\gamma}}\right)^2 \frac{1}{|\cot \phi_s|}$$

(T = same time scale factor used by Sørenssen)

$$K = \frac{\theta_o}{k}$$

$$C = k^3 \eta_o$$

where

$$k = \frac{3^{1/6}}{\pi^{1/2}} \Gamma(2/3) = 0.91749$$

$$\theta_o^3 = \frac{k^3}{2} \left(\frac{2}{\pi}\right)^{3/2} \frac{S_p^{3/2}}{\gamma^2} \left(\frac{\omega}{\dot{\gamma}}\right)^{1/2} \frac{1}{|\cot \phi_s|}$$

(θ_o = bunch half-length at transition without space charge)

$$\eta_o = \frac{3}{k^3} \left(\frac{\pi}{2}\right)^{3/2} \frac{r_p}{R} \frac{Ng}{S_p^{3/2}} \left(\frac{\omega}{\dot{\gamma}}\right)^{1/2}$$ = Sørenssen parameter $\eta_o(o)$

r_p = classical proton radius = 1.53×10^{-18} m

N = total number of protons

$$\begin{cases} S_p = \text{total phase space area in } \frac{p}{mc}\psi \text{ unit} = \frac{hS}{mcR} \\ \omega = \text{RF angular frequency} = \frac{hc\beta}{R} \end{cases}$$

Aside from the change in the sign-convention and the inclusion of $\dot{\gamma}_t$ (for treating γ_t-jump) Eq. (5) is identical to that given by A. Sørenssen. This equation is solved by numerical methods on a computer.

A. For $|x| \to \infty$ $\dot{\gamma}_t = 0$, $A = 1$ and we can neglect the derivative term $\left(\frac{y'}{x}\right)'$. This gives the algebraic equation

$$\delta y^4 + Cy - x = 0.$$

This equation gives the matched adiabatic solution of Eq. (5) at large $|x|$. Hence, starting with the solutions of this equation at some large values of $|x|$ as initial conditions and integrating Eq. (5) to $x = 0$ we can get the matched solutions for both below and above transition.

B. With $\eta_o = 0$ (no space charge), $A = 1$ (no γ_t-jump) and δ changing sign at $x = 0$ (RF phase jump at transition) the matched solution above transition is simply the reflection about the $x = 0$ axis of the matched solution below transition which is plotted in Figure 1 as Curve (0). Since $y'(x = 0) = 0$ the matched solutions above and below transition are automatically matched at transition.

C. With $\eta_o = 3.8$ (value for the NAL booster) and $A = 1$ (no γ_t-jump) the matched solutions below and above transition are shown as Curves (I) and (II) in Figure 1. The mismatch at transition caused by space charge effect is indicated by the values

$$y_I(x = 0) = 1.502, \qquad y_{II}(x = 0) = 0.$$

D. Starting the γ_t-jump before transition does not change $y(x = 0)$ significantly. For example starting a γ_t-jump with $A = 1 - \frac{\dot{\gamma}_t}{\dot{\gamma}} = 5$ at $x = x_1$ we get for various values of x_1

x_1	$y(x = 0)$
0	1.50214
-0.1	1.50210
-0.2	1.50245
-0.3	1.50537
-0.4	1.51455
-0.5	1.53433
-1.0	1.80331

The conclusion is that starting the γ_t-jump before transition does not significantly change the status of mismatch at $x = 0$. We have, therefore, confined ourselves to cases where the γ_t-jump starts at transition. (In practice, since the start of γ_t-jump cannot be abrupt we will have to start the γ_t-jump just slightly before transition.)

E. Because $y'(x = 0) = 0$ to integrate across $x = 0$ we have to invoke the continuity of $\frac{y'}{x}(x = 0)$ or $y''(x = 0)$. With $A = 1$ (no γ_t-jump) and δ changing sign at $x = 0$ (RF phase jump at transition) the continuation of y_I above transition is plotted as Curve (III) in Figure 1 which shows the familiar bunch length oscillation due to space-charge mismatching.

