Single Particle Dynamics in a Quasi-Integrable Nonlinear Accelerator Lattice

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Abstract

Fermilab is constructing the Integrable Optics Test Accelerator (IOTA) as the centerpiece of the Accelerator R&D Program towards high-intensity circular machines. One of the factors limiting the beam intensity in present circular accelerators is collective instabilities, which can be suppressed by a spread of betatron frequencies (tunes) through the Landau damping mechanism or by an external damper, if the instability is slow enough. The spread is usually created by octupole magnets, which introduce the tune dependence on the amplitude and, in some cases, by a chromatic spread (tune dependence on particle's momentum). The introduction of octupoles usually lead to a resonant behavior and a reduction of the dynamic aperture. One of the goals of the IOTA research program is to achieve a high betatron tune spread, while retaining a large dynamic aperture using conventional octupole magnets in a special but realistic accelerator configuration. In this report, we present results of computer simulations of an electron beam in the IOTA by particle tracking and the Frequency Map Analysis. The results show that the ring's octupole magnets can be configured to provide a betatron tune shift of 0.08 (for particles at large amplitudes) with the dynamical aperture of over 20 beam sigma for a 150-MeV electron beam. The influence of the synchrotron motion, lattice errors, and magnet imperfections is insignificant for the parameters and levels of tolerances set by the design of the ring. The described octupole insert could be beneficial for suppression of space-charge induced instabilities in high intensity machines.

Introduction

Modern circular accelerators are designed with linear focusing optics (also called lattice). In an ideal case, in linear optics the single particle motion is fully integrable, that is, there are as many independent conserved dynamic quantities, as the number of degrees of freedom. For an integrable system the motion is always "solvable" and regular with no chaotic behavior, independent of initial conditions. In reality, focusing lattices are far from being perfectly linear. If a system is linear by design, small nonlinear perturbations break the integrability and may lead

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to a resonant behavior, chaotic and unbounded motion. In accelerators such perturbations arise from high-order magnet errors and imperfections or are introduced on purpose (i.e. sextupole magnets for chromaticity correction), leading to a dependence of a single-particle stability on initial conditions [1].

If a dynamical system is integrable (linear or nonlinear), then according to the Kolmogorov-Arnold-Moser (KAM) theorem the dynamics of the Hamiltonian system remains stable under small perturbations [2]. An example of an integrable nonlinear system is the Kepler problem, where $U \propto 1/r$. The motion is regular and stable (not necessarily bounded), and for bounded orbits the oscillation frequency strongly depends on the amplitude: $\nu \propto a^{-3/2}$. Obviously, such a potential is of limited use in particles accelerators because of its singularity at the origin.

Generally, there has not been a lot of success in funding integrable nonlinear focusing lattices for circular accelerators. Nonetheless, in the late 1960's McMillan discovered a 1-dimensional nonlinear mapping – a thin kick of a special kind in an otherwise linear lattice leading to a conservation of an integral of motion, quadratic in particle's coordinate and momentum. This solution was later generalized to 2D-case of an uncoupled symmetric lattice [3]. The required nonlinear kick can be created by, for example, an electron lens [4] but difficult to implement with magnets.

Recently, in 2010 a new nonlinear accelerator focusing system, which can be implemented with magnets, was proposed [5]. It consist of a linear focusing lattice with a transfer matrix of a thin axially symmetric lens, followed by a nonlinear magnetic field region, which can be created by a special magnet, and, in 2D, yields a conservation of the Hamiltonian and the second invariant of motion, quadratic in momenta. In contrast to the traditional linear focusing, it provides large a amplitude-dependent tune spread, which is beneficial for Landau damping of collective instabilities, with single particle motion being unconditionally stable and non-chaotic at all amplitudes. This concept is to be experimentally tested at the Integrable Optics Test Accelerator (IOTA) at Fermilab [6].

The potential in [5] has two points of singularity, making the design, manufacturing, and testing of the special magnet rather challenging [7]. As a consequence, the first stage of the experiment will use conventional octupole magnets to create a corresponding term in the multipole expansion of the potential. In theory, this approach allows to achieve single particle dynamics with one invariant of motion (out of two) – a "quasi-integrable" case. Even a single integral of motion can significantly improve particle dynamics in terms of achievable tune shifts and dynamical aperture. For instance, it proved to be beneficial for achieving a record-high beam-beam tune shift with round colliding beams at VEPP-2000 (BINP, Russia) [8].

