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EXTRACTION AT A THIRD-INTEGRAL RESONANCE I

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We propose to derive some handy formulas applicable to third-integral resonant extraction, using approximate analytic techniques for solving the dynamical equations. For a first look at the problem, we note that the principal terms in the relevant Hamiltonian are

$$\underline{H} = (\nu - m/3)\rho + A\,(2\,\rho)^{3/2} \cos\,(3\,\underline{\gamma} + \eta), \tag{1}$$

where $\underline{\gamma}$, ρ are canonical polar coordinates in a suitable coordinate system. We will derive the above-Hamiltonian in a later paper in this series, and obtain the parameters A, η in terms of the strength and location of the sextupole magnets. We merely note here that if we introduce the rectangular canonical coordinates

$$X = (\,2\,\rho)^{\frac{1}{2}} \sin\,\underline{\gamma},\; P = (2\,\rho)^{\frac{1}{2}} \cos\,\underline{\gamma}\,, \tag{2}$$

then at the azimuth of the extraction septum, $x = \beta^{1/2}X$ is the radial betatron coordinate, and the radial betatron slope is $dx/d\theta = \beta^{1/2}(P + \alpha X)$. The independent variable here is the azimuthal angle θ. The beam is to be extracted at the $\nu = m/3$ resonance.

Let us take the case $\eta = \pi$, $\nu - \frac{m}{3} < 0$, $\alpha = 0$, as a typical case to study in detail. Any other choice of η merely rotates the phase plane; the choice $\eta = \pi$ will turn out to be the preferred choice for most purposes. The case $\nu - \frac{m}{3} > 0$ also follows without further calculation by reversing the signs of A and θ . If we make the transformation (2), we find

$$H = -\frac{1}{2}\,(\,\frac{m}{3} - \nu)\,(X^2 + P^2) - AP^3 + 3\,APX^2. \tag{3}$$

Since $\partial H/\partial\theta = 0$, H is a constant of the motion. Trajectories of constant H are sketched in Fig. 1. There are three fixed points, in addition to $X = P = 0$, corresponding to an orbit precisely on the resonance $\nu_{eff} = m/3$ which returns sequentially to the three fixed points, which are located at

$$X = \pm X_o, \ P = X_o/\sqrt{3}, \ \text{and} \ X = 0, \ P = -2\,X_o/\sqrt{3}, \tag{4}$$

where

$$X_o = \frac{(\frac{m}{3} - \nu)}{2\sqrt{3}\,A} \ . \tag{5}$$

The Hamiltonian has the value $H_o = -4\,AX_o^3/3\sqrt{3}$ at these points. For $H = H_o$, Eq. (3) factors into three straight lines:

$$(P - X_o/\sqrt{3})(P - \sqrt{3}\,X + 2\,X_o/\sqrt{3})(P + \sqrt{3}\,X + 2\,X_o/\sqrt{3}) = 0 \ . \tag{6}$$

Thus the separatrices are straight lines as sketched. The area of the triangle is

$$S_o = \sqrt{3}\,X_o^2 \ . \tag{7}$$

The equations of motion given by the Hamiltonian (3) are

$$\frac{d\gamma}{d\theta} = \frac{\partial H}{\partial\rho} = -\left(\frac{m}{3} - \nu\right) - 3\,A\,(2\,\rho)^{1/2}\cos 3\,\gamma \ , \tag{8}$$

$$\frac{d\rho}{d\theta} = -\frac{\partial H}{\partial\gamma} = -3\,A\,(2\,\rho)^{3/2}\sin 3\,\gamma \ . \tag{9}$$

Near the center of the triangle, the term on the right in Eq. (9) is small and ρ is nearly constant. If we take ρ to be constant in Eq. (8), we have for the effective tune ν_{eff} at amplitude ρ:

$$\int_0^{-2\pi} \frac{d\gamma}{(\frac{m}{3} - \nu) + 3\ A\ (2\ \rho)^{\frac{1}{2}} \cos 3\ \underline{\gamma}} = \frac{-\ 2\ \pi}{\frac{m}{3} - \nu_{eff}} \quad . \tag{10}$$

From Eq. (10) we have

$$\frac{m}{3} - \nu_{eff} = \left[(\frac{m}{3} - \nu)^2 - 9\ A^2 (2\rho)^{1/2} \right] . \tag{11}$$

This formula gives ν_{eff} for small amplitudes for which the phase trajectory is nearly a circle of radius $(2\ \rho)^{\frac{1}{2}}$. Formula (11) also gives the correct value $\nu_{eff} = \frac{m}{3}$ for $(2\ \rho)^{\frac{1}{2}} = (\frac{m}{3} - \nu)/3\ A$, the value at the corners of the triangle.

