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Unification of gauge, family, and flavor symmetries illustrated in gauged SU(12) models

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Abstract

To explain quark and lepton masses and mixing angles, one has to extend the standard model, and the usual practice is to put the quarks and leptons into irreducible representations of discrete groups. We argue that discrete flavor symmetries (and their concomitant problems) can be avoided if we extend the gauge group. In the framework of SU(12) we give explicit examples of models having varying degrees of predictability obtained by scanning over groups and representations and identifying cases with operators contributing to mass and mixing matrices that need little fine-tuning of prefactors. Fitting with quark and lepton masses run to the GUT scale and known mixing angles allows us to make predictions for the neutrino masses and hierarchy, the octant of the atmospheric mixing angle, leptonic CP violation, Majorana phases, and the effective mass observed in neutrinoless double beta decay.

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I. INTRODUCTION

Family and flavor symmetries of the observed quarks and leptons appear to be intimately related and remain much of a mystery today as to their precise structures. Although there is some ambiguity in the literature, here we make use of family symmetry to relate particles within a family of quarks and leptons as in the standard model (SM) or within some grand unified symmetry (GUTs) such as SU(5), SO(10) or E_6. Flavor symmetry, on the other hand, relates families which appear to be replicas of each other. The flavor symmetry may be continuous as in the case of SU(3), SU(2), U(1) or discrete as in the case of Z_2, $Z_2 \times Z_2$, S_3, A_4, S_4, etc. (For reviews see [1–3].) The conventional picture is to assume a direct product symmetry group, $\mathbf{G}_{\text{family}} \times \mathbf{G}_{\text{flavor}}$, where $\mathbf{G}_{\text{family}}$ is gauged but $\mathbf{G}_{\text{flavor}}$ is discrete. The necessity of $\mathbf{G}_{\text{flavor}}$ reflects the replication of families due to the fact that there are too few chiral, exotic free, irreducible representations (irreps) in the family groups for the observed chiral fermion families: just $\bar{\mathbf{5}}$ and $\mathbf{10}$ for SU(5), $\mathbf{16}$ for SO(10), and $\mathbf{27}$ for E_6.

Family and flavor unification requires a higher rank simple group. Some early attempts were based on SU(11), SU(8), SU(9) and SO(18), [4–8] but none were completely satisfactory [9]. More recently such unification has been proposed in the framework of string compactification, see [10]. Here we describe SU(12) models with interesting features that were constructed with the help of a Mathematica computer package written by one of the authors (RPF) called LieART [11]. This allows one to compute tensor products, branching rules, etc., and perform detailed searches for satisfactory models, although the predictions of such models are limited by the number of parameters needed to describe the data. We find that after all the known quark and lepton mass and mixing data are used to fit the data to our models, some predictions arise for the yet-unknown results for the neutrino mass hierarchy and individual masses, leptonic CP violation, octant for the atmospheric mixing angle, and the effective mass that can be observed in neutrinoless double beta decay.

Expanding the gauge group to eliminate all or part of the family and flavor symmetries has been discussed previously, see references [12–14]. An earlier version of an SU(12) model was previously published [15, 16], but subsequently several issues were found with some of the details, which are corrected here. In addition, we have adopted a new approach, made a more extensive study of the possibilities within this SU(12) framework, and present a more comprehensive treatment of these models, all of which are discussed below.

II. INGREDIENTS OF A UNIFICATION GROUP

Our starting point is a supersymmetric SU(N) unification group, where N must be large enough to assign chiral SU(N) matter families to a number of irreps without the need for a flavor symmetry to distinguish the families. In practice this requires $N \geq 8$, while models derived from orbifold compactifications of SO(32) and the heterotic string suggest $N \leq 14$ [17]. The larger SU(N) GUT group replaces both the conventional GUT and the flavor groups cited earlier.

A crucial issue then concerns the breaking of the large SU(N) group to a smaller GUT/family group such as SU(5) which we choose for the rest of this paper. We consider symmetry breaking that occurs in two possible ways. In the conventional approach, the symmetry is broken one step at a time with the help of the SU(N) adjoint scalar fields:

$$\mathrm{SU}(N) \to \mathrm{SU}(N-1) \times \mathrm{U}(1) \to ... \to \mathrm{SU}(5) \times \mathrm{U}(1)^{N-5}. \tag{1}$$

Then complex irreps are typically needed to break the U(1)'s and reduce the rank to 4. (This choice was improperly made in [15] and negates some of the results of that paper.) The other choice which we employ here reduces the rank in one step without any U(1)'s occurring, i.e., SU(N) $\to$ SU(5). This direct breaking preserves SUSY provided N is even, no SU(N) adjoint is present, and the F-flat and D-flat conditions hold. As shown in [18–20], a dramatic reduction in rank is possible provided the sum of the Dynkin weights vanishes for the vacuum expectation values (VEVs) involved in lowering the rank. This possibility exists for $N = 12$, as is easily demonstrated in Appendix A.

Other necessary conditions for a satisfactory unification group are the following. The matter fields must form anomaly-free sets of SU(N) and SU(5) irreps with three SU(5) families. Restrictions on the Higgs fields also must obtain. The SU(5) Higgs singlets must arise from SU(N) conjugate pairs to ensure D-flat directions, and they acquire SU(5) VEVs at the SU(5) GUT scale where the separation of scales is given by $M_{\mathrm{SU}(5)}/M_{\mathrm{SU}(N)} \sim 1/50$. With the SUSY GUT scale occurring around 2×10^{16} GeV, this implies the SU(N) scale can be as high as 10^{18} GeV, very close to the string scale. In addition, an SU(5) adjoint **24** should be present to break the SU(5) symmetry to the SM, but this adjoint should not be contained in an SU(N) adjoint which would spoil the desired symmetry-breaking pattern. One set or mixtures of two sets of Higgs doublets in **5** and $\overline{\mathbf{5}}$ of SU(5) must be available to

break the electroweak symmetry at the weak scale. The addition of massive matter pairs at the SU(N) scale will then allow one to introduce an effective operator approach.

