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0106

INJECTION CRITERIA STORAGE RING

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Since a definite storage ring lattice has not evolved as yet, the following arguments should be regarded as a generalized approach only. However, in order to come to the right order of magnitude of some of the parameters involved, the FODO lattice described earlier by L. C. Teng, with associated beta function and momentum function parameters has been used as a relevant lattice. Also a storage ring radius of 1/3 the main ring radius has been assumed here. In this case it is further assumed that there will be six beam crossing insertions in the two-ring structure. Two of these (120° apart) are needed for beam injection.

Since the intersection regions are specifically designed for low β_v, low or zero X_p in the intersection region, both parameters related to maximum obtainable luminosity, it is not a priori evident that these same domains are suitable for beam injection.

This aspect plus possible desirable modifications of an intersection region to suit better beam injection, will be the subject of the following discussion. Since a ratio of main ring radius to storage ring radius of a factor of three is assumed, in a first approach, only 3-turn betatron stacking will be taken into consideration (a storage ring ν value of integer + 2/3 is assumed).

For 3-turn "clean" injection, horizontal stacking into the storage ring, the well-known approach with two fast-kicker units separated by π phase degrees on either side (symmetrically) of an injection septum may be used.

The following basic fast kicker relationships and beam displacement at the septum magnet relationship are useful in establishing the lattice criteria or (modified) intersection region criteria for optimum injection.

$$N_k V_k \tau_r = (B\ell)_k w_k$$

$$I_k = f \frac{(B\ell)_k h_k}{\ell_k} ,$$

where $f \simeq 2$.

$$d_s \simeq \theta_k \sqrt{\beta_k \beta_s} \quad \sin(\Delta\Psi_{k,s}), \text{ displacement at septum,}$$

$$\theta_k = (B\ell)_k / (B\rho)_{particle} , \text{ bend angle kicker.}$$

Where w_k is the width of the kicker aperture, h_k is aperture height, ℓ_k is total length, N_k the number of sub-modules to achieve a suitable rise time τ_r, I_k kicker current, and V_k kicker voltage.

From these relationships it is possible to draw some preliminary conclusions:

(a) For maximum $(B\ell)_k / I_k$, h_k should be minimum or, at the kicker location, β_v should be minimum. Actually with minor manipulation of the above equations, together with basic beam size equations, it can be shown that I_k is approximately (for thin septum thickness) proportional to

$$\sqrt{A_h A_v} \quad \left(\sqrt{\frac{\beta_v}{\beta_h}} \right)_k \quad \frac{1}{\ell_k}$$

i.e., a larger value of β_h is also favorable at the kicker location.

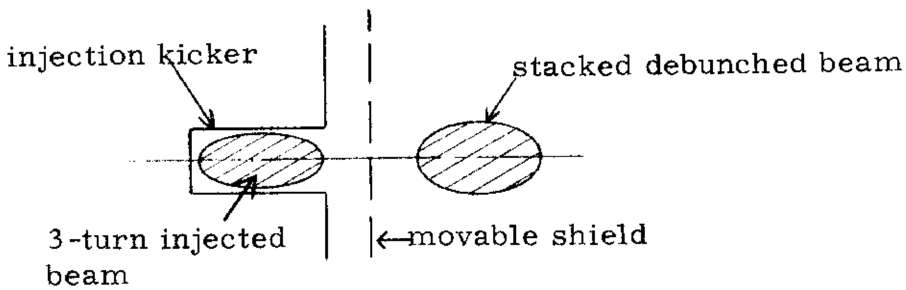
(b) In a FODO lattice, criterium (a) normally means maximum β_h at the same location. This would increase w_k and consequently tend to increase $N_k V_k$. (This may be modified by taking τ_r into account also.) Since the kicker rise time is not critical for storage ring stacking (as will be indicated below), and its fall time may be shortened by means of a "tailbiter" or by means of a compensating kicker at an adjacent azimuthal location, it follows that simultaneously with criterium (a), w_k small or β_h small is not a dominant necessity.*

(c) With horizontal stacking a larger β_h value at the septum location is favorable regarding $2a_s/t_{sept.}$ ratio, which should be large for minimum phase space dilution.

*Actually, if the magnitude of displacement at the septum is taken into account, as was pointed out by B. W. Montague, $N_k V_k \tau_r$ depends only weakly on the β value at the septum location (β_s) for a small septum thickness and proves to be independent of the β_h value at the kicker. See also some more generalized relationships given in NAL Internal Report FN-54, A. van Steenbergen, Sub B, August, 1967.

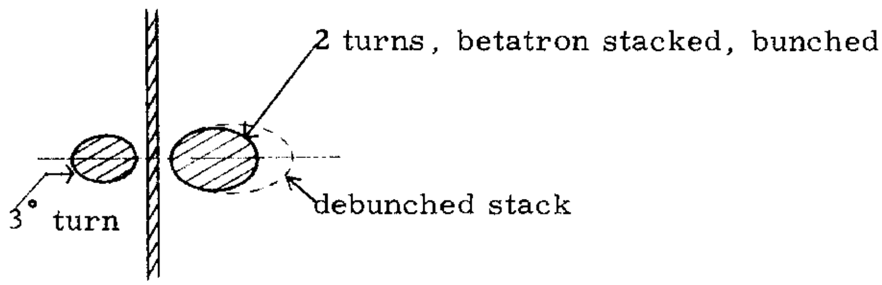
(d) Sin $(\Delta\Psi)_{k,s}$ should be $\simeq$ 1 (a refinement of K_1-S slightly larger than $\pi/2$ needs further elaboration). Consequently, the K_1-S phase advance should be $\simeq \pi/2$ and the phase advance between K_1-K_2 should be π. The latter criterium must hold exactly, otherwise either an additional kicker unit is required at or near the septum location (undesirable) or a penalty will be incurred in aperture requirements related to the betatron oscillation set up in the rest of the lattice structure outside the basic $(\lambda/2)$ bump. The required tune tolerance, with proper placement of K_1 and K_2, will be indicated below.

