

Inertial Spontaneous Symmetry Breaking and Quantum Scale Invariance

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Weyl invariant theories of scalars and gravity can generate all mass scales spontaneously, initiated by a dynamical process of “inertial spontaneous symmetry breaking” that does not involve a potential. This is dictated by the structure of the Weyl current, K_μ , and a cosmological phase during which the universe expands and the Einstein-Hilbert effective action is formed. Maintaining exact Weyl invariance in the renormalized quantum theory is straightforward when renormalization conditions are referred back to the VEV’s of fields in the action of the theory, which implies a conserved Weyl current. We do not require scale invariant regulators. We illustrate the computation of a Weyl invariant Coleman-Weinberg potential.

I. INTRODUCTION

The discovery of the Higgs boson with the appearance of a fundamental, point-like, scalar field, unaccompanied by a natural custodial symmetry, has led many authors, in search of a new organising principle, to turn to scale symmetry. In particular, Weyl symmetry [1] in conjunction with gravity may provide a modern context for fundamental scalar fields and a foundational symmetry for physics [2–5]. Scale or Weyl symmetry, like many of the flavour symmetries seen in nature, must be broken. Often the breaking is treated spontaneously, implemented for scale invariant potentials via the Coleman Weinberg (CW) mechanism of dimensional transmutation [6].

In this paper we emphasise that there is a new way to break scale symmetry that *does not employ a potential*. While this mechanism is implicit in many of the approaches taken to spontaneously generating the Planck scale, it seems not to have been made explicit prior to ref [3]. This mechanism is a direct consequence of the structure of the Weyl scale current. We call this *inertial spontaneous symmetry breaking*.

A crucial aspect of this mechanism is that quantum theory should not break scale symmetry. We believe this is generally possible. To understand this, it is important that one does not conflate the procedure of regularisation, which generally introduces arbitrary mass scales,

and renormalisation, which introduces counter-terms to define the final theory and its symmetries. Though it may be convenient, one need not deploy a regulator that is consistent with the symmetries of the renormalised theory. The nonexistence of a symmetry in the regulator does not imply the nonexistence of the symmetry in the renormalised theory. Furthermore, physics should not depend upon the choice of regulator [7].

In this view, Weyl symmetry is central and all mass scales must emerge by way of random initial conditions governing VEV’s (Vacuum Expectation Values) of fields that are entirely contained within the action. Essentially there exist no fundamental mass scales, and the mass of anything is defined only relative to field VEV’s in the theory. For this to be phenomenologically acceptable it is necessary to explain how the spontaneous breaking of Weyl symmetry can lead to a period of inflation followed by a reheat phase and transition in the infrared to a theory describing the fundamental states of matter and their interactions with an hierarchically large difference between the Planck scale and the electroweak breaking scale.

Remarkably it has been shown in a simplified model involving two scalar fields that this structure is possible [2, 3]. The model has a scale invariant scalar potential and non-minimal coupling of the scalar fields to the Ricci scalar. When the fields develop VEVs the Planck scale is generated spontaneously in the Brans-Dicke manner. For a wide range of the non-minimal couplings and scalar interactions, there is an initial period of “slow-roll” inflation that can give acceptable values for the slow-roll parameters. This is followed by a “reheat” phase and a flow of the field VEVs to an infrared fixed point at which

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the ratio of the scalar field VEVs are determined by the dimensionless couplings of the theory. Thus it is possible to arrange an hierarchically large ratio for the VEVs and, interpreting the second scalar as modelling the Standard Model Higgs boson, this large ratio corresponds to the ratio between the Planck scale and the electroweak scale.

In this paper we will concentrate on explaining how inertial symmetry breaking comes about. To clarify the discussion we use the 2-scalar model mentioned above for illustration but the mechanism and methods discussed immediately extend to the general case of Weyl invariant models involving fundamental scalars.

In section II we present the mechanism for inertial spontaneous symmetry breaking that results from the conservation of the Weyl current. As it does not involve a potential the mechanism opens a new pathway to generating spontaneous scale symmetry breaking and the associated spontaneous breaking of other symmetries. As such it may be useful for novel aspects of model building.