III. SU(12) UNIFICATION MODELS

After an extensive, but not exhaustive scan of possible SU(N) models, we have found a relatively economical set of models for $N = 12$. Thus, for the rest of this paper we confine our attention primarily to the SU(12) unification group. This group has twelve antisymmetric irreps, ten of which are complex, while the **924** and singlet are real:

$$\mathbf{12},\ \mathbf{66},\ \mathbf{220},\ \mathbf{495},\ \mathbf{792},\ (\mathbf{924}),\ \overline{\mathbf{792}},\ \overline{\mathbf{495}},\ \overline{\mathbf{220}},\ \overline{\mathbf{66}},\ \overline{\mathbf{12}},\ (\mathbf{1}) \tag{2}$$

which can be represented by Young diagrams with one to twelve blocks stacked vertically in a single column. These irreps contain no SU(5) exotics. Among the smaller anomaly-free sets containing exactly three families of SU(5) fermion matter are the following:

$$\begin{aligned}
&\mathbf{66} + \mathbf{495} + 2(\overline{\mathbf{220}}) + 2(\overline{\mathbf{12}}) \\
&\mathbf{220} + 3(\overline{\mathbf{66}}) + 3(\overline{\mathbf{12}}) \\
&3(\mathbf{12}) + \mathbf{220} + \mathbf{792} + \overline{\mathbf{495}} + 3(\overline{\mathbf{66}}) \\
&\mathbf{66} + 2(\mathbf{495}) + \overline{\mathbf{792}} + 2(\overline{\mathbf{220}}) + 8(\overline{\mathbf{12}}) \\
&\mathbf{220} + \mathbf{495} + \overline{\mathbf{792}} + 3(\overline{\mathbf{66}}) + 9(\overline{\mathbf{12}}) \\
&3(\mathbf{66}) + 3(\mathbf{792}) + 3(\overline{\mathbf{495}}) + 6(\overline{\mathbf{12}}) \\
&3(\mathbf{66}) + 2(\mathbf{792}) + 2(\overline{\mathbf{495}}) + 12(\overline{\mathbf{12}})
\end{aligned} \tag{3}$$

where we assume any complex conjugate pairs of irreps become massive at the SU(12) scale.

Two of the anomaly-free sets, the first and fourth, are of special interest for the third family top and bottom quarks are neatly contained in the **66** which has one 10-dimensional irrep in the SU(5) subgroup. For the fourth set, the rank-7 SU(12)/SU(5) factor group can be completely broken in one step while preserving supersymmetry with the aid of the SU(5)-singlet chiral superpartners of the fermions acquiring VEVs at the SU(12) unification scale. For the simplest first set, one needs the help of one additional scalar pair acquiring a VEV at the unification scale. Examples are illustrated in Appendix A.

With the aid of the SU(12) $\to$ SU(5) branching rules:

$$\begin{aligned}
\mathbf{66} &\to 7(\mathbf{5}) + (\mathbf{10}) + + 21(\mathbf{1}),\\
\mathbf{495} &\to 35(\mathbf{5}) + 21(\mathbf{10}) + 7(\overline{\mathbf{10}}) + (\overline{\mathbf{5}}) + 35(\mathbf{1}),\\
\overline{\mathbf{792}} &\to 7(\mathbf{5}) + 21(\mathbf{10}) + 35(\overline{\mathbf{10}}) + 35(\overline{\mathbf{5}}) + 22(\mathbf{1}),\\
\overline{\mathbf{220}} &\to (\mathbf{10}) + 7(\overline{\mathbf{10}}) + 21(\overline{\mathbf{5}}) + 35(\mathbf{1}),\\
\overline{\mathbf{12}} &\to (\overline{\mathbf{5}}) + 7(\mathbf{1}).
\end{aligned} \tag{4}$$

one can see that the two anomaly-free sets of interest break at the SU(5) scale to the following sets of SU(5) irreps according to

$$\begin{aligned}
\mathbf{66}+\mathbf{495}+2(\overline{\mathbf{220}})+2(\overline{\mathbf{12}}) &\to 3(\mathbf{10}+\overline{\mathbf{5}}+\mathbf{1}) + 21(\mathbf{10}+\overline{\mathbf{10}}) + 42(\mathbf{5}+\overline{\mathbf{5}}) + 137(\mathbf{1}),\\
\mathbf{66}+2(\mathbf{495})+\overline{\mathbf{792}}+2(\overline{\mathbf{220}})+8(\overline{\mathbf{12}}) &\to 3(\mathbf{10}+\overline{\mathbf{5}}+\mathbf{1}) + 63(\mathbf{10}+\overline{\mathbf{10}}) + 84(\mathbf{5}+\overline{\mathbf{5}}) + 236(\mathbf{1}),
\end{aligned} \tag{5}$$

where both have three chiral families containing the observed lefthanded quarks and leptons and lefthanded antiquarks and antileptons. The conjugate paired irreps all become massive at the SU(12) scale and are of no more interest to us.

The three SU(5) families can then be selected from among the following:

$$\begin{aligned}
(\mathbf{10})\mathbf{66} &= (2+0), & (\overline{\mathbf{5}})\mathbf{495} &= (4+0),\\
(\mathbf{10})\mathbf{495} &= (2+2), & (\overline{\mathbf{5}})\overline{\mathbf{792}} &= (4+3),\\
(\mathbf{10})\overline{\mathbf{792}} &= (2+5), & (\overline{\mathbf{5}})\overline{\mathbf{220}} &= (4+5),\\
(\mathbf{10})\overline{\mathbf{220}} &= (2+7), & (\overline{\mathbf{5}})\overline{\mathbf{12}} &= (4+7),
\end{aligned} \tag{6}$$

where by Eq. (5) up to two sets of $\mathbf{495}$ and $\overline{\mathbf{220}}$ and possibly more for $\overline{\mathbf{12}}$ are available for selection. For our purposes, no discrete symmetry is needed to distinguish them. We have chosen an SU(12) basis $(a+b)$, where the first number a in parenthesis refers to the number of SU(5) boxes placed on top of the second remaining number b of SU(12)/SU(5) boxes in the column of Young diagram boxes. If two columns are present in a diagram representing higher dimensional irreps, the two pairs of numbers will be separated by a comma.

Singlet Higgs conjugate pairs can be selected from among:

$$\begin{aligned}
(\mathbf{1})\mathbf{12}_{\rm H} &= (0+1), & (\mathbf{1})\overline{\mathbf{12}}_{\rm H} &= (5+6),\\
(\mathbf{1})\mathbf{66}_{\rm H} &= (0+2), & (\mathbf{1})\overline{\mathbf{66}}_{\rm H} &= (5+5),\\
(\mathbf{1})\mathbf{220}_{\rm H} &= (0+3), & (\mathbf{1})\overline{\mathbf{220}}_{\rm H} &= (5+4),\\
(\mathbf{1})\mathbf{495}_{\rm H} &= (0+4), & (\mathbf{1})\overline{\mathbf{495}}_{\rm H} &= (5+3),\\
(\mathbf{1})\mathbf{792}_{\rm H} &= (0+5) \text{ or } (5+0), & (\mathbf{1})\overline{\mathbf{792}}_{\rm H} &= (5+2) \text{ or } (0+7).
\end{aligned} \tag{7}$$

For simplicity we shall assume that the VEVs of the SU(5) Higgs singlets chosen in each model and their couplings to fermions are real and equal.