(e) For the present it has been assumed that the kicker aperture need only be sufficient to "transmit" the 100 BeV beam from the main ring, i.e., a "fractional" aperture kicker similar to the CERN ISR injection system kicker. Associated with this is the requirement that the "stacked" debunched beam needs to be shielded from the kicker fields (see sketch).



The shielding clearance, which adds to the storage ring aperture is proportional with the local X_p value. It follows that for minimum horizontal aperture it is required that (X_p) kicker location should be maximum. The shielding clearance may be avoided and more favorable aperture requirements may be obtained by use of a full aperture kicker. This will be further discussed below.

(f) At the septum location, X_p need only be large enough so that the stacked beam is displaced by $2a_{tot}$ (single turn)* + septum thickness (see sketch).



The injection requirements may then, with a fractional aperture kicker unit, be summarized as follows:

K_1 — — — — — — — — — — S — — — — — — —

phase advance $\pi/2 + \xi$

(phase advance K_1-K_2 = π, exact)

β_v small
X_p maximum (actual value, see pg. 7)
β_h large desirable, as related to I_k, see text, however, not too critical.

β_h { large, however, non critical related to $N_k V_k \tau_r$, but desirable related to transverse phase space dilution and I_k value. β_v small related to injected beam septum magnet aperture. X_p non critical, could be zero, see fig. on page 6.

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*$2a_{tot} = 2(a_\beta + a_s + CO)$ should be derived here from single-turn emittance, contrary to the kicker aperture which is based on the beam size derived from the three-turn (stacked) emittance including the associated dilution factor of $(2 + 1/3)^2/3$. This will be further indicated in the following.

Similarly, with full-aperture injection kickers, the storage ring lattice requirements in the injection domain would be modified as:

At Kicker Location

β_v small
X_p non critical, could be zero (see fig. pg. 8)
β_h as above

At Septum Location

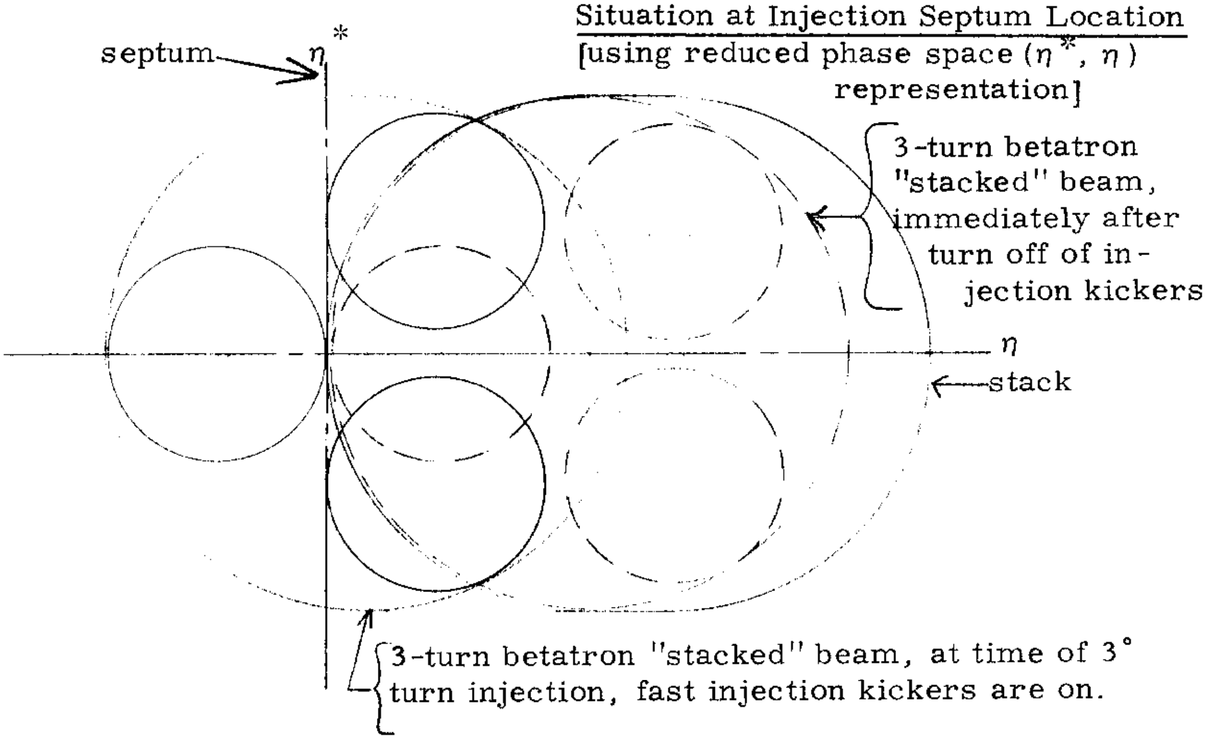
β_v and β_h, as above
X_p cannot be zero and must have a minimum value, i.e.,

$$\frac{X_p(p_{st} - p_{inj})}{p} \approx 12.5 \text{ mm}$$

(see fig. pg. 7)