With $A = 5$ and δ changing sign at $x = 0$ we get Curve (IV). Note that the first minimum of Curve (IV) is now above Curve (II). By adjusting A we obtain for $A = 2.446$ Curve (V) which is tangent to Curve (II) at $x = x_2 = 0.879$ near the first minimum. Thus, with $A = 2.446$ and turning off the γ_t-jump at $x_2 = 0.879$ the bunch length will be perfectly matched to Curve (II). For the NAL booster $\dot{\gamma} = 0.407 \times 10^3\ \mathrm{sec}^{-1}$ and $T = 0.281 \times 10^{-3}$ sec, this value of A gives

$$\dot{\gamma}_t = -1.446\dot{\gamma} = -0.589 \times 10^3\ \mathrm{sec}^{-1}$$

and this value of x_2 gives the corresponding real time

$$\Delta t = Tx_2 = 0.247 \times 10^{-3}\ \mathrm{sec.}$$

Together they give a total γ_t-jump of

$$\Delta\gamma_t = \dot{\gamma}_t \Delta t = -0.145$$

which is rather modest.

F. The strings of dots leading away from the first minima of Curves (III), (IV), and (V) in Figure 1 show the effects of delaying the RF phase jump. Delaying the RF phase jump to $x = x_3$ for Curve IV, for example, means integrating Eq. (5) with $A = 5$, $\delta = -1$ from $x = 0$ to $x = x_3$ and, then, with $A = 5$, $\delta = +1$ from $x = x_3$ on. The dots give the positions of the first minima of the respective curves for $x_3 = 0.1, 0.2, 0.3, \ldots$. The following conclusions can be drawn:

For $A = 5$ matching can be obtained with $x_3 \sim 0.96$ at $x_2 \sim 1.8$.

For A = 2.446 another matching exists with $x_3 \sim 0.77$ at $x_2 \sim 1.67$.

For A = 1 (no γ_t-jump) delaying RF phase jump cannot produce matching; however, it can reduce the amplitude of the bunch length oscillation. Detailed computation shows that at the optimum $x_3 = 0.92$ the maximum bunch length is reduced by about a factor of 1.4 from the case without RF phase jump delay ($x_3 = 0$).

In any case it is evident that delaying RF phase jump does not significantly relax the requirement on the γ_t-jump; in particular, it does not reduce the total jump $\Delta\gamma_t$.

To summarize for the NAL booster ($\eta_o = 3.8$) we should

1. Start the γ_t-jump at or near transition.
2. Jump the RF phase at transition.
3. The total γ_t-jump is $\Delta\gamma_t = -0.145$
4. The total time for the γ_t-jump is $\Delta t = 0.247$ msec.
5. After the γ_t-jump the jump quadrupoles can be turned off (γ_t returns to its value before the jump) slowly (adiabatically) some time (say, 5 msec) after transition.
6. With the small $\Delta\gamma_t$ the simple ν_x-jump (from $\nu_x = 6.7$ to $\nu_x = 6.555$) scheme is possible. However, the scheme of quadrupoles proposed in FN-207 giving a γ_t-jump without affecting ν_x is still desirable. Of course, the required gradient B' for the quadrupoles is much smaller than indicated in that report.

For the NAL Main Ring $\eta_o = 7.2$ the corresponding solutions of Eq. (5) are plotted in Figure 2. The parameters producing matching are

$$A = 2.180 \qquad x_2 = 0.864$$

The interesting point to note is that for $\eta_o \gtrsim 2.5$ the values of A and x_2 for matching do not change significantly with η_o. To understand this we observe that the effect of the γ_t-jump is to produce matching after the phase "ellipse" has rotated approximately 90°. In a crude manner of speaking the ψ-envelope before the γ_t-jump is matched onto the w-envelope after the γ_t-jump. One can, therefore, appreciate that although the mismatch in bunch length (ψ envelope) increases sharply with η_o, this particular mode of matching would not necessarily impose increasingly more stringent requirements on A and x_2 as η_o increases. However, the increased sharpness of the minima of Curves (III), (IV) and (V) in Figure 2 indicates that the precision of the γ_t-jump turn-off time x_2 required for matching becomes more critical with increasing η_o.