As emphasized earlier, a 24-plet Higgs, which must be present to break the SU(5) GUT symmetry down to the SM, can not be part of the SU(12) adjoint **143** in the one-step breaking of SU(12) to SU(5). Instead, we find it best to include the SU(5) adjoint in the complex pair of $(\mathbf{24})\mathbf{5148}_{\rm H}$ and $(\mathbf{24})\overline{\mathbf{5148}}_{\rm H}$ Higgs irreps which can develop VEVs at the GUT scale. In fact, the SU(12) breaking of this **5148** and $\overline{\mathbf{5148}}$ pair yields only one (**24**) each, as can be seen from the following decomposition for the **5148**,

$$\mathbf{5148} \to (\mathbf{24}) + 245(\mathbf{5}) + 147(\mathbf{10}) + 49(\overline{\mathbf{10}}) + 7(\overline{\mathbf{5}}) + 35(\mathbf{15}) + 21(\overline{\mathbf{40}}) + 7(\overline{\mathbf{45}}) + 224(\mathbf{1}), \quad (8)$$

and similarly for the conjugate irrep. The (24) Higgs contributions are represented by

$$(\mathbf{24})\mathbf{5148}_{\rm H} \;=\; (4+0, 1+0), \qquad (\mathbf{24})\overline{\mathbf{5148}}_{\rm H} \;=\; (4+7, 1+7). \quad (9)$$

Because these irreps represent complex pairs, we shall also assume that their VEVs are complex conjugates of each other and assign a common VEV to the quarks and a different common VEV to the leptons. This can be accomplished if their VEVs point in the $2B-L$ direction which is a linear combination of the λ_{15} and λ_{24} generators of SU(5):

$$\begin{aligned} 2B-L \;&=\; (5/12)\sqrt{6}\lambda_{15} + (1/4)\sqrt{10}\lambda_{24} \\ &=\; \mathrm{diag}(2/3,\ 2/3,\ 2/3,\ -1,\ -1) \end{aligned} \quad (10)$$

We then adopt the following notation for their VEVs:

$$\langle(\mathbf{24})\mathbf{5148}_{\rm H}\rangle = (2B-L)\kappa, \qquad \langle(\mathbf{24})\overline{\mathbf{5148}}_{\rm H}\rangle = (2B-L)\kappa^*, \quad (11)$$

Hence this choice provides a ready way in which to introduce complex phases into the mass matrices. The different VEVs generated from these Higgs fields will also prove useful to break the down-quark and charged-lepton mass spectral degeneracy.

In addition, we need a Higgs singlet to give mass to the lefthanded conjugate neutrinos. Since all families of such neutrinos are in SU(5) and SU(12) singlets, it is convenient to introduce a $(\mathbf{1})\mathbf{1}_{\mathbf{H}}$ Higgs singlet for this purpose. A dim-4 vertex mass diagram then requires that this Higgs singlet must change lepton number by two units, or $\Delta L = +2$.

In general, two sets of Higgs doublets which remain light down to the EW scale where they get VEVs can be formed from linear combinations of the **5**'s and $\overline{\mathbf{5}}$'s of SU(5):

$$a_1(\mathbf{5})\mathbf{12}_{\rm H} + a_2(\mathbf{5})\overline{\mathbf{495}}_{\rm H}, \quad \text{and} \quad b_1(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H} + b_2(\overline{\mathbf{5}})\mathbf{495}_{\rm H}. \quad (12)$$

In what follows in Sect. IV., it will become apparent that the $(\mathbf{5})\overline{\mathbf{495}}_{\rm H}$ must develop an EW VEV, while the $(\mathbf{5})\mathbf{12}_{\rm H}$ can get massive without requiring that it also develops an EW VEV. The situation is not so clear-cut for an EW $(\overline{\mathbf{5}})$ VEV generated from the $\mathbf{495}_{\rm H}$ or $\overline{\mathbf{12}}_{\rm H}$ Higgs. We shall consider both cases individually in our search for models and comment later on the results.

Renormalizable dim-4 operators can be formed from three-point vertices involving two fermions and a Higgs. This requires we identify the appropriate SU(5) and SU(12) singlet vertices. For this purpose, Young diagram product rules must be applied at every vertex, so that the SU(5) boxes are on top of the remaining SU(12)/SU(5) boxes. For example, $(\mathbf{10})\overline{\mathbf{220}}.(\overline{\mathbf{5}})\overline{\mathbf{12}}.(\overline{\mathbf{5}})\mathbf{495}_{\rm H} = (2+7).(4+7).(4+0) = (2+7).(5+7,3+0) = (5+7,5+7)$ is a proper SU(5)- and SU(12)-singlet vertex with two columns of 12 boxes with the 5 SU(5) boxes on top; on the other hand, the product $(\mathbf{10})\mathbf{792}.(\overline{\mathbf{5}})\overline{\mathbf{495}}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H} = (2+3).(4+4).(4+7)$ is not, for one can not carry out the product keeping the 5 SU(5) boxes on top of the remaining 7 SU(12) boxes in both columns without rearrangements.

Effective higher-dimensional operators can be formed by inserting Higgs and massive fermions in tree diagrams. With SUSY valid at the SU(5) scale, loop diagrams are highly suppressed. The massive intermediate fermion pairs at the SU(12) scale, which are formed from complex irreps and are obviously anomaly-free, can be selected from among the

$$\mathbf{12}\times\overline{\mathbf{12}},\ \mathbf{66}\times\overline{\mathbf{66}},\ \mathbf{220}\times\overline{\mathbf{220}},\ \mathbf{495}\times\overline{\mathbf{495}},\ \mathbf{792}\times\overline{\mathbf{792}} \tag{13}$$

pair insertions. In order to maintain the proper basis with the SU(5) boxes at the top of the Young diagrams, the only proper contractions of interest here involve the following:

$$\begin{array}{rcl@{\qquad}rcl}
(\mathbf{1})\mathbf{12} & \times & (\mathbf{1})\overline{\mathbf{12}}, & (\mathbf{5})\mathbf{12} & \times & (\overline{\mathbf{5}})\overline{\mathbf{12}}, \\
(\mathbf{1})\mathbf{66} & \times & (\mathbf{1})\overline{\mathbf{66}}, & (\mathbf{10})\mathbf{66} & \times & (\overline{\mathbf{10}})\overline{\mathbf{66}}, \\
(\mathbf{1})\mathbf{220} & \times & (\mathbf{1})\overline{\mathbf{220}}, & (\overline{\mathbf{10}})\mathbf{220} & \times & (\mathbf{10})\overline{\mathbf{220}}, \\
(\mathbf{1})\mathbf{495} & \times & (\mathbf{1})\overline{\mathbf{495}}, & (\overline{\mathbf{5}})\mathbf{495} & \times & (\mathbf{5})\overline{\mathbf{495}}, \\
(\mathbf{1})\mathbf{792} & \times & (\mathbf{1})\overline{\mathbf{792}}. & & &
\end{array} \tag{14}$$