Since the paper [5] considered only transverse motion, it was important to make sure that the introduction of the 3rd, longitudinal, degree of freedom does not destroy the integrability. Earlier works on simulations of dynamics in IOTA [6], [9] did not consider the quasi-integrable case, and treated the ring as a linear transfer matrix or disregarded the longitudinal motion. A thorough study of dynamics in a realistic machine lattice in the presence of coupling, chromaticity, magnet imperfections, and parasitic nonlinearities was required to estimate the influence of these effects. In this paper we present the results of 3D particle tracking simulations of single particle dynamics in a quasi-integrable optics experiment at IOTA.

Octupole experimental setup in IOTA

IOTA is a relatively small storage ring with the circumference of 40 m, designed to circulate either electrons or protons of the same momentum. In the electron mode, it will operate with short bunches of 150 MeV electrons injected from a superconducting linac [10]. The ring has a flexible linear optics, allowing the installation of one or two nonlinear magnets with the transfer matrix of the rest of the ring outside those magnets being an equivalent of a thin axially symmetric lens. The ring's focusing optics has a two-fold symmetry; its layout and optics functions are shown in FIG. *1*, and main parameters of the machine are summarized in Table I. A detailed description of the ring can be found in [9].

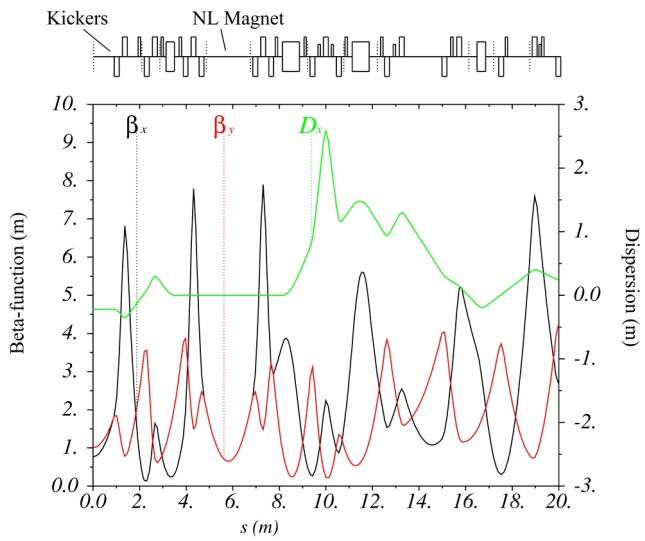

FIG. 1. The beta functions (horizontal – black line, vertical – red) are equal and the horizontal dispersion (green) is zero in the nonlinear insert. A half of the IOTA ring for the setup of one nonlinear insert is depicted; ring elements and their positions are shown on the top.

TABLE I. Main parameters of IOTA ring, setup with one nonlinear insert.

Electron energy	150 MeV
Number of bunches, particles per bunch	1, 10^9
Ring circumference	40 m
Synchrotron decay radiation damping time	1 s
Equilibrium emittance, x & y, RMS	0.074 mm-mrad
Betatron tunes, x & y	5.3, 5.3
Natural tune chromaticities, x & y	-11.4, -7.1
Synchrotron tune	$5.3\square 10^{\square 4}$
RF harmonic number and voltage	4, 10.13 kV
RMS bunch length	511 cm

At the first stage of its experimental program IOTA will study beam dynamics in a nonlinear lattice with one invariant of motion, proposed in [5]. The nonlinearity will be created by a set of octupole magnets, placed in one of the 1.8 m long drifts with equal beta-functions and zero dispersion (FIG. *1*). Their strength will vary inversely proportional to β^3, creating a potential:

$$V(x,y;s) = \frac{\alpha}{\beta(s)^3}\left(\frac{x^4}{4} + \frac{y^4}{4} - \frac{3x^2y^2}{2}\right) \tag{1}$$