In section III we discuss how Weyl invariance is maintained at the quantum level and thus preserves the inertial spontaneous symmetry breaking mechanism. The crucial aspect of this is the decoupling of the dilaton [8] and the appearance of the inertial spontaneous symmetry breaking scale. As a result the logarithmic corrections that normally break the scale invariance now automatically depend only on physically relevant ratios of field VEVs which preserve the underlying Weyl invariance of the theory. We compare this procedure to previous proposals for scale invariant regularisation that require an arbitrary choice of regulator, a function of the scalar fields.

Finally, in section IV, we present a summary of our results and the conclusions to be drawn.

II. INERTIAL SPONTANEOUS SYMMETRY BREAKING.

To illustrate the mechanism it is convenient to consider a simple example of a real scalar field theory action implementing Brans-Dicke gravity (our metric signature convention is $(1, -1, -1, -1)$),

$$S = \int \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - W(\phi) - \frac{1}{12} \alpha \phi^2 R \right) \quad (1)$$

where the scalar potential is given by:

$$W(\phi) = \frac{\lambda}{4!} \phi^4. \quad (2)$$

If $\alpha = 1$, this theory has the local Weyl invariance given by:

$$g_{\mu\nu} \rightarrow e^{-2\sigma(x)} g_{\mu\nu} \quad \phi \rightarrow e^{\sigma(x)} \phi(x) \quad (3)$$

In this case, redefining the metric as: $g_{\mu\nu} \rightarrow (\phi^2/M^2)\phi(x)\tilde{g}_{\mu\nu}$ to go from the Jordan to the Einstein frame, we can remove ϕ altogether from the action,

and we are left with an Einstein-Hilbert action in the “new metric” $\tilde{g}_{\mu\nu}$ with a cosmological constant λM^4 and a vanishing Weyl current [9] but with a *wrong-sign* $M^2 R$ term [10]). If, however, we consider the case that $\alpha < 0$, the field ϕ can acquire a vacuum expectation value (VEV), and this then generates a right-sign Planck mass, $M_P^2 = -(1/6)\alpha\langle\phi\rangle^2$ [3].

For $\alpha < 0$ the theory is only *globally* Weyl (scale) invariant. The associated conserved Noether current is given by:

$$K_\mu = \frac{\delta S}{\delta \partial^\mu \sigma} = (1 - \alpha)\phi \partial_\mu \phi \quad (4)$$

where:

$$D^\mu K_\mu = 4W(\phi) - \phi \frac{\delta}{\delta \phi} W(\phi) \quad (5)$$

For the scale invariant potential of eq.(2) we have that the *rhs* of eq.(5) vanishes and the K_μ current is then covariantly conserved (this can be seen from the combined Einstein and Klein-Gordon equations of motion, [3]).

The Noether current can be written in terms of a “Kernel,” K , given by:

$$K_\mu = \partial_\mu K, \quad K = \frac{1}{2}(1 - \alpha)\phi^2 \quad (6)$$

For the case of N scalars, ϕ_i , with action given by:¹

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{12} \sum_i^N \alpha_i \phi_i^2 R + \frac{1}{2} \sum_i^N \partial_\mu \phi_i \partial^\mu \phi_i - W(\vec{\phi}) \right] \quad (7)$$

with:

$$W(\vec{\phi}) = \sum_i^N \sum_j^N \phi_i^2 W_{ij} \phi_j^2 \quad (8)$$

the Noether current generalises to [3]:

$$K = \frac{1}{2} \sum_{i=1}^N (1 - \alpha_i) \phi_i^2. \quad (9)$$

Using the K -current we can easily understand the dynamics of this theory. If we take the ϕ_i to be functions of time t only, the current conservation equation implies:

$$\ddot{K} + 3 \left(\frac{\dot{a}}{a} \right) \dot{K} = 0. \quad (10)$$

¹ It is straightforward to extend this effective Lagrangian to matter and gauge fields [2, 8, 11].

