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Quarkonium wave functions at the origin: an update

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Using a newly developed interquark potential, we tabulate values of the radial Schrödinger wave function or its first nonvanishing derivative at zero quark–antiquark separation, for $c\bar{c}$, $c\bar{b}$, and $b\bar{b}$ levels that lie below, or just above, the flavor threshold. These quantities are required inputs for evaluating quarkonium production cross sections.

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Fragmentation of partons into quarkonium states has long been recognized as an important component of quarkonium production in high-energy collisions [1]. It is an essential element of the event generators now in common use to simulate $J/\psi, \Upsilon$, and B_c production [2].

The calculation of the production rate by fragmentation factorizes into a parton-level piece that can be evaluated using perturbative techniques for Quantum Chromodynamics times a hadronic piece expressed in terms of the quarkonium wave function. In a previous publication [3], we tabulated the values at the origin of (the absolute square of) the radial wave function (for s-wave levels), or its first nonvanishing derivative (for orbitally excited levels), for a selection of quarkonium potentials that gave reasonable accounts of the J/ψ and Υ spectra then known. There we examined the Cornell Coulomb-plus-linear potential [4], Martin's power-law potential [5], Richardson's QCD-inspired potential [6], and a second QCD-inspired potential due to Buchmüller and Tye [7].

The QCD-inspired potentials incorporate running of the strong coupling constant α_s using the perturbative-QCD evolution equation at leading order and beyond. But at distances relevant for confinement, perturbation theory ceases to be a reliable guide. It is now widely held, following Gribov [8], that α_s approaches a critical, or frozen, value at long distances (low energy scales), as a result of quantum screening. Recently we incorporated the spirit of this insight into a new version of the Coulomb-plus-linear form that we call the *frozen-α_s potential* [9], which we employed in a prospectus for the $(c\bar{b})$ spectrum. The purpose of this note is to record calculations for the wave functions at the origin in all three of the quarkonium families, $J/\psi, B_c$, and Υ.

For bound states in a central potential, the Schrödinger wave function separates neatly into radial and angular pieces, as $\Psi_{n\ell m}(\vec{r}) = R_{n\ell}(r)Y_{\ell m}(\theta,\phi)$, where n is the principal quantum number, ℓ and m are the orbital angular momentum and its projection, $R_{n\ell}(r)$ is the radial wave function, and $Y_{\ell m}(\theta,\phi)$ is a spherical harmonic with normalization $\int d\Omega\, Y^*_{\ell m}(\theta,\phi)Y_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\,\delta_{mm'}$. From the normalization condition $\int d^3\vec{r}\,|\Psi_{n\ell m}(\vec{r})|^2 = 1$, it follows that $\int_0^\infty r^2 dr|R_{n\ell}(r)| = 1$. The value of the radial wave function, or its first nonvanishing derivative, at the origin,

$$R^{(\ell)}_{n\ell}(0) \equiv \left.\frac{d^{\,\ell} R_{n\ell}(r)}{dr^\ell}\right|_{r=0} \quad , \tag{1}$$

is required to evaluate meson decay constants and the parton-fragmentation contribution to production rates.

The long-range part of the frozen-α_s potential has the standard Cornell linear form. To obtain the Coulomb piece, we converted the four-loop running of $\alpha_s(q)$ in momentum space [10] to the behavior in position space using the method of [11], and enforced saturation of $\alpha_s(r)$ at long distances, as detailed in Ref. [9]. Values of $\alpha_s(r)$ in a form convenient for interpolation are presented in the Appendix to that article.

For each quarkonium level, we show in Table I the computed centroid of the mass along with the wave function at the origin. For each family, we have included not only the narrow levels, but also states above flavor threshold that might be useful in assessing cascade contributions or the production of discrete excited levels. Comparing with Tables I, II, and III of Ref. [3], we find that the new values lie comfortably within the range of our earlier estimates [12]. We recommend the new results presented here as reliable modern reference values.

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[1] E. Braaten, S. Fleming, and T. C. Yuan, Production of heavy quarkonium in high-energy colliders, Ann. Rev. Nucl. Part. Sci. **46**, 197 (1996), arXiv:hep-ph/9602374 [hep-ph]; M. Krämer, Quarkonium production at high-energy colliders, Prog. Part. Nucl. Phys. **47**, 141 (2001), arXiv:hep-ph/0106120 [hep-ph]; §4 of N. Brambilla *et al.*, Heavy quarkonium: progress, puzzles, and opportunities, Eur. Phys. J. **C71**, 1534 (2011), arXiv:1010.5827 [hep-

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TABLE I. Absolute squares of radial wave functions or their first nonvanishing derivatives at the origin for quarkonium states. The masses are centroids of the indicated states, calculated in the frozen-α_s potential with $m_c = 1.84$ GeV and $m_b = 5.19$ GeV.