In the normalized coordinates $U = \beta(s)V(x,y;s)$, and the resulting Hamiltonian (neglecting the longitudinal motion) is

$$H = H_0 + U = \frac{1}{2}\left(P_x^2 + P_y^2 + x_N^2 + y_N^2\right) + \alpha\left(\frac{x_N^4}{4} + \frac{y_N^4}{4} - \frac{3x_N^2 y_N^2}{2}\right), \tag{2}$$

where x_N, y_N, P_x, and P_y are the normalized coordinates and momenta. H is independent of the longitudinal position s and thus is an invariant of motion. Figure 2 shows the distribution of octupole strength in the channel; B_3 is the normalized octupole gradient:

$$B_3 = \frac{1}{B\rho}\frac{\partial^3 B}{\partial x^3} = 6\alpha\frac{1}{\beta(s)^3} \tag{3}$$

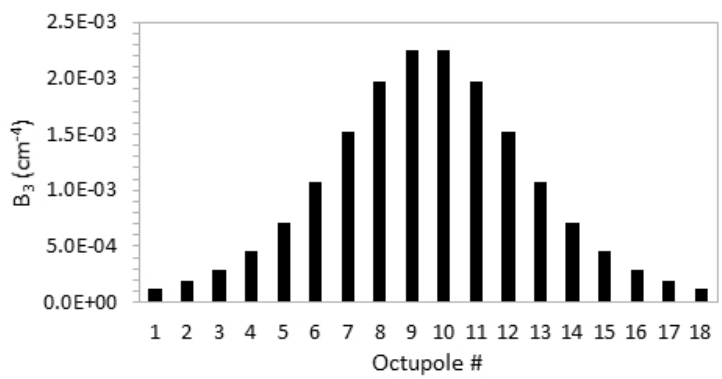

FIG. 2. Octupole strength is maximal in the middle of the nonlinear section and rapidly decreases towards the ends. Magnetic length of each octupole – 10 cm.

Numerical model

Our model of the ring incorporated dipoles with their fringe fields, quadrupoles, and an RF cavity – all ring elements, except the injection magnet, kickers, and corrector magnets. The nonlinear potential (1) was created by a set of 10-cm-long octupole magnets. The strengths of the

magnets were governed by the expression (3). Thus, we simulated 3D particle dynamics in a real machine with betatron and synchrotron motion coupled through dispersion and chromaticity. To speed up the computations thick elements were represented by a number of thin kicks and drifts in a symplectic manner.

In the planned experiment the electron bunch will be kicked in the transverse plane using a horizontal and a vertical kicker. The bunch trajectory then will be recorded turn-by-turn by the beam position monitors to create a Poincare map of the phase space. This procedure was simulated by giving the electrons an initial displacement in 2D (x and y) and tracking their positions on each turn at the center of the nonlinear section.

Numerical simulations were performed using Lifetrac particle tracking code [11] running on Fermilab's Wilson cluster. The code was initially developed for beam-beam simulations but can also be used for 3D tracking of particles in the presence of an arbitrary nonlinear potential. Two methods were used to study particle dynamics: tracking and Frequency Map Analysis. FMA, originally proposed by Laskar in [12], thanks to its high computing performance allows to rapidly scan through the phase space and determine frequency shifts as functions of betatron oscillation amplitudes, identify potentially dangerous resonances, and find the boundary of chaotic region [13].

FMA finds only the single dominant frequency of particle motion and may fail to resolve multiple Fourier peaks, which arise, when a particle is close to a resonance. So in addition to FMA we performed tracking for a time of synchrotron radiation decay(~ 10^6 turns) to treat such situations and determine the dynamic aperture of the machine exactly. A smaller subset of initial amplitudes was used for this purpose. Initial amplitude both in x and y was increased until a particle loss was observed.

Results and discussion

FMA footprints for a case of ideal lattice with one octupole section are shown in FIG. 3.3. A wide resonance line at 45 degrees in FIG. 3. (a) (red) is the difference resonance $Q_x - Q_y = 0$. Tracking shows that particles in this region remain stable due to the absence of crossings with other resonances, which could lead to transfer of energy between degrees of freedom and eventually particle loss. Dynamical aperture for this case is 21σ of the electron beam. Phase space footprint corresponding to the stable region is depicted in FIG. *3*.3 (b). Tune shifts in Q_x

and Q_y range from -0.05 to +0.03, creating a total maximum tune spread of 0.08 for a beam occupying the entire stable region.