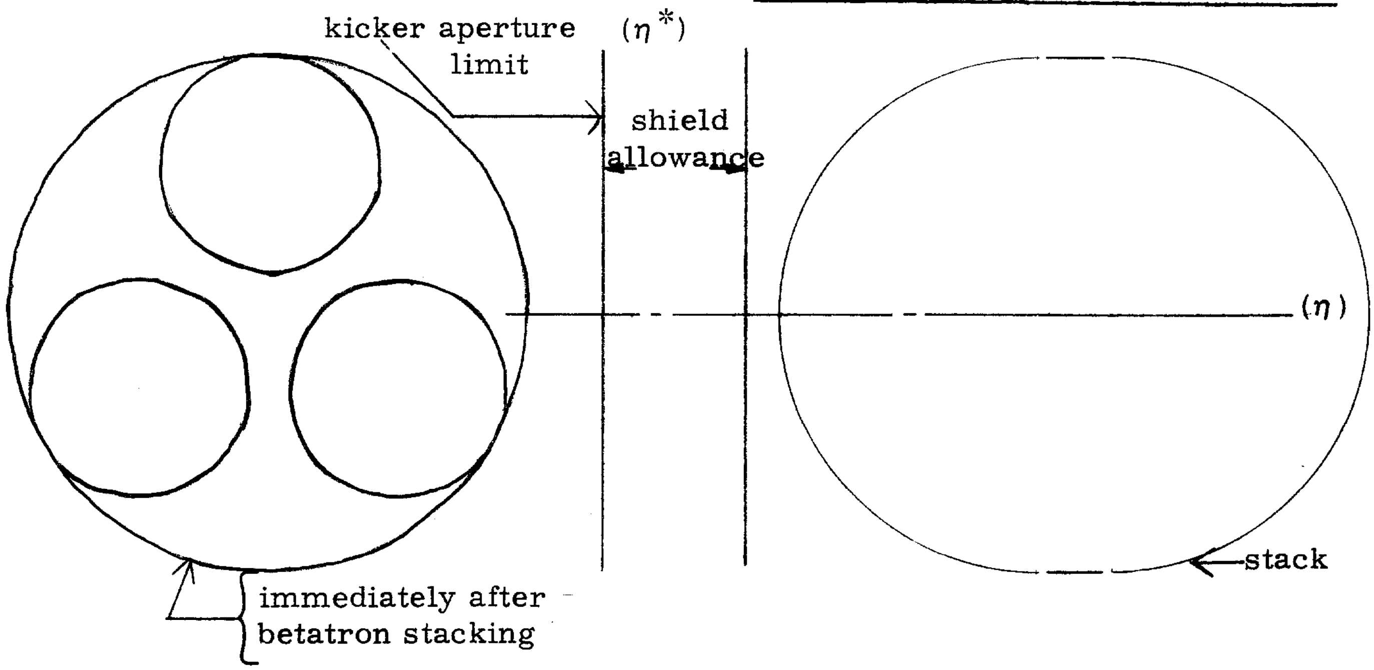
Fractional Aperture Kicker

Situation at Injection Septum Location
[using reduced phase space (η^*, η) representation]



Evident, that at the injection septum, X_p could be zero and, related to the actual magnitude of X_p $(\Delta p/p)_{stack}$, X_p is non critical.

Situation at Injection Kicker Location

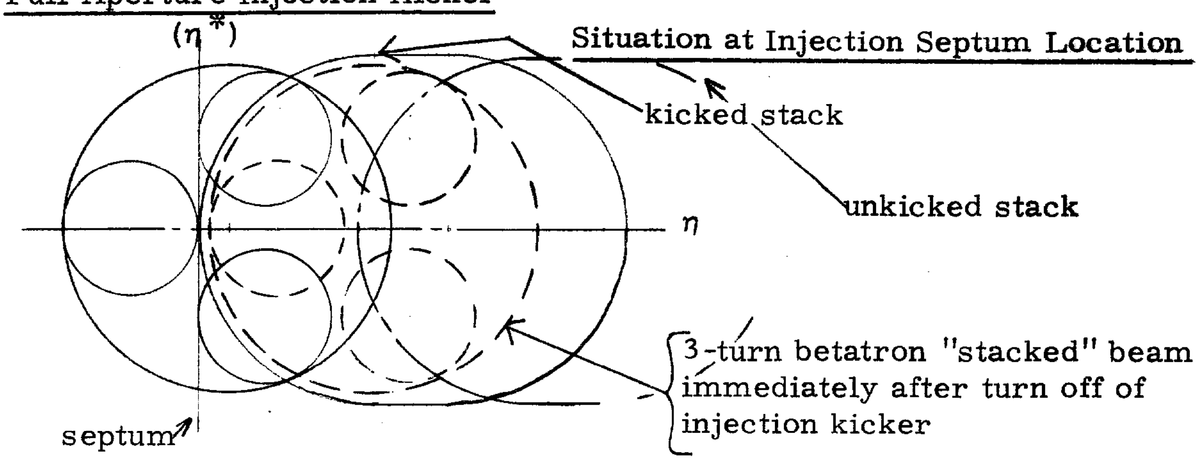


Evident that at injection kicker location X_p is critical and should be maximum. [First estimate of actual value, see below,

$$X_p \frac{\left(p_{stack} - p_{inj}\right)}{p} \approx (25 + 15) \text{ mm.}]$$