For the Main Ring $\dot{\gamma} = 0.135 \times 10^3$ sec^{-1} and $T = 2.44 \times 10^{-3}$ sec. The above values of A and x_2 give

$$\dot{\gamma}_t = -1.180\,\dot{\gamma} = -0.159 \times 10^3 \text{ sec}^{-1}$$

$$\Delta t = Tx_2 = 2.11 \times 10^{-3} \text{ sec}$$

$$\Delta\gamma_t = \dot{\gamma}_t \Delta t = -0.336$$

which is, again, not excessive.

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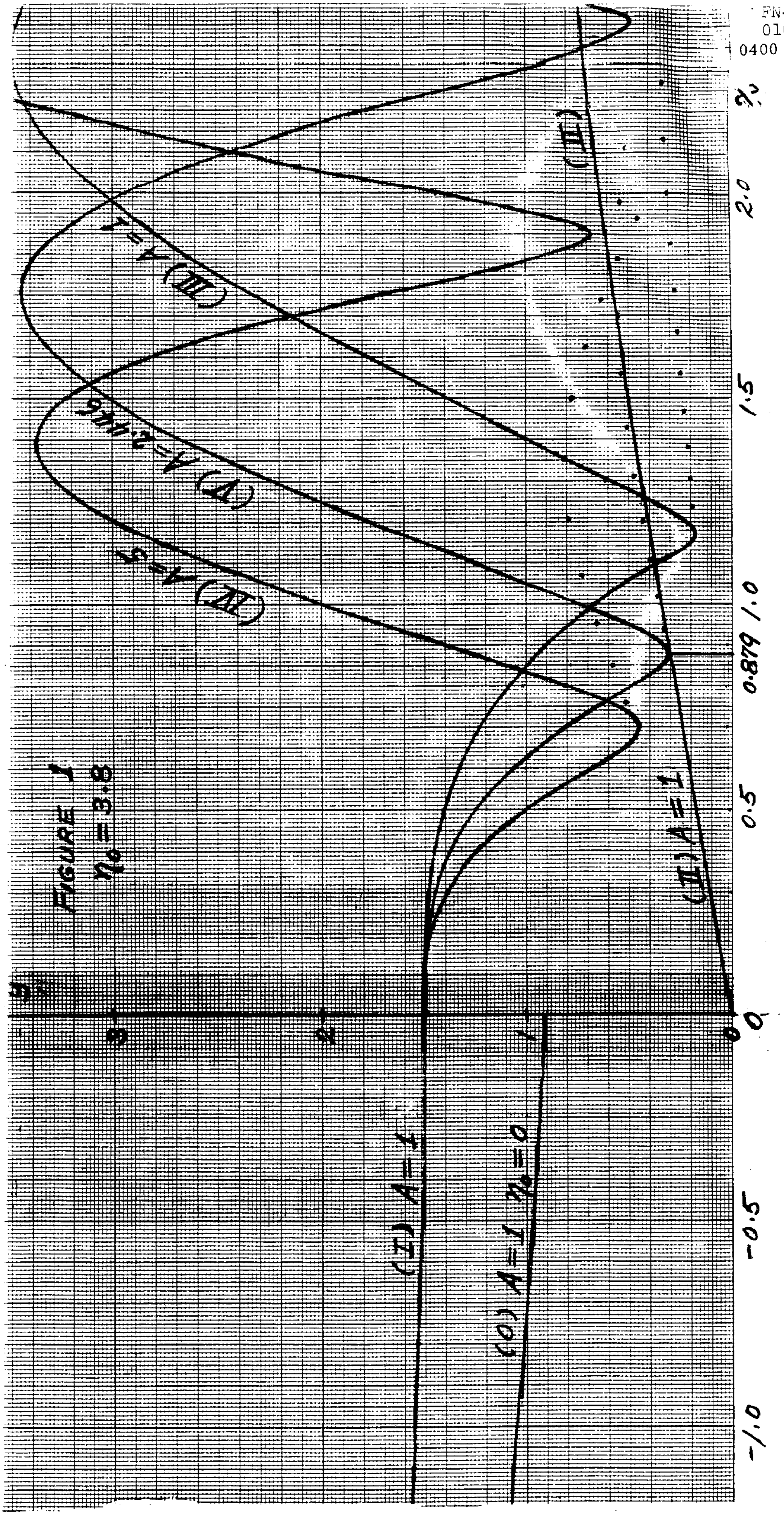