In a Friedman-Robertson-Walker universe ($g_{\mu\nu} = [1, -a^2(t), -a^2(t), -a^2(t)]$) this can be readily solved to give:

$$K = c_1 + c_2 \int \frac{dt}{a^3(t)}. \quad (11)$$

where $c_{1,2}$ are constants. Therefore in an expanding universe K will evolve to a constant value, $K \rightarrow \bar{K}$. In other words, the scalar fields will rapidly evolve such that their values will be constrained to lie on the N -dimensional ellipsoid given by eq(9).² Since K has dimension of (mass)², a constant vacuum value of K implies a spontaneous breaking of the scale symmetry in the theory has occurred. Note that, unlike the CW mechanism, this does not employ a potential but is driven solely by the initial conditions. K is the order parameter of inertial spontaneous symmetry breaking.

In the single scalar case, as $K \rightarrow \bar{K}$, the Jordan theory flows to the Einstein frame theory with parameters $\Lambda = \frac{\lambda \bar{K}^2}{(1-\alpha)^2}$, $M_P^2 = -\frac{\alpha \bar{K}}{3(1-\alpha)}$, $f^2 = \bar{K}$, explicitly demonstrating how the equivalence between the theories defined in the two different frames is achieved dynamically. In a multi scalar theory the flow $K \rightarrow \bar{K}$ does not fix the relative values of the scalar field VEVs, which initially end up at some random point on the ellipse. It is here that the potential becomes important.

In the infrared (IR) the fields, attached to the ellipse, flow towards an IR fixed point in which the ratios of the field VEVs are determined by the potential terms alone [3]. For the case that the potential has a flat direction, the vacuum energy vanishes at the minimum, corresponding to vanishing cosmological constant. The IR fixed point is then the intersection of the potential's flat direction with the ellipsoid. The ratios of the VEV's is then determined by the scalar potential couplings, but constrained by the requirement the fields lie on the N -dimensional ellipsoid.

For the case that the potential is positive definite, the IR fixed point corresponds to an eternally inflating de-Sitter solution in which the ratio of the field VEV's is determined by the scalar potential couplings together with the couplings, α_i , of the scalars to the Ricci scalar. Thus we see that inertial spontaneous symmetry breaking is responsible for triggering the spontaneous breaking in all sectors of the theory without the need for dimensional transmutation. As such it opens new possibilities for model building.

III. QUANTUM SCALE INVARIANCE AND REGULARISATION

Up to now our discussion has been confined to the classical action. For the scenario of inertial spontaneously broken scale symmetry to work, and lead to a stable Planck mass, it is essential that the Weyl current be identically conserved at the quantum level:

$$D^\mu K_\mu = 0. \quad (12)$$

In what follows we will refer to nonzero contributions coming from loops to the rhs of eq.(12) as ‘‘Weyl anomalies.’’ The trace anomalies of the scale current determined by diffeomorphisms are identical to those in K for the scalar sector of the theory.

Scale and Weyl symmetry of a theory appears ab initio to be broken by quantum loops. Loop divergences are subtle, however, and are often confused with physics. Here we adopt an operating principle that has been espoused by W. Bardeen [7]: The allowed symmetries of a renormalised quantum field theory are determined by anomalies, (or absence thereof). Quantum loop divergences are essentially unphysical artefacts of the method of calculation.

Weyl or scale symmetry is permitted if the renormalised theory has no Weyl anomalies. Since trace anomalies come from triangle diagrams they are necessarily associated with dimension-4 operators. Hence there is no Weyl anomaly in the Standard Model of the form $H^\dagger H$ where the Higgs mass is $m^2 H^\dagger H$. Thus there are no Weyl anomalies associated with quadratic or quartic divergences in quantum field theory in four dimensions. Another way of saying this is that divergent terms and counter terms are not separately measurable, only the renormalised mass is physical. In a variation of the Standard Model with no gravity, no grand unification and no Landau poles in the far UV the Higgs mass would be technically natural with no hierarchy problem!