FMA also shows numerous thin resonance lines (green). Those are high order resonances (typical order in the range 10 - 20) involving both betatron and synchrotron motion. They are unlikely to be excited in the real machine and consequently would not disturb the particle motion.

In the ideal case described above, the particles started with no initial synchrotron motion: $dE/E = 0$. It arose due to coupling between the horizontal and longitudinal degrees of freedom. The average amplitude of synchrotron oscillations increased with betatron amplitude up to $3\sigma_s$ at the boundary of the stable region.

Figure 4 shows relative error in the integral of motion, Hamiltonian (2), as a function of betatron amplitudes. One can see that despite the presence of synchrotron motion the Hamiltonian conserves on a level of 10^{-4}-10^{-3} up to the amplitudes of 20σ.

In a non-ideal case, for the particles with initial synchrotron oscillation amplitudes $A_s < 1\sigma_s$, the resulting frequency maps differ from the ideal picture insignificantly. In those the cases the initial synchrotron amplitude is much smaller than the amplitude, arising from the betatron motion due to betatron-synchrotron coupling. However, for $A_s > 6\sigma_s$ the synchrotron motion starts to destroy the dynamics significantly, and the dynamical aperture can be reduced by 15% or more.

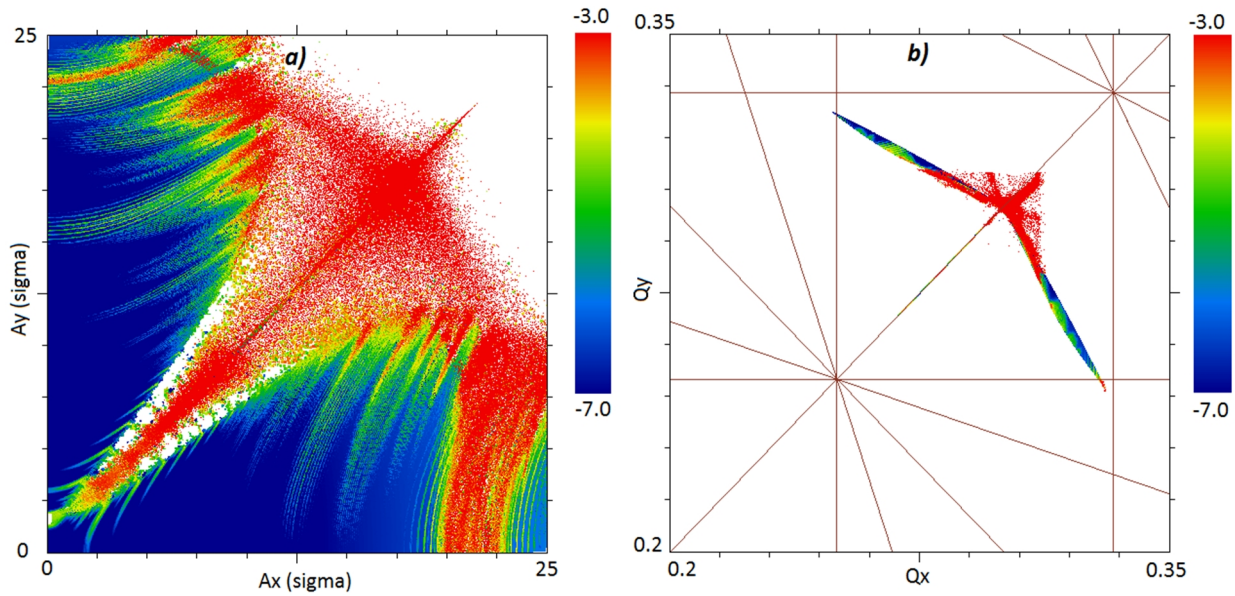

FIG. 3. FMA footprints of phase space in average betatron amplitudes – a) and betatron frequencies – b).