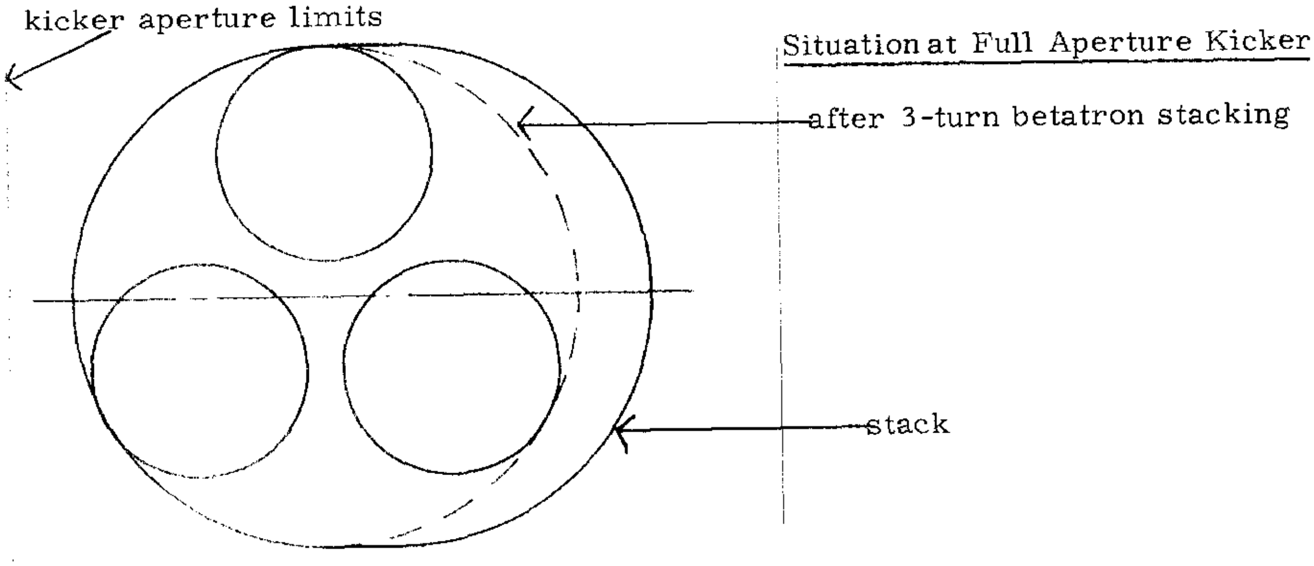
Full Aperture Injection Kicker

Situation at Injection Septum Location



Evident, that at the injection septum X_p cannot be zero and must have a minimum value, i.e.,

$$X_p \frac{\left(p_{st} - p_{inj}\right)}{p} \approx 12.5 \text{ mm}, \quad \text{see below.}$$



Evident, that at the injection kicker location X_p could be zero [related to actual $(\Delta p/p)_{stack}$ values], X_p is not critical.

It is obvious from the foregoing that the criteria associated with a full-aperture kicker, approach much more closely the desirable criteria for a maximum luminosity intersection region, i.e., β_v small, X_p zero. Thus, it is not excluded at this point that an unmodified intersection region would suit optimum injection as well.

Some relevant parameters related to injection kicker and septum-magnet parameters and kicker-aperture requirements, considering a fractional-aperture kicker or, alternatively, a full-aperture kicker, have been evaluated in Appendices A, B, and C.

These numerical results indicate:

(a) A full-aperture kicker leads to a practicable set of parameters for the injection-kicker parameters. In what follows it will be tentatively assumed that a full-aperture kicker will be utilized.

(b) Related to the criteria for the injection straights, indicated in the foregoing, it also follows that it is still true that at the kicker location X_p preferably equals zero, its actual value is, for any practicable lattice, not critical, related to the magnitude of X_p $(\Delta p/p)_{stack}$ comparable with the betatron amplitude of the stacked beam.

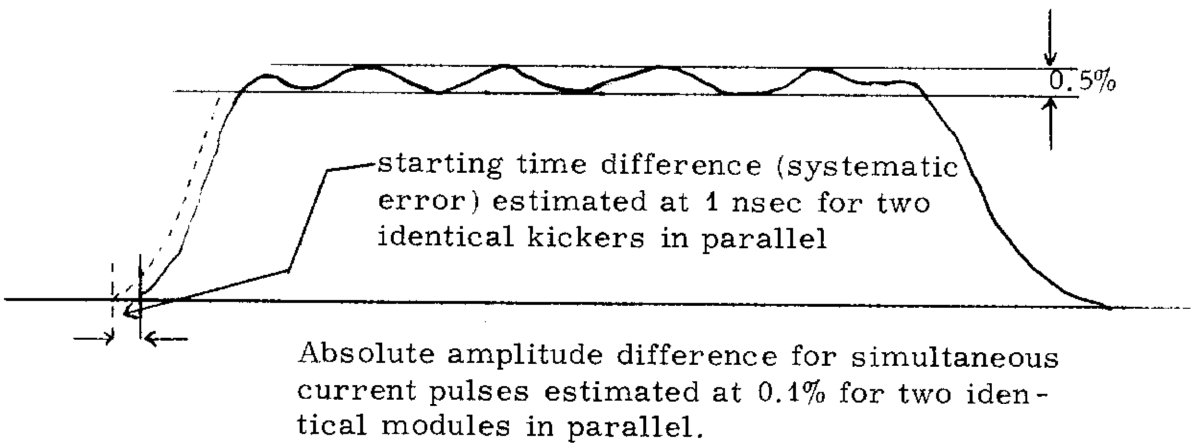
(c) With a full-aperture kicker the stacked beam will be kicked by the two fast kickers, separated by π phase degrees, i.e., $\lambda/2$ bump. Deviation from equality of the two kicker amplitudes and error in phase advance between the two kickers results in a net betatron amplitude growth, which, since between successive injection cycles from the main ring, coherence cannot be maintained in the storage ring, in a first approach leads to an additive growth of the betatron amplitude due to the kicker errors, i.e., for N-turn synchrotron stacking (main ring filling cycles) the addition to the betatron amplitude is given by $N\,(\Delta\hat{X})_{inj}$ per stacking cycle. This last point will be further examined here.