We can now proceed to construct the most general Higgs and Yukawa superpotentials preserving R-parity, where the Higgs superfields and the matter superfields are assigned R-parity +1. The Higgs superpotential with the three-point couplings involving all Higgs

fields which appear in SU(12) has the following SU(5) and SU(12) singlet terms:

$$
\begin{aligned}
W_{\rm Higgs} &= (\mathbf{1})\mathbf{12}_{\rm H}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} + (\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} + (\mathbf{1})\mathbf{220}_{\rm H}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H} + (\mathbf{1})\mathbf{495}_{\rm H}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H} \\
&+ (\mathbf{1})\mathbf{792}_{\rm H}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H} + (\mathbf{24})\mathbf{5148}_{\rm H}.(\mathbf{24})\overline{\mathbf{5148}}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H}.(\mathbf{5})\overline{\mathbf{495}}_{\rm H}.(\mathbf{24})\mathbf{5148}_{\rm H} \\
&+ (\overline{\mathbf{5}})\mathbf{495}_{\rm H}.(\mathbf{5})\mathbf{12}_{\rm H}.(\mathbf{24})\overline{\mathbf{5148}}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H}.(\mathbf{5})\overline{\mathbf{495}}_{\rm H}.(\mathbf{1})\mathbf{792}_{\rm H} \\
&+ (\overline{\mathbf{5}})\mathbf{495}_{\rm H}.(\mathbf{5})\mathbf{12}_{\rm H}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H} + (\mathbf{1})\mathbf{12}_{\rm H}.(\mathbf{1})\mathbf{12}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} + (\mathbf{1})\mathbf{12}_{\rm H}.(\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H} \\
&+ (\mathbf{1})\mathbf{12}_{\rm H}.(\mathbf{1})\mathbf{220}_{\rm H}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H} + (\mathbf{1})\mathbf{12}_{\rm H}.(\mathbf{1})\mathbf{495}_{\rm H}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H} + (\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H} \\
&+ (\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\mathbf{220}_{\rm H}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H} + (\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\mathbf{792}_{\rm H}.(\mathbf{1})\mathbf{792}_{\rm H} + (\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} \\
&+ (\mathbf{1})\mathbf{220}_{\rm H}.(\mathbf{1})\mathbf{495}_{\rm H}.(\mathbf{1})\mathbf{792}_{\rm H} + (\mathbf{1})\mathbf{220}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} + (\mathbf{1})\mathbf{495}_{\rm H}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} \\
&+ (\mathbf{1})\mathbf{495}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} + (\mathbf{1})\mathbf{792}_{\rm H}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} + (\mathbf{1})\mathbf{792}_{\rm H}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} \\
&+ (\mathbf{1})\overline{\mathbf{792}}_{\rm H}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} + (\mathbf{1})\overline{\mathbf{792}}_{\rm H}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H}
\end{aligned}
\tag{15}
$$

The corresponding Yukawa superpotential has the following structure:

$$
W_{\rm Yukawa} = W_{(\mathbf{24})} + W_{(\mathbf{5})} + W_{(\overline{\mathbf{5}})} + W_{(\mathbf{1})}, \tag{16}
$$

where

$$
\begin{aligned}
W_{(\mathbf{24})} &= (\overline{\mathbf{10}})\overline{\mathbf{66}}.(\mathbf{10})\overline{\mathbf{220}}.(\mathbf{24})\mathbf{5148}_{\rm H} + (\overline{\mathbf{10}})\mathbf{220}.(\mathbf{10})\mathbf{66}.(\mathbf{24})\overline{\mathbf{5148}}_{\rm H} \\
&+ (\overline{\mathbf{5}})\overline{\mathbf{12}}.(\mathbf{5})\overline{\mathbf{495}}.(\mathbf{24})\mathbf{5148}_{\rm H} + (\overline{\mathbf{5}})\mathbf{495}.(\mathbf{5})\mathbf{12}.(\mathbf{24})\overline{\mathbf{5148}}_{\rm H},
\end{aligned}
$$

$$
\begin{aligned}
W_{(\mathbf{5})} &= (\mathbf{10})\overline{\mathbf{220}}.(\mathbf{10})\mathbf{66}.(\mathbf{5})\mathbf{12}_{\rm H} + (\overline{\mathbf{10}})\mathbf{220}.(\mathbf{5})\overline{\mathbf{495}}.(\mathbf{5})\mathbf{12}_{\rm H} + (\overline{\mathbf{10}})\overline{\mathbf{66}}.(\mathbf{5})\mathbf{12}.(\mathbf{5})\mathbf{12}_{\rm H} \\
&+ (\overline{\mathbf{5}})\overline{\mathbf{66}}.(\mathbf{1})\mathbf{12}.(\mathbf{5})\mathbf{12}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{220}}.(\mathbf{1})\mathbf{66}.(\mathbf{5})\mathbf{12}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{495}}.(\mathbf{1})\mathbf{220}.(\mathbf{5})\mathbf{12}_{\rm H} \\
&+ (\overline{\mathbf{5}})\overline{\mathbf{792}}.(\mathbf{1})\mathbf{495}.(\mathbf{5})\mathbf{12}_{\rm H} + (\overline{\mathbf{5}})\mathbf{495}.(\mathbf{1})\overline{\mathbf{792}}.(\mathbf{5})\mathbf{12}_{\rm H} + (\mathbf{10})\mathbf{66}.(\mathbf{10})\mathbf{66}.(\mathbf{5})\overline{\mathbf{495}}_{\rm H} \\
&+ (\overline{\mathbf{10}})\mathbf{220}.(\mathbf{5})\mathbf{12}.(\mathbf{5})\overline{\mathbf{495}}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{12}}.(\mathbf{1})\mathbf{792}.(\mathbf{5})\overline{\mathbf{495}}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{220}}.(\mathbf{1})\overline{\mathbf{792}}.(\mathbf{5})\overline{\mathbf{495}}_{\rm H} \\
&+ (\overline{\mathbf{5}})\overline{\mathbf{495}}.(\mathbf{1})\overline{\mathbf{495}}.(\mathbf{5})\overline{\mathbf{495}}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{792}}.(\mathbf{1})\overline{\mathbf{220}}.(\mathbf{5})\overline{\mathbf{495}}_{\rm H} \\
&+ (\overline{\mathbf{5}})\mathbf{792}.(\mathbf{1})\overline{\mathbf{12}}.(\mathbf{5})\overline{\mathbf{495}}_{\rm H},
\end{aligned}
$$