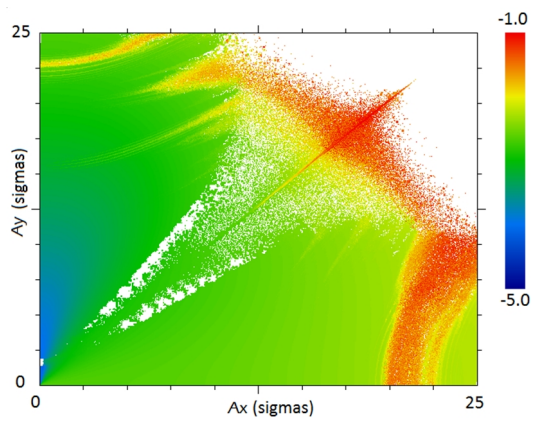

FIG. 4. Jitter in the Hamiltonian is of the order of or below 10^{-3} for most amplitudes of stable betatron oscillations.

Apart from the synchrotron motion, another factor affecting the integrability is the chromaticity of betatron tunes. It introduces a first order perturbation to the Hamiltonian if x- and y-chromaticities are not equal [14]. To obtain numerical estimates of this effect a part of the ring was replace by a transfer matrix with phase advances in x and y depending on p_z. By changing this artificially introduced chromaticity we were able to get a dependence of dynamic aperture on the difference of betatron frequencies $\Delta Q_{xy} = (\xi_x - \xi_y) \cdot \delta p / p$, depicted in FIG. 5.5. One can also see that the dynamical aperture is the greatest when $\xi_x \approx \xi_y$, as predicted in [14]. The natural

chromaticities of IOTA: -11, -7 are close to this condition. Dynamical aperture decreases as the difference in tunes, created by unequal chromaticities, shifts further away from 0. For $\left|\Delta Q_{xy}\right| > 10^{-3}$ it is less than ½ of the maximum value.

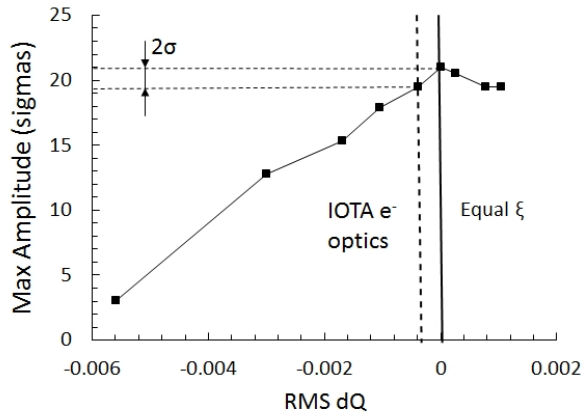

FIG. 5. Tune mismatch, created by chromaticity, leads to reduction of dynamic aperture. For electron operation the decrease is about 2 RMS beam sizes.

Since most of the particles in IOTA electron bunch will have dp/p less or of the order of $(dp/p)_{RMS}$, the longitudinal motion does not affect the transverse dynamics significantly. Though it might become an issue for the proton experiment at IOTA because of the larger momentum spread of the proton beam (~ 10^{-3}).

To study the influence of errors in the linear lattice, the strengths of several quadrupole magnets were manually varied to detune Q_y from its design value 0.3, while leaving Q_x unperturbed. Previous similar studies of fully integrable potential show that it requires a precise matching of beta functions and tune to the design values. An error of more than 10^{-3} in phase advance over the ring breaks the integrability [9]. In contrast to the fully integrable case, the quasi-integrable optics is more robust against errors in linear focusing: errors of the order of 10^{-3} in betatron tunes produce no visible effect on particle dynamics, $\Delta Q_y = 10^{-2}$ ($\Delta\beta/\beta$ in the middle of the insert of 0.06) leads to a reduction of dynamic aperture of ~ 10%, and for $\Delta Q_y = 10^{-1}$ ($\Delta\beta/\beta = 0.2$) dynamic aperture shrinks by a half (FIG. 6.6).

To estimate the influence of imperfections in the nonlinear potential on particle dynamics, a normally distributed error was added to each of the octupole magnets. The simulations show that for a 10% r.m.s. error in octupole strength, the reduction of dynamical aperture is negligible.