With the proper choice of kicker location in the lattice the π phase advance can be maintained adequately by ν value control. Therefore, it will be assumed here for the present (see further Appendix D) that only kick-amplitude errors exist. For the tentative lattice considered, with, what would normally be assumed 2% deviation between the two kicker amplitudes, it follows with $(\Delta\hat{X})_{inj} \approx \left[(\theta_1 - \theta_2)/2\right]\sqrt{\beta_i\beta_{max}}$ that $(\Delta\hat{X})_{inj} \approx 0.2$ mm for the parameters considered in the foregoing and the appendices. With N = 30-turn stacking this would lead to a non-negligible beam half amplitude growth of 6 mm, i.e., the betatron amplitude could in this case be increased by approximately 50%.

Tighter tolerances on the equality of the two injection kickers can be obtained by arranging each individual module of the downstream kicker unit in parallel with an identical unit of the upstream injection kicker and driving both modules, via matched long lines of equal length, from a single pulsed-storage line.

It is actually necessary to examine the kicker pulse structure in detail since, as will be indicated below, the dominant growth of the betatron amplitude per injected stacking cycle applies only to a fraction of the orbit azimuth.

To illustrate this further, consider a typical current pulse of a fast injection kicker (see sketch).



The pulse structure at full amplitude reflect the finite number of sections of the storage line. Assuming three-turn injection (betatron stacking) in the storage ring the pulse length required is 20.8 μsec nominal. Simplifying the current pulse structure at its maximum value as eight full sinusoidal waves during this time, leads, for a starting time difference between the two kicker units of 1 nsec, to $\delta/\theta \approx 1.25 \times 10^{-5}$, where δ is the kick amplitude difference and θ is the full nominal kick value per kicker. This amplitude difference during kicker "flat top" due to starting time difference, is obviously negligible compared with the "absolute" amplitude difference estimated at 0.1%.

More pronounced is the kick amplitude difference during rise and fall time as a result of the starting time difference as indicated. This is given by:

$$\frac{\delta}{\theta} = \frac{1}{\tau_r} \quad \text{during rise time or}$$

$$\frac{\delta}{\theta} = \frac{1}{\tau_f} \quad \text{during fall time}$$

where τ is in nanoseconds.

Since the rise time can be arbitrarily long of the two injection kickers, δ/θ can be kept sufficiently small during the kicker rise time,

$$\frac{\delta}{\theta} \simeq 0.1\% \text{ for } \tau_r \simeq 1\ \mu\text{sec}.$$

This would imply an "artificially" stretched rise time since the kicker characteristic rise and fall time must remain short.

In order to avoid scraping the septum magnet after the three-turn stacking the fall time will be taken as either < 20 nsec or < 180 nsec, based on the following argument. With 13 shots injected into the main ring from the booster and an orbit ratio of 1/3 of main ring to storage ring no advantage can be taken of the empty buckets, nominally 2, between booster shots. Consequently the injection kicker fall time (99% to < 1%) needs to be, with 50 MHz rf frequency, smaller than 20 nsec.

The alternative is to inject 12 booster shots into the main ring which would permit a fall time of approximately 180 nsec,* although it would reduce the transferred charge in the ratio of 12/13.

With a 1 nsec starting time difference of the two parallel kicker units and a 20 nsec fall time, δ/θ (during fall time) = 0.05, leading to $(\Delta\hat{X})_{inj}$ per charge transfer of 0.5 mm. However, this would result in a betatron amplitude growth, for a single charge transfer from the main ring, for only 0.003 part of the orbit length (orbit time 6.93 μsec). For N charge transfers the probability of additive amplitude growth would be 0.003 N or approximately 10%.

Similarly for a 170 μsec fall time, δ/θ = 0.005 (during fall time), leading to $(\Delta\hat{X})_{inj}$, per charge transfer = 0.05 mm. This would affect 0.024 part of the orbit length. Again, for N main ring cycles for charge transfer the probability of additive amplitude growth (i.e., at least an increase of the betatron amplitude by $2(\Delta\hat{X})_{inj}$ would be 75%.

Short of a more complete analysis, the aperture allowance for betatron amplitude growth of the stacked beam due to stack perturbation by the full aperture kickers can be kept less than a few mm's, as indicated above.

In conclusion it is suggested that full-aperture kickers be utilized

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*The present mode of filling the main ring is Booster, h = 84; main ring rf buckets $(13 \times 83)_{filled} + (12 \times 2)_{empty} + (10)_{empty}$.

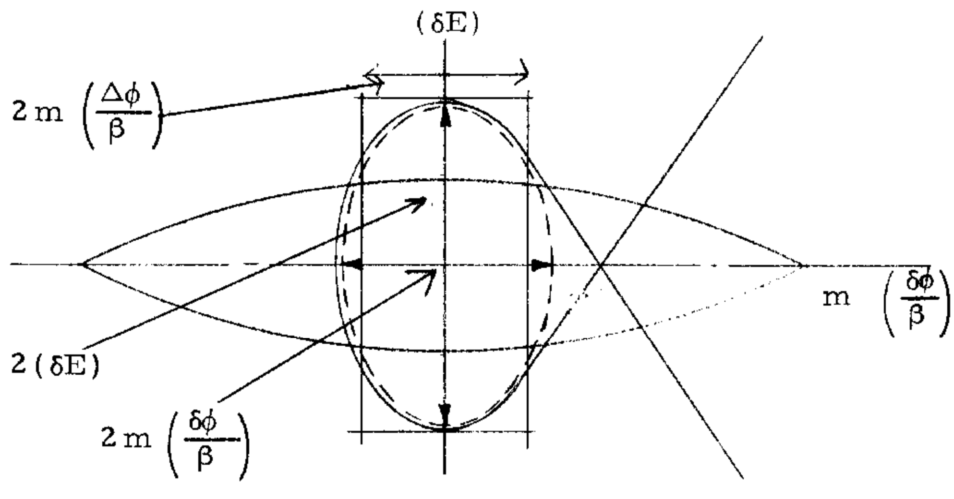
for the storage ring injection system reducing thereby the required total aperture significantly when compared with a system whereby a fractional aperture kicker would be used. Betatron amplitude growth due to kicker error stack perturbation can be kept small.