$$
\begin{aligned}
W_{(\overline{\mathbf{5}})} &= (\overline{\mathbf{10}})\overline{\mathbf{66}}.(\overline{\mathbf{10}})\overline{\mathbf{66}}.(\overline{\mathbf{5}})\mathbf{495}_{\rm H} + (\mathbf{10})\overline{\mathbf{220}}.(\overline{\mathbf{5}})\overline{\mathbf{12}}.(\overline{\mathbf{5}})\mathbf{495}_{\rm H} + (\mathbf{5})\overline{\mathbf{792}}.(\mathbf{1})\mathbf{12}.(\overline{\mathbf{5}})\mathbf{495}_{\rm H} \\
&+ (\mathbf{5})\mathbf{792}.(\mathbf{1})\mathbf{220}.(\overline{\mathbf{5}})\mathbf{495}_{\rm H} + (\mathbf{5})\mathbf{495}.(\mathbf{1})\mathbf{495}.(\overline{\mathbf{5}})\mathbf{495}_{\rm H} + (\mathbf{5})\mathbf{220}.(\mathbf{1})\mathbf{792}.(\overline{\mathbf{5}})\mathbf{495}_{\rm H} \\
&+ (\mathbf{5})\mathbf{12}.(\mathbf{1})\overline{\mathbf{792}}.(\overline{\mathbf{5}})\mathbf{495}_{\rm H} + (\overline{\mathbf{10}})\overline{\mathbf{66}}.(\overline{\mathbf{10}})\mathbf{220}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H} + (\mathbf{10})\overline{\mathbf{220}}.(\overline{\mathbf{5}})\mathbf{495}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H} \\
&+ (\mathbf{10})\mathbf{66}.(\overline{\mathbf{5}})\overline{\mathbf{12}}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H} + (\mathbf{5})\overline{\mathbf{495}}.(\mathbf{1})\mathbf{792}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H} + (\mathbf{5})\mathbf{792}.(\mathbf{1})\overline{\mathbf{495}}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H} \\
&+ (\mathbf{5})\mathbf{495}.(\mathbf{1})\overline{\mathbf{220}}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H} + (\mathbf{5})\mathbf{220}.(\mathbf{1})\overline{\mathbf{66}}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H} + (\mathbf{5})\mathbf{66}.(\mathbf{1})\overline{\mathbf{12}}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H},
\end{aligned}
$$