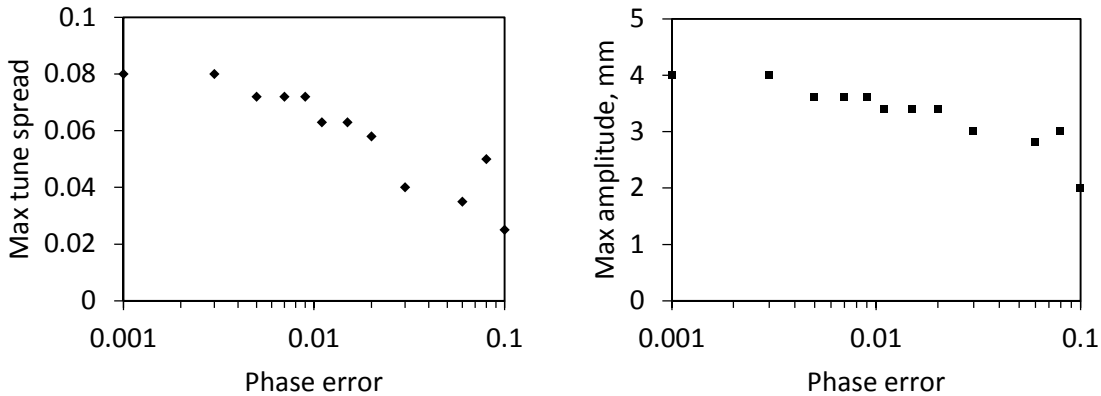

FIG. 6. Detuning of linear lattice leads to decrease of maximum tune spread (a) and dynamic aperture (b).

FIG. 7.7 shows the dependence of maximum tune shift and betatron amplitude on the strength of octupole potential. For small strength parameters α, the amplitude is limited by physical aperture and the maximum tune shift increases with α. For sufficiently big α, the dynamical aperture is limited by resonances (FIG. 3.3(b)), and consequently the maximum tune shift is independent of the potential strength. Thus there is a value of α, which yields the greatest dynamical aperture for the maximum possible tune shift. Our modelling shows that in IOTA this value is $\alpha = 100$ cm^{-1}.

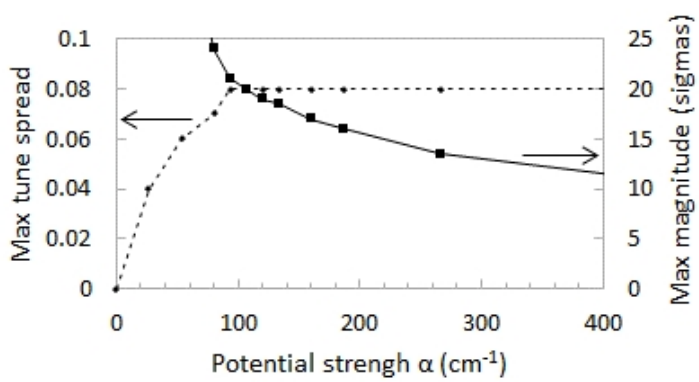

FIG. 7. Maximum tune spread is achieved at the strength of octupole potential $\alpha = 100$ cm^{-1}; further increase of nonlinearity leads only to reduction of dynamical aperture.

Since the octupole focusing strength is quadratic in betatron amplitude and the maximum octupole tune shift is independent of the strength above a certain value, there is a tradeoff between the dynamical aperture and r.m.s. tune shift:

$$dQ_{RMS} = \frac{dQ_{\max}}{A_{\max}^2} \tag{4}$$

FIG. 8.8 depicts a dependence of maximum tune shift on the phase advance over the drift section. For the purpose of this simulation the rest of the ring was treated as an axially-symmetric focusing transfer matrix with an integer phase advance. One can see that the dependence is close to linear and can be approximated as $dQ_{\max} \sim 0.4\Delta\Psi$, where $\Delta\Psi$ stands for the phase advance in the drift. Since the phase advance in the ring outside of the nonlinear section shall always be a multiple of 2π, the ring tunes change with $\Delta\Psi$ and when it gets close to a low order resonance the maximum tune spread decreases (point 0.35 on the plot).