A. The origin of Weyl anomalies

Our problem of maintaining Weyl symmetry requires that we build a theory that has no anomaly in K_μ . To understand this problem, and its solution, we turn to the CW potential. In computing CW potentials for massless scalar fields we encounter an infrared divergence that must be regularised [6, 12]. To do so we often introduce explicit ‘‘external’’ mass scales into the theory by hand. These are mass scales that are not part of the defining action of the theory, and essentially define the RG trajectories of coupling constants. These externally injected mass scales lead directly to the Weyl anomaly.

We can see this in eq.(3.7) of CW [6] where, to renormalise the quartic scalar coupling constant, λ , in an effective potential at one loop level, $W(\phi)$, they introduce a mass scale M . Once one injects M into the theory, one

² In the 2-scalar model discussed below we have checked numerically that the initial rate of approach to the ellipsoid is very rapid and thereafter the fields precisely track the ellipsoid corresponding to constant K . This is true for a wide range of initial conditions and readily allows for an inflationary period to commence.

has broken scale and Weyl symmetry, and the effective potential in the large $\frac{\phi}{M}$ limit then takes the form

$$W(\phi) = \frac{\beta_1}{4!} \phi^4 \ln \left(\frac{\phi}{M} \right) \quad (13)$$

Here β_1 is the one-loop renormalisation group coefficient, $d\lambda(\mu)/d\mu = \beta_1$. The manifestation of this is seen in the trace of the improved stress tensor [13], and in the divergence of the K_μ current:

$$\partial^\mu K_\mu = 4W(\phi) - \phi \frac{\delta}{\delta \phi} W(\phi) = -\frac{\beta_1}{4!} \phi^4 \quad (14)$$

Of course, there is nothing wrong with the CW potential, or with this procedure, if one is only treating the effective potential as a subsector of the larger theory. If, however, Weyl symmetry is to be maintained as an exact invariance of the world, then M must be replaced by an internal mass scale that is part of action, i.e. M must then be the VEV of a field, χ , or some combination of the fields, appearing in the extended action. We would then have the Coleman-Weinberg potential:

$$W(\phi, \chi) = \frac{\beta_1}{4!} \phi^4 \ln \left(\frac{\phi}{\chi} \right) \quad (15)$$

and, because we now have no external mass scales, the current divergence vanishes:

$$\partial^\mu K_\mu = 4W(\phi, \chi) - \phi \frac{\delta W(\phi, \chi)}{\delta \phi} - \chi \frac{\delta W(\phi, \chi)}{\delta \chi} = 0. \quad (16)$$

This defines the basic idea for maintaining scale symmetry in the quantum theory. It simply implements the notion that there are no fundamental mass scales, and masses are determined only as dimensionless ratios involving VEV's of scalar fields.

B. Weyl Invariant Coleman-Weinberg Calculation

How might we derive such a result as in eq.(15) from first principles? We do so via a computation of a Coleman-Weinberg (CW) effective potential. It is important to realise that CW effective potentials themselves must have the full symmetry of the underlying theory. The symmetry is then broken spontaneously by the minimum of the potential.

In fact it is straightforward to show that the usual regularisation procedure applied to the Weyl invariant theory of eq.(7) *does* have a Weyl invariant form. For the simple two scalar case, $N = 2$, with fields $\phi = \phi_1$ and $\chi = \phi_2$, it reduces to that of eq.(15) when the ratio of VEV's is small, but the general form is applicable for arbitrary values of the ratio.

1. The two scalar action

The case, $N = 2$, is the simplest model with “realistic” phenomenological properties. For reasonable parameter choices and initial conditions it can have an initial inflationary period followed by a “reheat” phase and subsequent evolution to an IR stable fixed point in which the ratio of the field VEVs is determined by the fundamental couplings of the theory. We will illustrate the regularisation procedure applied to this model but we emphasise that the procedure immediately generalises to the case with arbitrary N and indeed to the inclusion of fundamental fermions and vectors.