Along the separatrix $P = X_o/\sqrt{3}$, the motion is given by

$$\frac{dX}{d\theta} = \frac{\partial H}{\partial \rho} = 3\ A\ (X^2 - X_o^2) \quad . \tag{12}$$

The solution is

$$X = X_o \operatorname{ctnh} \left[\frac{1}{2} \sqrt{3} (\frac{m}{3} - \nu)\ (\theta_1 - \theta) \right] . \tag{13}$$

Note that $X = \infty$ in a finite time, when $\theta = \theta_1$, so that the amplitude grows faster than exponentially. The step size per three revolutions is given approximately by Eq. (12):

$$\Delta X \doteq 18\ \pi\ A\ (X^2 - X_o^2) \quad . \tag{14}$$

This is valid provided $\Delta X << X - X_o$, which will be roughly true at the ex traction point where $\Delta X \lesssim \frac{1}{2} (X - X_o)$.

In order to find phase curves near the separatrices, let $H = H_o + \delta H$. We substitute in Eq. (3), remembering Eq. (6):

$$(P - X_o/\sqrt{3})(P - \sqrt{3}\ X + 2\ X_o/\sqrt{3})(P + \sqrt{3}\ X + 2\ X_o/\sqrt{3}) = -\delta\ H/A. \tag{15}$$

We now look for curves near the horizontal separatrix, and put $P = X_o/\sqrt{3} + \delta P$, to obtain, to first order in δP:

$$\delta P \doteq \frac{\delta H}{3\ A\ (X^2 - X_o^2)} \quad . \tag{16}$$

This formula is valid for $X^2 > X_o^2$ or $X^2 < X_o^2$. At $X = X_o$, we must go to second order:

$$\delta P \doteq \left[\frac{\delta H}{2\sqrt{3}\ AX_o}\right]^{1/2} ,\ X \doteq X_o \quad . \tag{17}$$

The rate of flow of phase area per radian of revolution, between the separatrix and the curve $H_o + \delta H$ is

$$S' = \frac{dS}{d\theta} = |\delta P|\, dX/d\theta = |\delta H| \quad . \tag{18}$$

The emittance of this particular component of extracted beam is given then by Eqs. (16) and (14):

$$E_c = \Delta X . \delta P = 6\ \pi S' \quad , \tag{19}$$

as also follows immediately from Liouville's theorem.

Suppose the beam has an initial amplitude $\pm\ x_i$ in centimeters at the extraction azimuth, and an angular amplitude $\pm\ \psi_i$. Then

$$\psi_i = \frac{1}{R}\frac{dx_i}{d\theta} \quad , \tag{20}$$

where R is the mean machine radius ($R\ d\theta = dS$ along the orbit). In terms of capital variables,

$$X_i = \beta^{-1/2}\ x_i ,\ P_i = R\ \beta^{+1/2}\ \psi_i \quad . \tag{21}$$

The phase area is

$$S_i = \pi X_i P_i = \pi R x_i \psi_i . \tag{22}$$

We are assuming that initially we are far from the resonance. Since $X_i = P_i$, we also have

$$S_i = \pi \beta^{-1} x_i^2 . \tag{23}$$

If the beam is to be extracted uniformly during N turns, then

$$S' = \frac{S_i}{2 \pi N} , \tag{24}$$

so that the emittance of any component of the extracted beam is

$$E_c = \frac{3 \pi \beta^{-1} x_i^2}{N} , \tag{25}$$

made up of a source size

$$\Delta x = \beta^{1/2} \Delta X = 18 \pi A \beta^{+1/2} (X^2 - X_o^2) , \tag{26}$$

and an angular divergence

$$\Delta \psi = R^{-1} \beta^{-1/2} \delta P = \frac{R^{-1} \beta^{-3/2} x_i^2}{6 NA (X^2 - X_o^2)} . \tag{27}$$

Extraction begins when the triangular separatrix has shrunk to the area given by Eq. (23), so that, in view of Eq. (7):

$$X_{o_i} = \left(\frac{\pi}{\beta\sqrt{3}}\right)^{1/2} x_i . \tag{28}$$

In centimeters, the corner of the triangle is displaced from the center of the beam by

$$x_{o_i} = \left(\frac{\pi}{\sqrt{3}}\right)^{\frac{1}{2}} x_i . \tag{29}$$

Since X_1 shrinks to zero during extraction, it is desirable to extract at a point X_e where $X_e^2 >> X_o^2$ so that ΔX in Eq. (14), and also δP, remain substantially constant during extraction. Presumably $X_e > 3\ X_{oi}$ or $x_e > 3\ x_{oi}$ is adequate. We will specify the step size at extraction Δx, so that by Eq. (14),

$$\Delta x = \beta^{1/2} \Delta X = 18\ \pi\ A\ \beta^{1/2} x_e^2 \quad . \tag{30}$$

This fixes the constant A.

The frequency shift at which extraction begins is given by Eqs. (5) and (28):

$$\frac{m}{3} - \nu = \frac{3^{\frac{1}{4}}}{18\ \pi^{\frac{1}{2}}} \ \frac{x_i\ \Delta x}{x_e^2} \quad . \tag{31}$$

The frequency ν must then go smoothly to $\nu = m/3$ when extraction ends, for any particular beam component. However, if there is an energy spread in the beam, and ν depends on energy, beam components of different energy will have different ν values at any one time.