$$
\begin{aligned}
W_{(\mathbf{1})} \;=\;& (\overline{\mathbf{10}})\mathbf{220}.(\mathbf{10})\overline{\mathbf{495}}.(\mathbf{1})\mathbf{12}_{\rm H} + (\overline{\mathbf{10}})\overline{\mathbf{220}}.(\mathbf{10})\mathbf{66}.(\mathbf{1})\mathbf{12}_{\rm H} + (\overline{\mathbf{10}})\mathbf{220}.(\mathbf{10})\overline{\mathbf{792}}.(\mathbf{1})\mathbf{66}_{\rm H} \\
+\;& (\overline{\mathbf{10}})\overline{\mathbf{495}}.(\mathbf{10})\mathbf{66}.(\mathbf{1})\mathbf{66}_{\rm H} + (\overline{\mathbf{10}})\overline{\mathbf{792}}.(\mathbf{10})\mathbf{66}.(\mathbf{1})\mathbf{220}_{\rm H} + (\overline{\mathbf{10}})\mathbf{220}.(\mathbf{10})\mathbf{792}.(\mathbf{1})\mathbf{495}_{\rm H} \\
+\;& (\overline{\mathbf{10}})\mathbf{220}.(\mathbf{10})\mathbf{495}.(\mathbf{1})\mathbf{792}_{\rm H} + (\overline{\mathbf{10}})\mathbf{792}.(\mathbf{10})\mathbf{66}.(\mathbf{1})\mathbf{792}_{\rm H} + (\overline{\mathbf{10}})\overline{\mathbf{66}}.(\mathbf{10})\overline{\mathbf{220}}.(\mathbf{1})\mathbf{792}_{\rm H} \\
+\;& (\overline{\mathbf{10}})\mathbf{220}.(\mathbf{10})\mathbf{66}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H} + (\overline{\mathbf{10}})\overline{\mathbf{495}}.(\mathbf{10})\overline{\mathbf{220}}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H} + (\overline{\mathbf{10}})\overline{\mathbf{66}}.(\mathbf{10})\overline{\mathbf{792}}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H} \\
+\;& (\overline{\mathbf{10}})\overline{\mathbf{792}}.(\mathbf{10})\overline{\mathbf{220}}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H} + (\overline{\mathbf{10}})\overline{\mathbf{66}}.(\mathbf{10})\mathbf{792}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H} + (\overline{\mathbf{10}})\mathbf{792}.(\mathbf{10})\overline{\mathbf{220}}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} \\
+\;& (\overline{\mathbf{10}})\overline{\mathbf{66}}.(\mathbf{10})\mathbf{495}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} + (\overline{\mathbf{10}})\mathbf{495}.(\mathbf{10})\overline{\mathbf{220}}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} + (\overline{\mathbf{10}})\overline{\mathbf{66}}.(\mathbf{10})\mathbf{220}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} \\
+\;& (\overline{\mathbf{5}})\mathbf{495}.(\mathbf{5})\overline{\mathbf{792}}.(\mathbf{1})\mathbf{12}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{66}}.(\mathbf{5})\mathbf{12}.(\mathbf{1})\mathbf{12}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{220}}.(\mathbf{5})\mathbf{12}.(\mathbf{1})\mathbf{66}_{\rm H} \\
+\;& (\overline{\mathbf{5}})\mathbf{495}.(\mathbf{5})\mathbf{792}.(\mathbf{1})\mathbf{220}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{495}}.(\mathbf{5})\mathbf{12}.(\mathbf{1})\mathbf{220}_{\rm H} + (\overline{\mathbf{5}})\mathbf{495}.(\mathbf{5})\mathbf{495}.(\mathbf{1})\mathbf{495}_{\rm H} \\
+\;& (\overline{\mathbf{5}})\overline{\mathbf{792}}.(\mathbf{5})\mathbf{12}.(\mathbf{1})\mathbf{495}_{\rm H} + (\overline{\mathbf{5}})\mathbf{495}.(\mathbf{5})\mathbf{220}.(\mathbf{1})\mathbf{792}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{12}}.(\mathbf{5})\overline{\mathbf{495}}.(\mathbf{1})\mathbf{792}_{\rm H} \\
+\;& (\overline{\mathbf{5}})\mathbf{495}.(\mathbf{5})\mathbf{12}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{220}}.(\mathbf{5})\overline{\mathbf{495}}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{495}}.(\mathbf{5})\overline{\mathbf{495}}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H} \\
+\;& (\overline{\mathbf{5}})\overline{\mathbf{12}}.(\mathbf{5})\mathbf{792}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{792}}.(\mathbf{5})\overline{\mathbf{495}}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{12}}.(\mathbf{5})\mathbf{495}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H} \\
+\;& (\overline{\mathbf{5}})\overline{\mathbf{12}}.(\mathbf{5})\mathbf{220}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} + (\overline{\mathbf{5}})\mathbf{792}.(\mathbf{5})\overline{\mathbf{495}}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} + (\overline{\mathbf{5}})\overline{\mathbf{12}}.(\mathbf{5})\mathbf{66}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} \\
+\;& (\mathbf{1})\mathbf{12}_{\rm H}.(\mathbf{1})\mathbf{12}.(\mathbf{1})\overline{\mathbf{66}} + (\mathbf{1})\mathbf{12}_{\rm H}.(\mathbf{1})\mathbf{66}.(\mathbf{1})\overline{\mathbf{220}} + (\mathbf{1})\mathbf{12}_{\rm H}.(\mathbf{1})\mathbf{220}.(\mathbf{1})\overline{\mathbf{495}} \\
+\;& (\mathbf{1})\mathbf{12}_{\rm H}.(\mathbf{1})\mathbf{495}.(\mathbf{1})\overline{\mathbf{792}} + (\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\mathbf{66}.(\mathbf{1})\overline{\mathbf{495}} + (\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\mathbf{220}.(\mathbf{1})\overline{\mathbf{792}} \\
+\;& (\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\mathbf{792}.(\mathbf{1})\mathbf{792} + (\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\overline{\mathbf{12}}.(\mathbf{1})\overline{\mathbf{12}} + (\mathbf{1})\mathbf{220}_{\rm H}.(\mathbf{1})\mathbf{495}.(\mathbf{1})\mathbf{792} \\
+\;& (\mathbf{1})\mathbf{220}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}}.(\mathbf{1})\overline{\mathbf{12}} + (\mathbf{1})\mathbf{495}_{\rm H}.(\mathbf{1})\overline{\mathbf{220}}.(\mathbf{1})\overline{\mathbf{12}} + (\mathbf{1})\mathbf{495}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}}.(\mathbf{1})\overline{\mathbf{66}} \\
+\;& (\mathbf{1})\mathbf{792}_{\rm H}.(\mathbf{1})\overline{\mathbf{495}}.(\mathbf{1})\overline{\mathbf{12}} + (\mathbf{1})\mathbf{792}_{\rm H}.(\mathbf{1})\overline{\mathbf{220}}.(\mathbf{1})\overline{\mathbf{66}} + (\mathbf{1})\overline{\mathbf{792}}_{\rm H}.(\mathbf{1})\overline{\mathbf{792}}.(\mathbf{1})\overline{\mathbf{66}} \\
+\;& (\mathbf{1})\overline{\mathbf{792}}_{\rm H}.(\mathbf{1})\overline{\mathbf{495}}.(\mathbf{1})\overline{\mathbf{220}} + (\mathbf{1})\mathbf{12}.(\mathbf{1})\mathbf{12}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}} + (\mathbf{1})\mathbf{12}.(\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\overline{\mathbf{220}} \\
+\;& (\mathbf{1})\mathbf{12}.(\mathbf{1})\mathbf{220}_{\rm H}.(\mathbf{1})\overline{\mathbf{495}} + (\mathbf{1})\mathbf{12}.(\mathbf{1})\mathbf{495}_{\rm H}.(\mathbf{1})\overline{\mathbf{792}} + (\mathbf{1})\mathbf{66}.(\mathbf{1})\mathbf{66}_{\rm H}.(\mathbf{1})\overline{\mathbf{495}} \\
+\;& (\mathbf{1})\mathbf{66}.(\mathbf{1})\mathbf{220}_{\rm H}.(\mathbf{1})\overline{\mathbf{792}} + (\mathbf{1})\mathbf{66}.(\mathbf{1})\mathbf{792}_{\rm H}.(\mathbf{1})\mathbf{792} + (\mathbf{1})\mathbf{66}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H}.(\mathbf{1})\overline{\mathbf{12}} \\
+\;& (\mathbf{1})\mathbf{220}.(\mathbf{1})\mathbf{495}_{\rm H}.(\mathbf{1})\mathbf{792} + (\mathbf{1})\mathbf{220}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H}.(\mathbf{1})\overline{\mathbf{12}} + (\mathbf{1})\mathbf{495}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H}.(\mathbf{1})\overline{\mathbf{12}} \\
+\;& (\mathbf{1})\mathbf{495}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}} + (\mathbf{1})\mathbf{792}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H}.(\mathbf{1})\overline{\mathbf{12}} + (\mathbf{1})\mathbf{792}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}} \\
+\;& (\mathbf{1})\overline{\mathbf{792}}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H}.(\mathbf{1})\overline{\mathbf{66}} + (\mathbf{1})\overline{\mathbf{792}}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H}.(\mathbf{1})\overline{\mathbf{220}} + (\mathbf{1})\mathbf{12}.(\mathbf{1})\mathbf{12}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} \\
+\;& (\mathbf{1})\mathbf{12}.(\mathbf{1})\mathbf{66}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H} + (\mathbf{1})\mathbf{12}.(\mathbf{1})\mathbf{220}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H} + (\mathbf{1})\mathbf{12}.(\mathbf{1})\mathbf{495}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H} \\
+\;& (\mathbf{1})\mathbf{66}.(\mathbf{1})\mathbf{66}.(\mathbf{1})\overline{\mathbf{495}}_{\rm H} + (\mathbf{1})\mathbf{66}.(\mathbf{1})\mathbf{220}.(\mathbf{1})\overline{\mathbf{792}}_{\rm H} + (\mathbf{1})\mathbf{66}.(\mathbf{1})\mathbf{792}.(\mathbf{1})\mathbf{792}_{\rm H} \\
+\;& (\mathbf{1})\mathbf{66}.(\mathbf{1})\overline{\mathbf{12}}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} + (\mathbf{1})\mathbf{220}.(\mathbf{1})\mathbf{495}.(\mathbf{1})\mathbf{792}_{\rm H} + (\mathbf{1})\mathbf{220}.(\mathbf{1})\overline{\mathbf{66}}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} \\
+\;& (\mathbf{1})\mathbf{495}.(\mathbf{1})\overline{\mathbf{220}}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} + (\mathbf{1})\mathbf{495}.(\mathbf{1})\overline{\mathbf{66}}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} + (\mathbf{1})\mathbf{792}.(\mathbf{1})\overline{\mathbf{495}}.(\mathbf{1})\overline{\mathbf{12}}_{\rm H} \\
+\;& (\mathbf{1})\mathbf{792}.(\mathbf{1})\overline{\mathbf{220}}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} + (\mathbf{1})\overline{\mathbf{792}}.(\mathbf{1})\overline{\mathbf{792}}.(\mathbf{1})\overline{\mathbf{66}}_{\rm H} + (\mathbf{1})\overline{\mathbf{792}}.(\mathbf{1})\overline{\mathbf{495}}.(\mathbf{1})\overline{\mathbf{220}}_{\rm H}.
\end{aligned}
$$