We start with the action given in eq.(7) with $N = 2$. The Weyl invariance of the theory is spontaneously broken by the VEVs of the fields giving a massless Goldstone boson, the dilaton, σ . It was shown in [8] that the dilaton decouples and so, of the two initial scalar degrees of freedom, only one interacting one remains. To see how this happens in practice, we change variables to:

$$\begin{aligned} \phi_i &= e^{-\sigma/f} \hat{\phi}_i \\ g_{\mu\nu} &= e^{2\sigma/f} \hat{g}_{\mu\nu} \end{aligned} \quad (17)$$

where $\hat{\phi}_i$ are constrained to lie on the ellipse given by:

$$2\bar{K} = \sum_{i=1}^N (1 - \alpha_i) \hat{\phi}_i^2 = f^2 \quad (18)$$

where f^2 is a constant. It is important to note that f is invariant under scale transformations as the dilaton dependence of the original fields has been factored out.

To illustrate the regularisation procedure it is sufficient to calculate the CW potential resulting from the $\frac{\lambda}{4!} \phi_1^4$ term in the potential. We first re-parameterise the fields by:

$$\hat{\phi}_1 = \frac{f}{\sqrt{1 - \alpha_1}} \sin \theta, \quad \hat{\phi}_2 = \frac{f}{\sqrt{1 - \alpha_2}} \cos \theta \quad (19)$$

After scaling out the dilaton, the relevant terms of eq.(7) become:

$$\begin{aligned} S = \int d^4x \sqrt{-\hat{g}} & \left[\frac{1}{2} f^2 \left(\frac{\cos^2 \theta}{(1 - \alpha_1)} + \frac{\sin^2 \theta}{(1 - \alpha_2)} \right) \partial_\mu \theta \partial^\mu \theta \right. \\ & \left. - \frac{\lambda}{4} f^4 \frac{\sin^4 \theta}{(1 - \alpha_1)^2} \right] \end{aligned} \quad (20)$$

Performing the further redefinition $\Theta = F(\theta)$ where:

$$F(\theta) = \int_0^\theta \sqrt{\frac{\cos^2 \theta'}{(1 - \alpha_1)} + \frac{\sin^2 \theta'}{(1 - \alpha_2)}} d\theta' \quad (21)$$

the action becomes:

$$S = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2} f^2 \partial_\mu \Theta \partial^\mu \Theta - \frac{\lambda}{4!} f^4 \frac{\sin^4 F^{-1}(\Theta)}{(1 - \alpha_1)^2} \right]. \quad (22)$$

For the case θ is small the action approximates to the simpler form:

$$S \approx \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{\lambda}{4!} \Phi^4 \right]. \quad (23)$$

where $\Phi = f\Theta$ and $\Theta \approx \frac{\theta}{\sqrt{1-\alpha_1}}$.

2. The CW potential

Here we demonstrate the derivation of the Weyl invariant CW potential for the case $\frac{\phi_1}{\phi_2} \ll 1$, starting with the action of eq(23). Adding a classical source term, $-J\Phi$, to the Lagrangian induces a shift in the Φ field:

$$\Phi = \Phi_c + \hbar^{1/2} \hat{\Phi} \quad (24)$$

where $\hat{\Phi}$ is the small fluctuation about the classical minimum. Thus the potential has the form:

$$W(\Phi) = \frac{\lambda}{4!} \Phi_c^4 + \hbar \frac{\lambda}{4} \Phi_c^2 \hat{\Phi}^2 + \dots \quad (25)$$

where the linear term cancels due to the classical source term. Treating the quadratic term in $\hat{\Phi}$ as an interaction the 1-loop potential with $\hat{\Phi}$ the propagating field is given by:

$$\begin{aligned} W_{eff} &= \Omega + i \int \frac{d^4k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{2n} \left(\frac{\frac{1}{2} \lambda \Phi_c^2}{k^2 + i\varepsilon} \right)^n \\ &= \Omega + \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln \left(1 + \frac{\lambda \Phi_c^2}{2k^2} \right) \\ &= \Omega + \frac{\lambda \Lambda^2}{128\pi^2} \Phi_c^2 - \frac{\lambda^2 \Phi_c^4}{256\pi^2} \ln \left(\frac{\frac{1}{2} \lambda \Phi_c^2 + \Lambda^2}{\frac{1}{2} \lambda \Phi_c^2} \right) \\ &\quad + \frac{\Lambda^4}{64\pi^2} \ln \left(\frac{\frac{1}{2} \lambda \Phi_c^2 + \Lambda^2}{\Lambda^2} \right) \end{aligned} \quad (26)$$