Appendix A. Beam Momentum Spread in Storage Ring, 100 GeV

To calculate $(\Delta p/p)_{100\ \mathrm{GeV}}$ of the beam to be injected into the storage ring use will be made of the conjugate variables δE and $(\delta\phi/\beta)$, together with the longitudinal phase space invariant $A_2 = m\,\pi(\delta E)(\delta\phi/\beta)$; where A_2 will be expressed in eV-sec, with $m = 0.3\ 10^{-8}$ sec. Actually,

$$m = \frac{R}{hc} = \frac{1}{c}\left(\frac{R}{h}\right)_{\text{Booster}} = \frac{1}{c}\left(\frac{R}{h}\right)_{\text{MR}} .$$

Further $\delta\phi$ is defined by $\Delta\phi = (\pi/4)\ \delta\phi$, with the "bunching factor" $B = 2\Delta\phi/2\pi$ (see sketch).



Area Ellipse $\pi(\delta E)m\left(\frac{\delta\phi}{\beta}\right) = 4(\delta E)m\left(\frac{\Delta\phi}{\beta}\right)$

From synchrotron theory one finds the damping relationships

$$\left(\frac{\delta\phi}{\beta}\right) = \frac{1}{\beta}\left(C\,\frac{\eta}{E}\right)^{1/4}$$

and

$$(\delta E) = \beta \left(\frac{E}{C'\eta} \right)^{1/4} \qquad \text{(note *)}$$

Numerical values are obtained by using (≃) 200 MeV booster parameters.

$$\frac{\Delta p}{p} = \pm 1.8 \; 10^{-3}$$

$$\delta E = \pm 0.6566 \text{ MeV}$$

$$\frac{\delta\phi}{\beta} = \pm 3.158 \text{ rad}$$

$$B = 0.44$$

(These values were calculated by S. C. Snowdon for the bunched booster beam near injection.)

It follows for the booster

$$\frac{\delta\phi}{\beta} = \frac{1}{\beta} \left(18.03 \; 10^3 \; \frac{\eta}{E} \right)^{1/4} \quad \text{and} \quad \pi m \left(\frac{\delta\phi}{\beta} \right) (\delta E) = 1.95 \; 10^{-2} \text{ eV-sec.}$$

$$(\delta E) = \beta \left(\frac{E}{0.976 \; 10^3 \; \eta} \right)^{1/4}$$

With a phase space dilution factor of 2 at transition the following parameters result at 10 GeV.

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*Since $\eta = 1/\gamma^2 - 1/\gamma_{tr}^2$, the constants C, C' will be different for booster and main ring related to the γ_{tr} values.

$$\delta E = 6.348 \text{ MeV}$$

$$\frac{\Delta p}{p} = 0.584\ 10^{-3}$$

$$\frac{\delta\phi}{\beta} = 0.653 \text{ rad}, \ B = 0.16$$

Assuming further, a transfer dilution, booster to main ring, of a factor of 2 (related to this, no further dilution will be assumed at transition in the main ring, this was suggested by E. D. Courant), the following parameters are valid at injection energy for the main ring:

$$\delta E = 8.977 \text{ MeV}$$

$$\frac{\Delta p}{p} = 0.827\ 10^{-3}$$

$$\frac{\delta\phi}{\beta} = 0.924 \text{ rad}, \ B = 0.23$$

It follows for the main ring

$$\frac{\delta\phi}{\beta} = 2\,\frac{1}{\beta}\left(10.216\ 10^{4}\ \frac{\eta}{E}\right)^{1/4}$$

and

$$(\delta E) = 2\,\beta\left(\frac{E}{\eta}\ \frac{1}{0.553\ 10^{4}}\right)^{1/4}$$

where the factor of 2 stems from

$$\sqrt{d_{B,\ \text{transition}} \cdot d_{B\text{-MR},\ \text{transfer}}}$$

From this the longitudinal parameters for the 100 BeV proton beam are

$$\delta E = 18.487 \text{ MeV}$$

$$\frac{\Delta p}{p} = \pm 1.83\ 10^{-4} \text{ (bunched)}$$

$$\frac{\delta\phi}{\beta} = 0.448 \text{ rad, } B = 0.112$$

Assuming no further dilution, it follows for the debunched beam, in the storage ring, $(\Delta p/p)_{\text{debunched}} = \pm 0.2\ 10^{-4}$. (No longitudinal stacking has been assumed yet.)

Appendix B. Transverse Phase Space Parameters in Storage Ring, 100 GeV

The transverse beam emittance areas in the booster, at 200 MeV are (V $\times$ H) $18\,\pi \times 54\,\pi$ (μrad-m)^2. This includes already a factor of 2 vertical dilution and a factor of 1.5 horizontal dilution when taking into account the nominal linac values. Therefore, no further dilution will be assumed in the booster. The booster to main ring transfer errors with resulting dilution are "lumped" in the assumption of an injection kicker with 2% kick amplitude error for the vertical injection with a concomitant injection "kick" error in the horizontal plane of 1/4 that magnitude. Using actual values for kick amplitude and beam sizes, with $\Delta b \simeq (\Delta\theta)\,(\hat{\beta}_v \beta_{v,\,inj})^{1/2}$ it follows that for the booster to main ring transfer dilution the factors ($d_v \times d_h$) equal to 1.4×1.1 must be taken into account.