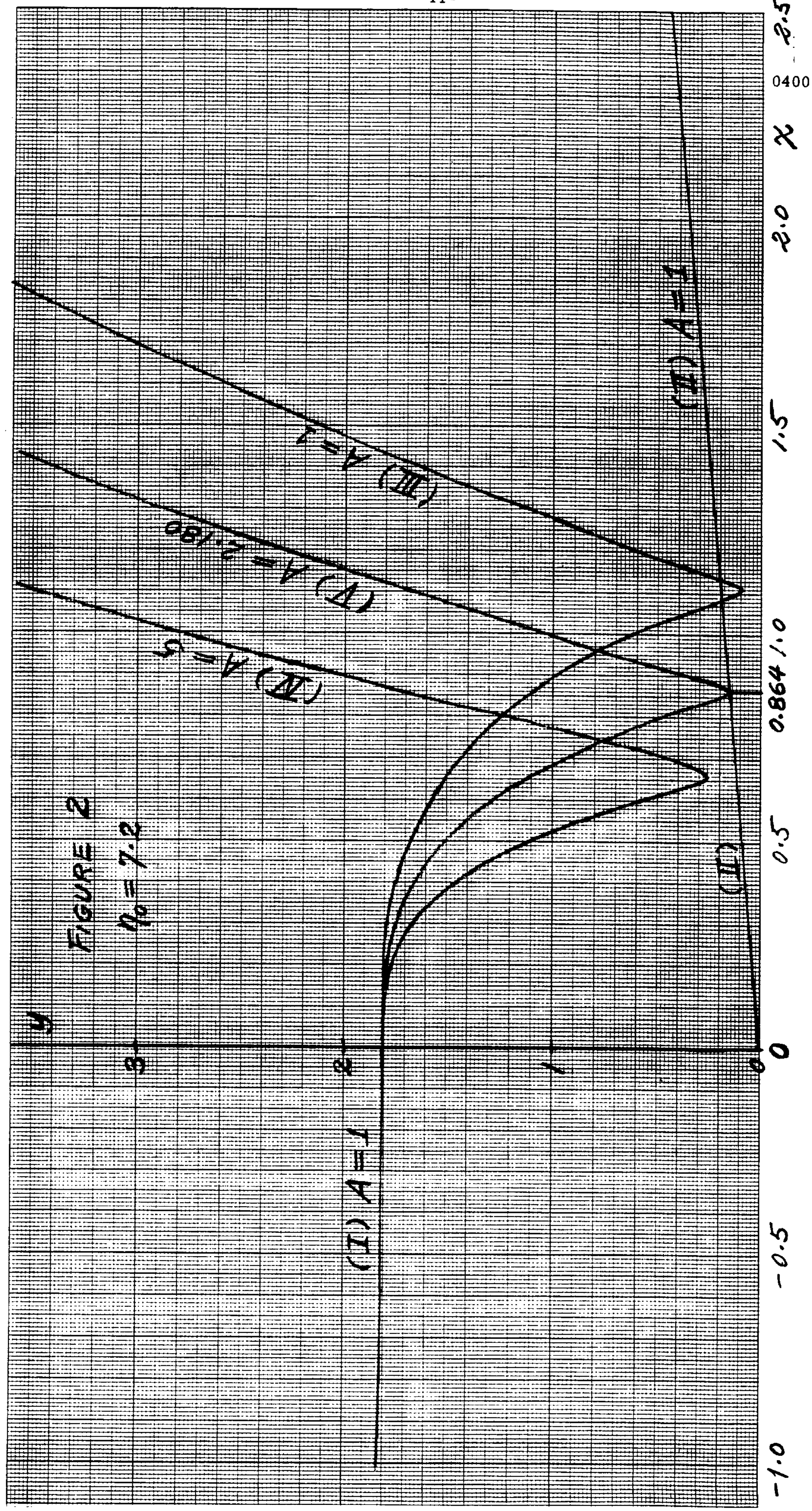
FIGURE 2
$\eta_0 = 7.2$
y
x
(I) A = I
(II) A = I
(II)
(III) A = 1
(IV) A = 4
(V) A = 2.180
0.864
-1.0
-0.5
0
0.5
1.0
1.5
2.0
2.5
1
2
3
0400