where:

$$\Omega = \frac{\lambda}{4!} \Phi_c^4 - \frac{1}{2} B \Phi_c^2 - \frac{\lambda}{4!} C \Phi_c^4 \quad (27)$$

Note, at the intermediate stage the UV divergences are regulated by introducing a cut-off, Λ^2 , when performing the k^2 integration. Thus, in the $\Lambda \rightarrow \infty$ limit, we have the CW result:

$$W_{eff} = \Omega + \frac{\lambda \Lambda^2}{64\pi^2} \Phi_c^2 + \frac{\lambda^2 \Phi_c^4}{256\pi^2} \left(\ln \frac{\lambda \Phi_c^2}{2\Lambda^2} - \frac{1}{2} \right) \quad (28)$$

Following CW, the renormalisation conditions are:

$$\left. \frac{d^2 W_{eff}}{d\Phi_c^2} \right|_{\Phi_c=0} = 0, \quad \left. \frac{d^4 W_{eff}}{d\Phi_c^4} \right|_{\Phi_c=M} = \lambda, \quad Z|_{\Phi_c=M} = 1 \quad (29)$$

Here CW renormalise at an “external” mass scale, M , to avoid the IR singularity. Implementing these conditions³ determines the counter terms and gives the final CW result:

$$W = \frac{\lambda}{4!} \Phi_c^4 + \frac{\lambda^2 \Phi_c^4}{256\pi^2} \left(\ln \frac{\Phi_c^2}{M^2} - \frac{25}{6} \right) \quad (30)$$

In terms of the original fields $\Phi = f\Theta$, $\Theta \approx \frac{\theta}{\sqrt{1-\alpha_1}}$ and $\theta \approx \hat{\phi}_1/\hat{\phi}_2$, the potential is given by:

$$W \approx \frac{\lambda}{4!} \hat{\phi}_1^4 + \frac{\lambda^2 \hat{\phi}_1^4}{256\pi^2} \left(\ln \left(\frac{C \hat{\phi}_{1c}^2}{\hat{\phi}_{2c}^2} \right) - \frac{25}{6} \right) \quad (31)$$

where $C = \frac{f^2}{M^2} \frac{1}{1-\alpha_2}$ is a constant invariant under scale changes. This is the Weyl invariant CW potential written in terms of the variables $(\hat{\phi}_1, \hat{\phi}_2)$ which are constrained by eq.(19). In addition there is a dilaton, σ , with an isolated kinetic term. By performing a Weyl transformation that is the inverse of eq.(17), we can relax the constraint eq.(19) and obtain,

$$W \approx \frac{\lambda}{4!} \phi_1^4 + \frac{\lambda^2 \phi_1^4}{256\pi^2} \left(\ln \left(\frac{C \phi_{1c}^2}{\phi_{2c}^2} \right) - \frac{25}{6} \right) \quad (32)$$

which is Weyl invariant, and the the fields $(\phi_1, \phi_2) = \exp(-\sigma/f)(\hat{\phi}_1, \hat{\phi}_2)$ are independent variables.

The reason Weyl invariance has been preserved is because the inertial spontaneous symmetry breaking has introduced the mass scale, f , that compensates for the appearance of the renormalisation scale M under the log, leaving the logarithmic terms invariant. Note that the usual renormalisation group equations still apply as a change in the renormalisation scale M (a change in C in eq.(31)) is compensated by a change in the couplings and wave function factors in the usual way.

3. Scale invariant regularisation

The standard regularisation described above clearly preserves Weyl invariance even away from the small $\frac{\phi_1}{\phi_2}$ limit because, on dimensional grounds, the spontaneous scale breaking factor, f , always compensates for the renormalisation scale factor to give an overall constant under the log, together with a function of the scale invariant field $\Theta = f\Phi$.