With these ingredients in mind, we can now construct SU(12) models whose renormalizable and effective higher-dimensional operators determine the elements of the quark and lepton mass matrices. The fitting procedure to be described later then allows us to deter-

mine which models are viable and acceptable in describing the quark and lepton mass and mixing data.

IV. SU(12) MODEL CONSTRUCTION WITH EFFECTIVE OPERATORS

Starting with either the first or fourth anomaly-free sets of Eq. (5), we can assign SU(12) irreps for the three SU(5) (**10**) family members defining the up quark mass matrix ($M_{\rm U}$), the three ($\overline{\mathbf{5}}$) family members required in addition to define the down quark mass matrix ($M_{\rm D}$), and the additional three singlets defining the Dirac neutrino ($M_{\rm DN}$) and Majorana neutrino ($M_{\rm MN}$) mass matrices. Because of the greater arbitrariness in making these family assignments for the fourth anomaly-free set, we shall concentrate our attention from now on to the simplest first anomaly-free set of Eq. (5). The contributions to the matrix elements for the Yukawa matrices involving the up quarks (**Uij**), down quarks (**Dij**), charged leptons (**Lij**), and Dirac neutrinos (**DNij**), as well as the Majorana matrix for the heavy righthanded neutrinos (**MNij**), can arise from renormalizable dim-4 operators as well as higher dimensional effective operators involving SU(5) (**1**) scalar singlets and (**24**) scalar adjoints appearing in external lines, along with a (**5**) or ($\overline{\mathbf{5}}$) EW Higgs scalar in the case of the Yukawa matrices. Each effective operator diagram must be constructed according to the Young diagram multiplication rules illustrated in the previous section, where each vertex of the diagram represents a term in the superpotential of Eq. (16).

A. Possible Sets of Assignments for the Chiral Fermion Families

We begin with the 33 component of the up quark Yukawa matrix (**U33**) and strive for a dim-4 renormalizable contribution, as that will represent the largest source for the top quark mass. Scanning through the four possible (**10**) matter families and (**5**) Higgs assignments in Eqs. (6) and (12), it becomes clear that only one possibility exists for a proper 3-point vertex singlet, namely,

$$\begin{aligned}\mathbf{U33}:\ & (\mathbf{10})\mathbf{66_3}.(\mathbf{5})\overline{\mathbf{495}}_{\rm H}.(\mathbf{10})\mathbf{66_3}\\ &= (2+0).(1+7).(2+0) = (5+7) \sim (\mathbf{1})\mathbf{1}.\end{aligned} \tag{17}$$

For all other **U** matrix elements, the effective operators will be dim-5 or higher, with one or more singlet and adjoint Higgs fields, as well as the $(\mathbf{5})\overline{\mathbf{495}}_{\rm H}$ Higgs field attached to the

fermion line. We shall assume the other $(\mathbf{5})\mathbf{12}_{\rm H}$ is inert and does not develop a VEV.

For the simplest anomaly-free set of Eqs. (3) and (5), the possible assignments of the $(\mathbf{10})$ family members are $(\mathbf{10})\mathbf{495}$, $(\mathbf{10})\overline{\mathbf{220}}$, $(\mathbf{10})\mathbf{66}_3$ and its permutation of the first and second family assignments, along with $(\mathbf{10})\overline{\mathbf{220}}$, $(\mathbf{10})\overline{\mathbf{220}}$, $(\mathbf{10})\mathbf{66}_3$. The three $(\overline{\mathbf{5}})$ family members can then be selected from the four possibilities given in Eq. (6), consistent with the anomaly-free set in question in Eq. (5). We list below the permissible $(\mathbf{10})$ and $(\overline{\mathbf{5}})$ family combinations,

$$\begin{array}{llll} (\mathbf{10})\mathbf{495},\ (\mathbf{10})\overline{\mathbf{220}},\ (\mathbf{10})\mathbf{66_3}; & & (\overline{\mathbf{5}})\overline{\mathbf{220}},\ (\overline{\mathbf{5}})\overline{\mathbf{12}},\ (\overline{\mathbf{5}})\overline{\mathbf{12}}; \\ (\mathbf{10})\overline{\mathbf{220}},\ (\mathbf{10})\overline{\mathbf{220}},\ (\mathbf{10})\mathbf{66_3}; & & (\overline{\mathbf{5}})\mathbf{495},\ (\overline{\mathbf{5}})\overline{\mathbf{12}},\ (\overline{\mathbf{5}})\overline{\mathbf{12}}, \end{array} \tag{18}$$

where only one SU(5) family is assigned to each of the SU(12) irreps in the set. It is to be understood that aside from the third $(\mathbf{10})$ family member being associated with $(\mathbf{10})\mathbf{66_3}$, all permutations of the family assignments are allowed.

Since all the non-trivial SU(12) irreps in the set have already been assigned, the conjugate lefthanded (or heavy righthanded) neutrinos must appear in stand alone SU(12) singlet irreps, i.e., $(\mathbf{1})\mathbf{1}$'s, one for each massive Majorana family: $(\mathbf{1})\mathbf{1_1}$, $(\mathbf{1})\mathbf{1_2}$, $(\mathbf{1})\mathbf{1_3}$ with the assumption of three families of righthanded singlet neutrinos.