The angular divergence of any beam component is given by Eq. (27):

$$\Delta\psi = \frac{3\ \pi \quad x_i^2}{N\beta R\ \ \Delta x} \quad . \tag{32}$$

However, the beam emerges intially at an angle given by the momentum of the separatrix $P_o = X_o/\sqrt{3}$:

$$\psi_i = \left(\frac{\pi}{3\sqrt{3}}\right)^{\frac{1}{2}} \frac{x_i}{\beta R} \quad . \tag{33}$$

When the triangle shrinks to zero, $\psi = 0$. Thus the overall angular spread of the beam is given by Eq. (33) and not by Eq. (32). With an energy spread, beam may emerge at all angles $0 < \psi < \psi_i$ simultaneously, and the effective emittance is increased over the ideal value (25) by a factor

$$\frac{\psi_i}{\Delta\psi} = \frac{1}{3\ (3\ \ \sqrt{3}\pi)^{\frac{1}{2}}} \ \frac{N\ \Delta x}{x_i} \quad . \tag{34}$$

This factor could in principle be recovered later by a suitable combination of lenses and prisms, since there is a correlation between ψ and energy. In principle, an energy dependent orbit bump might be devised inside the accelerator so that the equilibrium orbit has an angle $-\psi(E)$ at the extraction septum which just cancels the angular divergence $\psi(E)$ of the extraction separatrix.

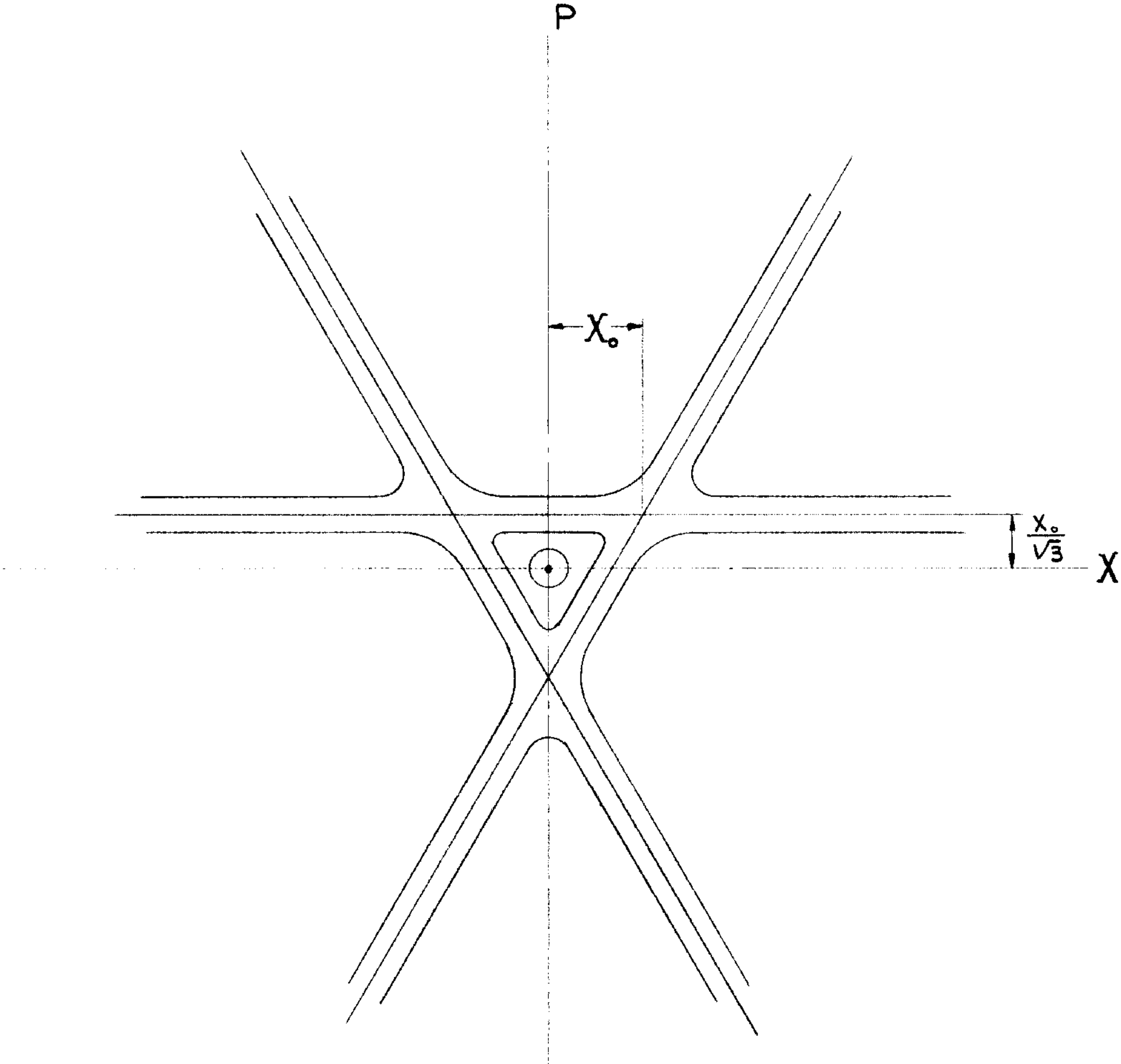

Fig. 1. Phase plane near third-integral Resonance