Level	$c\bar{c}$ Mass [GeV]	$c\bar{c}$ $\lvert R_{n\ell}^{(\ell)}\rvert^2$	$c\bar{b}$ Mass [GeV]	$c\bar{b}$ $\lvert R_{n\ell}^{(\ell)}\rvert^2$	$b\bar{b}$ Mass [GeV]	$b\bar{b}$ $\lvert R_{n\ell}^{(\ell)}\rvert^2$
$1S$	3.0687	1.0952 GeV^3	6.3155	1.9943 GeV^3	9.4425	5.8588 GeV^3
$2P$	3.5029	0.1296 GeV^5	6.7517	0.3083 GeV^5	9.8827	1.6057 GeV^5
$2S$	3.6790	0.6966 GeV^3	6.8860	1.1443 GeV^3	10.0159	2.8974 GeV^3
$3D$	3.7983	0.0329 GeV^7	7.0179	0.0986 GeV^7	10.1448	0.8394 GeV^7
$3P$	3.9554	0.1767 GeV^5	7.1539	0.3939 GeV^5	10.2550	1.8240 GeV^5
$3S$	4.1079	0.5951 GeV^3	7.2732	0.9440 GeV^3	10.3639	2.2496 GeV^3
$4F$	4.0419	0.01317 GeV^9	7.2348	0.0493 GeV^9	10.3454	0.6643 GeV^9
$4D$	4.1873	0.06923 GeV^7	7.3630	0.1989 GeV^7	10.4461	1.5572 GeV^7
$4P$	4.3297	0.2106 GeV^5	7.4864	0.4540 GeV^5	10.5451	1.9804 GeV^5
$4S$	4.4685	0.5461 GeV^3	7.5963	0.8504 GeV^3	10.6421	1.9645 GeV^3
$5G$	4.2573	0.00750 GeV^{11}	...	...	10.5153	0.7392 GeV^{11}
$5F$	4.3937	0.03740 GeV^9	...	...	10.6100	1.7228 GeV^9
$5D$	4.5284	0.1074 GeV^7	...	...	10.7031	2.2324 GeV^7
$5P$	4.6607	0.2389 GeV^5	...	...	10.7947	2.1175 GeV^5
$5S$	4.7900	0.5160 GeV^3	...	...	10.8843	1.7990 GeV^3
$6H$	...	...	...	...	10.6668	1.1071 GeV^{13}
$6G$	...	...	...	...	10.7568	2.4623 GeV^{11}
$6F$	...	...	...	...	10.8457	3.0936 GeV^9
$6D$	...	...	...	...	10.9332	2.8903 GeV^7
$6P$	...	...	...	...	11.0194	2.2430 GeV^5
$6S$	...	...	...	...	11.1036	1.6885 GeV^3
$7I$	...	...	...	...	10.8058	2.1639 GeV^{15}
$7H$	...	...	...	...	10.8919	4.50055 GeV^{13}
$7G$	...	...	...	...	10.9771	5.3196 GeV^{11}
$7F$	...	...	...	...	11.0614	4.7389 GeV^9
$7D$	...	...	...	...	11.1446	3.5411 GeV^7
$7P$	...	...	...	...	11.2266	2.3600 GeV^5
$7S$	...	...	...	...	11.3066	1.6080 GeV^3

ph].

[2] H.-S. Shao, HELAC-Onia 2.0: an upgraded matrix-element and event generator for heavy quarkonium physics, Comput. Phys. Commun. **198**, 238 (2016), arXiv:1507.03435 [hep-ph], `http://helac-phegas.web.cern.ch/helac-phegas/helac-onia.html`. For integration into PYTHIA, see `http://home.thep.lu.se/~torbjorn/pythia82html/HelacOniaProcesses.html`; C.-H. Chang, X.-Y. Wang, and X.-G. Wu, BCVEGPY2.2: A newly upgraded version for hadronic production of the meson B_c and its excited states, Comput. Phys. Commun. **197**, 335 (2015), arXiv:1507.05176 [hep-ph].

[3] E. J. Eichten and C. Quigg, Quarkonium wave functions at the origin, Phys. Rev. **D52**, 1726 (1995), arXiv:hep-ph/9503356 [hep-ph].

[4] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T.-M. Yan, Charmonium: The Model, Phys. Rev. **D17**, 3090 (1978), [Erratum: Phys. Rev. **D21**, 313 (1980)]; Charmonium: Comparison with Experiment, Phys. Rev. **D21**, 203 (1980).

[5] A. Martin, A Fit of Upsilon and Charmonium Spectra, Phys. Lett. **93B**, 338 (1980).

[6] J. L. Richardson, The Heavy Quark Potential and the Υ, J/ψ Systems, Phys. Lett. **82B**, 272 (1979).

[7] W. Buchmüller and S. H. H. Tye, Quarkonia and Quantum Chromodynamics, Phys. Rev. **D24**, 132 (1981).

[8] V. N. Gribov, The theory of quark confinement, Eur. Phys. J. **C10**, 91 (1999), arXiv:hep-ph/9902279 [hep-ph].

[9] E. J. Eichten and C. Quigg, Mesons with Beauty and Charm: New Horizons in Spectroscopy, Phys. Rev. **D99**, 054025 (2019), arXiv:1902.09735 [hep-ph].

[10] K. G. Chetyrkin, Four-loop renormalization of QCD: Full set of renormalization constants and anomalous dimensions, Nucl. Phys. **B710**, 499 (2005), arXiv:hep-ph/0405193 [hep-ph]; M. Czakon, The Four-loop QCD beta-function and anomalous dimensions, Nucl. Phys. **B710**, 485 (2005), arXiv:hep-ph/0411261 [hep-ph].

[11] M. Jeżabek, M. Peter, and Y. Sumino, On the relation between QCD potentials in momentum and position space, Phys. Lett. **B428**, 352 (1998), arXiv:hep-ph/9803337 [hep-ph].

[12] The strong Coulomb term of the original Cornell potential is reflected in spatially compact states, and large values of the wave functions at the origin.