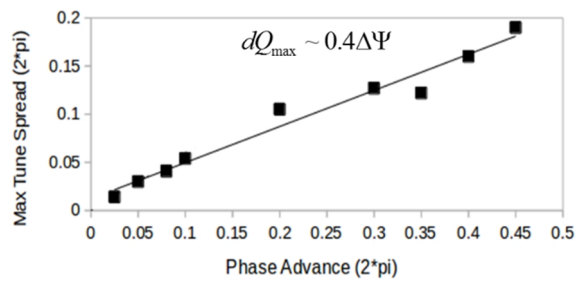

FIG. 8. Maximum tune spread increases linearly with the phase advance in the insert.
One insert in the ring, the rest of the ring was modelled by a linear transfer matrix.

The tune spread, created by the octupole channel, can be used to create Landau damping of collective instabilities in a high intensity beam. Note that there are other ways to create the spread of betatron tunes: for example, tune chromaticity. But linear chromaticity provides Landau damping only in the cases a coasting beam or very long bunches, where the period of synchrotron oscillations is significantly greater than the characteristic growth time of the instability: $T_s \gg \tau_{inst}$. In a more common case of short bunches with relatively low T_s the betatron tune shift due to chromaticity, averaged over the synchrotron period, is zero to the first order: $\langle \xi\, dp/p \rangle_s = 0$ for every particle in the bunch, and thus there is no damping. Although it does not provide damping (at the first order), tune chromaticity affects the instability through its growth rate. Correcting the chromaticity reduces the growth rate or even stabilizes the most unstable collective mode, which is typically the lowest frequency one [15]. The amount of correction one can achieve without too much damage to dynamic aperture and beam lifetime is specific for every accelerator as it depends on the working point, beam intensity, etc. In some cases it may be

more beneficial to create additional Landau damping, introducing nonlinear tune spread with octupoles.

Knowing the tune spread one can estimate the strength of Landau damping. According to Burov [15], the damping rate of space charge driven collective instabilities is quadraticaly proportional to RMS nonlinear tune spread:

$$\Lambda_x = \frac{\Delta Q_{xx}^2}{\Delta Q_{SC}} F(a_{xx}, a_{xy}),$$
$$\Delta Q_x = a_{xx} J_x + a_{xy} J_y = \Delta Q_{xx} + \Delta Q_{xy} \tag{5}$$

where ΔQ_{SC} is a space charge tune shift and F is a damping factor. For the proposed parameters of PIP-II Recycler upgrade [16], a single octupole insert is capable of creating the tune spread up to $5*10^{-3}$, but the downside is a relatively small resulting dynamic aperture: about 4 sigma in ideal case with no imperfections. A possible solution for achieving a greater spread may be using more than one octupole insert. We shall also notice that [15] gives only a qualitative estimate of the damping rate, and a detailed numerical study of Landau damping in the presence of nonlinear tune spread is required. Such research is currently under way at FNAL, and the first simulation results are consistent with the theory [17].

Conclusion

We have studied single particle dynamics in the Integrable Optics Test Accelerator using a realistic 3D model of the ring. The numerical simulation allows to determine nonlinear tune shifts and dynamical aperture in the presence of lattice errors and magnet imperfections. The results show that a tune spread of 0.08 can be achieved with conventional octupole magnets. The effect of longitudinal motion and lattice chromaticity on the dynamics of electrons in the ring are insignificant, although they may create certain challenges for proton operation due to higher momentum spread.

We have found that for octupole potential the tolerances to imperfections can be higher than for the fully integrable case. Whereas for fully integrable optics the IOTA ring needs a precise control of lattice errors at the level of 10^{-3}, for a quasi-integrable case errors of up to 0.01 in betatron tunes, 6% in beta-functions, and 10% in the potential itself lead to less than 15% reduction of dynamical aperture. This level of precision of can be realistically achieved in future high-intensity circular accelerators. They can benefit from employing the described “quasi-

integrable" optics by creation of significant nonlinear tune spreads to provide Landau damping of collective beam instabilities.

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