Therefore, main ring injected beam V $\times$ H is $1.55\,\pi \times 3.6\,\pi$ (μrad-m)^2 resulting in values of $0.17\,\pi \times 0.39\,\pi$ (μrad-m)^2 for the 100 BeV beam. Conservatively, these figures are rounded off and, somewhat arbitrarily, an additional factor of 2 dilution for both phase space projections will be assumed, leading to ($A_{2,v} \times A_{2,h}$) of $0.4\,\pi \times 1.0\,\pi$ (μrad-m)^2. Following the suggestion by E. Keil, transverse phase space interchange (as originally proposed for a different application by A. Maschke), will be used in order to maintain a desirable aspect ratio of the beam after three-turn horizontal transverse stacking in the storage ring.

This leads to:

The transfer channel aperture should be based on $(A_{2,v} \times A_{2,h})$ equal $0.4\pi \times 1.0\pi\ (\mu\text{rad-m})^2$ and $1.0\pi \times 0.4\pi\ (\mu\text{rad-m})^2$. The injection septum aperture should be based on $1.0\pi \times 0.4\pi\ (\mu\text{rad-m})^2$. A fractional aperture fast kicker aperture should be based on $1.0\pi \times 2.2\pi\ (\mu\text{rad-m})^2$, since with three-turn horizontal "clean" stacking an additional dilution factor, for zero septum thickness of $(2 + 1/3)^2/3$ is required.

Since no final storage ring lattice parameters are available as yet the following values have been assumed to calculate the fractional aperture kicker parameters: At the kicker location $\beta_h = 35\text{m}$, $\beta_v = 10\text{m}$, $X_p = 1.6$ m. (Somewhat based on L. C. Teng's preliminary FODO lattice.)

With this, the results from Appendix A and the calculated values for closed-orbit deviations (again from L. C. Teng) the following minimum beam aperture dimensions are obtained at the location of the kicker:

$$a_{tot} = a_\beta + a_{CO} = 12.5 \text{ mm} \quad \text{(3 turn betatron stacked beam)}$$

$$b_{tot} = b_\beta + b_{CO} = 4.6 \text{ mm}$$

i.e., the fractional kicker aperture window would be approximately 10 mm height by 25 mm width.

This has been used to calculate some preliminary parameters (Appendix C) for a fractional aperture fast injection kicker.

Similarly the aperture requirements for a full-aperture kicker may be found as follows:

Assuming 30-turn longitudinal stacking, with 75% stacking efficiency the full width amplitude $X_p\left(\frac{\Delta p}{p}\right)_{stacked} = 2.5$ mm. Further, the three-turn betatron stacked beam required 25 mm as before. Consequently, a full-aperture kicker requires approximately 10 mm height by 30 mm width. The required fractional aperture or full-aperture kick amplitude follows from the displacement required at the septum location, as follows: $2(a_\beta + a_S) + 2$ (clearance) + (septum thickness) $\simeq 11$ mm. With this the required $\theta_k \simeq 0.3$ mrad and $(B\ell)_k \simeq 1$ kG-m, for a 100 GeV beam. These values are indeed quite practicable. For completeness sake, the septum magnet aperture has been calculated as nominally 1 cm $\times$ 1 cm aperture.

Appendix C. Injection Kicker Parameters

Fractional Aperture Kicker

Using the well-known relationships for kicker voltage and current,

$$N_k V_k = \frac{(B\ell)_k w_k}{\tau_r}$$

and

$$I_k = \frac{f\,(B\ell)_k h_k}{\mu_o \ell_k}$$

with $f \simeq 2$, it follows, with the numerical values from Appendix B,

$N_k V_k = 125$ kV, consequently $N_k = 2$, with $V_{\text{charging voltage}}$ $= 125$ kV ($V_{ch} = 2 V_k$).

Rather than a priori fixing kicker length, a practicable characteristic impedance Z_k of 50 Ω is assumed, leading to

$$Z_k = 50\ \Omega$$

$$I_k = 1250\ \text{A}$$

$$\ell_k = 1.27\ \text{m}$$

In summary, first estimate fractional aperture, storage ring, injection kicker

2 modules	pulse length = 20.9 μsec
$V_k = 62.5$ kV	$\tau_r = 20\ 10^{-9}$ sec

$I_k = 1250$ A

$\ell_k = 1.27$ m (total)

$(B\ell)_k = 0.1$ Tm

$\theta_k = 0.31$ mrad

$Z_k = 50\,\Omega$

good field region = full size

$w_k = 0.025$ m

$h_k = 0.010$ m

$Z_k = 50\,\Omega$

field in gap $\simeq$ 800 G

Full-Aperture Kicker

Compared with the above case only the kicker w_k value would be increased from 25 mm to 30 mm. Further, in the previous case real width and "good field" (pure dipole component) width were assumed to be the same. Because of single traversal of the kicker field typically a gradient perturbation would not be serious for the injected beam. For the full-aperture kicker the total stacked beam would be perturbed repeatedly. Therefore, in this case, it will be assumed that the "good field" aperture width is 30 mm, and the actual kicker aperture width is 45 mm (i.e., $w_{kicker} \simeq w_k$ (good field) + $1.5\, h_k$). This results in $N_k V_k = 225$ kV, consequently $N_k = 4$, with $V_{charging\ volt.} = 112.5$ kV. ($V_{ch} = 2\, V_k$). Consequently:

$I_k = 1250$ A

$Z_k = 45\,\Omega$

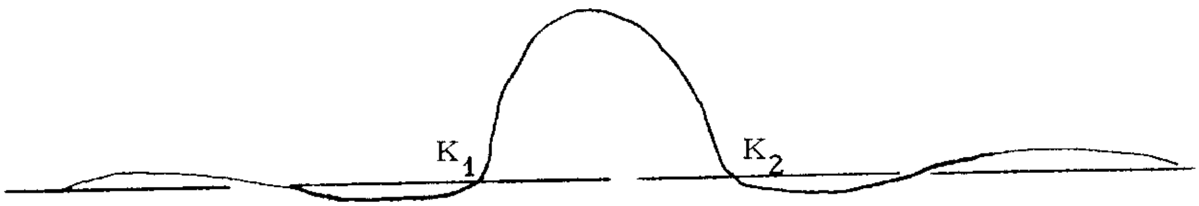
$\ell_k = 1.27$ m

In summary, first estimate full-aperture injection kicker storage rings:

4 modules	$Z_k = 45\,\Omega$
$V_k = 56.25$ kV	good field region = 0.030 m
$I_k = 1250$ A	$w_k = 0.045$ m
$\ell_k = 1.27$ m	$h_k = 0.010$ m
$(B\ell)_k = 0.1$ Tm	pulse length = 20.9 μsec
$\theta_k = 0.31$ mrad	$\tau_r = 20\ 10^{-9}$ sec
	field in gap = 800 G

Appendix D. Consequences of Injection Kicker Errors

The stationary deformed orbit, for the case of two injection kickers may be obtained using the Courant-Snyder perturbation theory and applying matrix transformations, as follows:



x_2, x_2' immediately downstream from K_2 may be found from

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \left[M_{12} \begin{pmatrix} 0 \\ \theta_1 \end{pmatrix} + M_{21} \begin{pmatrix} x_2 \\ x_2' \end{pmatrix} + \begin{pmatrix} 0 \\ \theta_2 \end{pmatrix} \right]$$

with

$$M_{12} = \begin{pmatrix} \sqrt{\dfrac{\beta_1}{\beta_2}} \left(\cos \Delta\psi_{12} + \alpha_1 \sin \Delta\psi_{12} \right) & \sqrt{\beta_1 \beta_2} \sin \Delta\psi_{12} \\ \dfrac{(\alpha_1 - \alpha_2)\cos \Delta\psi_{12} - (1 + \alpha_2\alpha_1)\sin \Delta\psi_{12}}{\sqrt{\beta_1\beta_2}} & \sqrt{\dfrac{\beta_1}{\beta_2}} \left(\cos \Delta\psi_{12} - \alpha_2 \sin \Delta\psi_{12} \right) \end{pmatrix}$$

$$M_{orbit} = M_{22} = \begin{pmatrix} \cos 2\pi\nu + \alpha_2 \sin 2\pi\nu & \beta_2 \sin 2\pi\nu \\ \dfrac{-(1 + \alpha_2)^2 \sin 2\pi\nu}{\beta_2} & \cos 2\pi\nu - \alpha_2 \sin 2\pi\nu \end{pmatrix}$$

$$M_{21} = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_2}}\left[\cos(2\pi\nu - \Delta\psi_{12}) + \alpha_2 \sin(2\pi\nu - \Delta\psi_{12})\right] & \sqrt{\beta_2\beta_1}\ \sin(2\pi\nu - \Delta\psi_{12}) \\ \frac{(\alpha_2-\alpha_1)\cos(2\pi\nu - \Delta\psi_{12}) - (1+\alpha_1\alpha_2)\sin(2\pi\nu - \Delta\psi_{12})}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_2}{\beta_1}}\begin{pmatrix}\cos(2\pi\nu - \Delta\psi_{12}) - \\ -\alpha_1 \sin(2\pi\nu - \Delta\psi_{12})\end{pmatrix} \end{pmatrix}$$

The standard process is to solve the equations for x_2 and x_2' and subsequently to find the maximum excursion from

$$\begin{pmatrix} x_m \\ x_m' \end{pmatrix} = M_{2,s} \begin{pmatrix} x_2 \\ x_2' \end{pmatrix} \quad \text{where s refers to}$$

a location $(\pi/2 + n\pi)$ phase degrees downstream from K_2 (n is integer) with

$$\theta_2 = \theta_1 + \delta \quad \text{and} \quad (\delta/\theta) < 0.02$$

and

$$\Delta\psi_{12} = \pi + \epsilon \quad \text{with} \quad \epsilon < 10^{-3}$$

The closed orbit deformation due to errors δ and ϵ can thus be found outside the basic $(\lambda/2)$ bump. Since the process of evaluation is a rather lengthy one, it will be assumed that $\epsilon \simeq 0$. Actually the implied ν value control is given by $\delta\nu = \nu\epsilon/\pi$, with $\epsilon \simeq 10^{-3}$, $\delta\nu < 0.005$, which seems possible to achieve.

In this case, rather than still following the lengthy recipe indicated above, it is more straightforward to find a superimposed deformed stationary orbit due to δ alone, superimposed on the unperturbed orbit plus an "ideal" $\lambda/2$ bump ($\delta = 0$, $\epsilon = 0$). This may be treated than as a single dipole perturbation at the location of K_2. Using the same approach as above, it follows that immediately downstream from K_2

$$x_2 = \frac{1}{2} \beta_2 \delta \operatorname{cotg}(\pi\nu)$$

$$x_2' = \frac{1}{2} \delta (1 - \alpha_2 \cot \pi\nu)$$

Consequently at ψ phase degrees downstream from K_2,

$$x_s = \frac{\delta}{2} \sqrt{\beta_2 \beta_s} \frac{\operatorname{cotg}(\pi\nu - \psi)}{\sin \pi\nu} \qquad \text{and with } \psi = \pi/2 .$$

$$x_s = \frac{\delta}{2} \sqrt{\beta_2 \beta_s}$$

Depending on the β function behavior for the storage ring lattice the maximum value should be taken at present, until storage ring lattice details are known, as

$$x_s = (\Delta \hat{X})_{inj} = \frac{\delta}{2} \sqrt{\beta_i \beta_{max}} .$$

This has been used in the text.