Expanding eq.(20) beyond leading order leads to higher order terms in θ but these non-renormalisable terms are small. The reason is that Planck scale is predominantly due to the VEV of ϕ_2 whereas the VEV of ϕ_1 , which models the SM Higgs, is at the electroweak scale so that the non-renormalisable terms are Planck

³ There is no wave-function renormalisation at 1-loop order

suppressed. In order to generate the hierarchy in the VEV's at the IR fixed point it is necessary that the only large coupling is λ while the other couplings associated with the other scale invariant quartic interactions are hierarchically small and can be neglected when calculating the radiative corrections.

Of course there will be further terms when the gravitational interactions are included. Gravitational corrections require the addition of the Weyl tensor, W^2 , and R^2 terms, which are induced by matter loops and have logarithmically running coefficients. An analysis of the full renormalization group equations appears in [13]. While the Weyl tensor term is locally invariant, the R^2 term is only globally invariant. Hence we expect to maintain a conserved current, K'_μ , however the current will be modified by the addition of a new term, $K'_\mu = K_\mu + c' \partial_\mu R / f_0^2$ in the notation of [13]. We expect that this is a small correction to the above scenario of a fixed ellipse, but may have some phenomenological implications that will be pursued elsewhere.

Another potentially challenging consequence of the gravitational corrections is that the λ_i become locked to the α_i by the renormalization group. This may necessitate some large fine-tunings to maintain a small cosmological constant and/or flat potentials. We feel that this requires a more sophisticated fundamental analysis since the RG equations computed in flat geometries amount to a “gauge choice” for the Weyl symmetry and do not admit analysis of the Weyl transformation.

Finally, it is possible to maintain the local Weyl symmetry without choosing special values of the α_i , but rather by introducing the Weyl vector potential. When this is done, the dilaton is “eaten” to become the longitudinal part of a massive Weyl vector potential. The relationship of this to gravitational corrections and our general framework is unexplored.

4. Scale invariant dimensional regularisation

Of course regularisation should not depend on the method used to control the intermediate divergences. Up to now we have used a momentum space cut-off but it is straightforward to use dimensional regularisation. In this case one first continues the theory to d -dimensions and introduces an external mass scale, μ , to relate the $4-D$ dimensionless couplings to the dimension-full ones in d -dimensions. For the 2-scalar theory discussed above, dimensional regularisation leads straightforwardly to the form of eq(31) with M_{non} replaced by μ . In this case the quartic and quadratic terms are automatically absent. The dependence on the mass parameter, μ , needed to continue away from four dimensions, will always appear in the scale invariant ratio μ/f giving eq(31) as before.

5. Relation to previous regularisation proposals

Scale invariant dimensional regularisation that differs from the one just described has been considered by several authors - see [5] and references therein. The origin of the difference is that the analyses were performed in flat space and so the decoupling of the dilaton through redefinition of the metric did not apply. Thus, to maintain scale invariance in radiative order, it is necessary to replace μ by a function of the scalar fields, $\mu \rightarrow \mu(\phi_i)$, with the appropriate scaling behaviour. In this case the d -dimensional tree level potential $\tilde{V}$ has the form

$$\tilde{V}(\phi, \chi) \equiv \mu(\phi, \chi)^{4-d} V(\phi, \chi). \quad (33)$$

As a result the tree level potential introduced in eq(33) has *additional* interactions of the form

$$\tilde{W}(\phi, \chi) - W(\phi, \chi) = (4-d) W(\phi, \chi) \ln \mu(\phi, \chi) + O(4-d)^2. \quad (34)$$

Although these interactions vanish in 4 dimensions, they give a finite correction to W_{eff} at 1-loop order because the underlying divergence in 4-dimensions cancels the $4-d$ factor in the additional term in eq(34). Thus, due to the additional interaction terms in eq(34) that depend on the choice of $\mu(\phi, \chi)$, the scale invariant d -dimensional theory is *not* the same as that defined purely in 4-dimensions. As a result the final regulated theory in 4-dimensions has additional terms that depend on the precise choice of the regulator $\mu(\phi, \chi)$. For the 2-scalar case with potential given by eq(33) and the choice $\mu(\phi, \chi) = \chi$ the additional term at one-loop is of the form ϕ^6/χ^2 . While this is still scale invariant it means the resulting 4-dimensional potential is different from that obtained by the regularisation procedure discussed above.

In summary, we have shown that the standard regularisation procedure preserves scale invariance. It does not involve the introduction of an arbitrary regularisation function and, although it involves non-renormalisable interactions, these are well defined. Of course it is possible to add additional non-polynomial terms to the theory while preserving scale invariance but we see no reason so to do.

IV. SUMMARY AND CONCLUSIONS

We have discussed how inflation and Planck scale generation can emerge from a dynamics associated with global Weyl symmetry and its current, K_μ . In the pre-inflationary universe, the Weyl current density, K_0 , is driven to zero by general expansion. However, K_μ has a kernel structure, i.e., $K_\mu = \partial_\mu K$ and, as $K_0 \rightarrow 0$, the kernel evolves as $K \rightarrow \bar{K}$, constant. This resulting constant $\bar{K}$, that does not depend on the scalar potential, is the order parameter of the Weyl symmetry breaking; indeed, $\bar{K}$ directly defines the Planck mass.

In multi-scalar-field theories K has the form $K = \frac{1}{2} \sum_{i=1}^N (1 - \alpha_i) \phi_i^2$. As this is driven to a constant by

gravity, it defines an ellipsoidal constraint on the scalar field VEVs. An inflationary slow-roll period is then associated with the field VEVs migrating along the ellipse. In the flow to the IR the potential ultimately sculpts the structure of the vacuum (together with any quantum effects that may distort the K ellipse) fixing the relative value of the scalar field VEVs through quartic terms only. There is a harmless massless dilaton associated with the dynamical symmetry breaking.

Any Weyl symmetry breaking effect at the quantum level is intolerable and will show up as a nonzero divergence in the K_μ current. We showed how, due to the decoupling of the dilaton, these quantum effects actually preserve the Weyl symmetry using the normal momentum space cut-off or dimensional regularisation schemes. The potential scale dependence introduced by the “external” mass scale needed to regulate the logarithmic divergences is cancelled by the scale invariant order parameter responsible for spontaneous breaking of the Weyl symmetry.

A strong motivation for considering such Weyl invariant theories is as a solution to the hierarchy problem of the Standard Model. In the absence of gravity or very massive states associated with the Landau pole of the Standard Model or of an extension of the Standard Model such as Grand or string unification, the Standard Model is natural in the sense that the quadratic divergence found in radiative corrections to the Higgs mass is unphysical and is cancelled by the mass counter term. Requiring scale invariance ensures that the Higgs is massless but, of course, some mechanism to spontaneously break the scale symmetry is needed.

If gravity is included via the Weyl invariant extension discussed here the Standard Model *plus* gravity is natural in the sense just discussed⁴. Moreover the scale symmetry is now automatically spontaneously broken by the inertial mechanism. To obtain the hierarchy between the Planck scale and the electroweak breaking scale it is necessary to have hierarchically large ratios of the dimensionless couplings of the scalar potential but, in the absence of gravitational radiative corrections, these ratios are only multiplicatively changed by radiative corrections and thus are natural. This may be seen from the underlying shift symmetry of the Weyl invariant Higgs potential.

This symmetry is broken by the Higgs coupling to the Ricci scalar. To determine whether the hierarchy is ultimately preserved requires a calculation of the gravitational radiative corrections which is beyond the scope of the present paper. In a Weyl invariant variation of the Standard Model with no gravity, no grand unification and no Landau poles in the far UV the Higgs mass is technically natural with no hierarchy problem!

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⁴ Of course it is still necessary that there be no massive states strongly coupled to the Higgs with masses much larger than the electroweak scale.

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