B. Construction of the Mass Matrix Elements

We now have all the necessary ingredients to assemble the renormalizable and effective operator contributions to the four Dirac and one Majorana mass matrices. We begin the actual mass matrix constructions with the $\mathbf{U}$ matrix where, as noted earlier, the only suitable dim-4 contribution arises for the 33 element which involves the $(\mathbf{5})\overline{\mathbf{495}}_{\rm H}$ EW Higgs, which we repeat here,

$$\mathbf{U33} : (\mathbf{10})\mathbf{66_3}.(\mathbf{5})\overline{\mathbf{495}}_{\rm H}.(\mathbf{10})\mathbf{66_3}. \tag{19}$$

In order to obtain an appropriate hierarchy for the $\mathbf{Uij}$ mass matrix elements, all other matrix elements must arise from dim-5 or higher contributions involving the $(\mathbf{5})\overline{\mathbf{495}}_{\rm H}$ and at least one singlet or adjoint Higgs field, and one or more massive fermion pairs. From the structure of the $M_{\rm U}$ matrix which involves only light $(\mathbf{10})$ SU(5) chiral fermion families, it is clear that only $(\mathbf{10})$ and $(\overline{\mathbf{10}})$ massive fermions can contribute in the intermediate states. From Eq. (14) it is then obvious that the only possible mass insertions will involve

$(\mathbf{10})\mathbf{66} \times (\overline{\mathbf{10}})\overline{\mathbf{66}}$ and/or $(\mathbf{10})\overline{\mathbf{220}} \times (\overline{\mathbf{10}})\mathbf{220}$ irreps. As for the Higgs singlet vertices or those involving a Higgs ($\mathbf{24}$), these will be determined by the list of Higgs fields considered, the light fermion families involved, and the proper Young diagram product rules explained earlier. In any case, we retain only the lowest-order contributions to each matrix element.

We now turn to the $M_{\rm D}$ mass matrix which connects the ($\mathbf{10}$) lefthanded down quarks and the ($\overline{\mathbf{5}}$) lefthanded conjugate quarks with either the $(\overline{\mathbf{5}})\mathbf{495}_{\rm H}$ or $(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H}$ EW Higgs. The same considerations will apply to the $M_{\rm L}$ mass matrix connecting the ($\overline{\mathbf{5}}$) lefthanded charged leptons and the ($\mathbf{10}$) lefthanded charged conjugate leptons, where the diagrams are the transpose of those for the down quarks. Whether or not dim-4 contributions appear in these mass matrices depends on which of the two possible down-type Higgs are chosen for the models to be illustrated, $(\overline{\mathbf{5}})\mathbf{495}_{\rm H}$ or $(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H}$, as well as the light family assignments. In the first instant, a dim-4 vertex may be present involving

$$\mathbf{Dij}: \quad (\mathbf{10})\overline{\mathbf{220}}_{\mathbf{i}}.(\overline{\mathbf{5}})\mathbf{495}_{\rm H}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\mathbf{j}}, \tag{20}$$

while in the second instance, a dim-4 vertex may be present involving either of the two possibilities

$$\begin{aligned} \mathbf{D3j}: &\ (\mathbf{10})\mathbf{66}_{\mathbf{3}}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\mathbf{j}}, \\ \mathbf{Dij}: &\ (\mathbf{10})\overline{\mathbf{220}}_{\mathbf{i}}.(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H}.(\overline{\mathbf{5}})\mathbf{495}_{\mathbf{j}}. \end{aligned} \tag{21}$$

The transverse conditions apply for the corresponding **Lji** Yukawa matrix elements.

Higher dimensional contributions can involve not only ($\mathbf{10}$) and ($\overline{\mathbf{10}}$) intermediate states as for the $M_{\rm U}$ mass matrix, but also ($\mathbf{5}$) and ($\overline{\mathbf{5}}$) states. Again from Eq. (14), we see the latter choices are just $(\mathbf{5})\mathbf{12} \times (\overline{\mathbf{5}})\overline{\mathbf{12}}$ and $(\mathbf{5})\overline{\mathbf{495}} \times (\overline{\mathbf{5}})\mathbf{495}$. The same considerations as in the previous paragraphs also apply for the ($\mathbf{1}$) and ($\mathbf{24}$) Higgs vertices.

For the $M_{\rm DN}$ mass matrix connecting the SU(5) ($\overline{\mathbf{5}}$) lefthanded neutrinos with the ($\mathbf{1}$) lefthanded conjugate neutrinos, we assume that the same $(\mathbf{5})\overline{\mathbf{495}}_{\rm H}$ EW Higgs is involved as for the up quark sector. A dim-4 contribution to the $M_{\rm DN}$ mass matrix is possible, if one of the ($\overline{\mathbf{5}}$) family states arises from the $\mathbf{495}$ as in the second family assignment of Eq. (18), but not so otherwise. For the higher dimensional contributions to the $M_{\rm DN}$ mass matrix, the massive fermion insertions involve the same ($\mathbf{5}$) and ($\overline{\mathbf{5}}$) possibilities as for the $M_{\rm D}$ and $M_{\rm L}$ mass matrices. As with the other three Dirac matrices, singlet ($\mathbf{1}$) and adjoint ($\mathbf{24}$) Higgs scalars can appear in the $M_{\rm DN}$ matrix elements.

Finally, for the $M_{\rm MN}$ heavy righthanded Majorana mass matrix, since only $(\mathbf{1})\mathbf{1}$ fermion

singlets are involved, any mass insertions must involve only $(\mathbf{1})$ singlets. This fact then negates the appearance of $(\mathbf{24})\mathbf{5148}_{\rm H}$ or $(\mathbf{24})\overline{\mathbf{5148}}_{\rm H}$ Higgs contributions which would allow complex VEVs. Hence the Majorana matrix in all models discussed here will be real. The simplest dim-4 mass contribution is then given by

$$\mathbf{MNij}: \quad (\mathbf{1})\mathbf{1_i}.(\mathbf{1})\mathbf{1_H}.(\mathbf{1})\mathbf{1_j} \tag{22}$$

for all i and j. Note that this Higgs singlet must carry lepton number $L = 2$, in order to balance the two $L = -1$ lefthanded conjugate neutrino assignments. When this Higgs singlet obtains a VEV, L is broken by two units, and the Majorana mass matrix element obtains a mass Λ_R. The mass matrix then corresponds to a democratic matrix, aside from $O(1)$ prefactors which make the matrix non-singular.

In general, a restricted set of Higgs singlets and/or massive fermions may provide just one contribution to each mass matrix element. Allowing more and more Higgs singlets and massive fermion insertions may lead to many contributions of the same, higher, or even lower order for certain matrix elements. Since only the lowest-dimensional contributions per matrix element are of interest, the more contributing tree diagrams that appear, the flatter the hierarchy will tend to be for any given mass matrix.

C. Illustrated Structure for One Model of Interest

We have selected one model leading to interesting mixing results as a way of illustrating the steps involved to form the mass matrices and their consequent mixing matrices and mixing parameters. The model in question has the following family structure, massive fermions, and Higgs fields:

$$\begin{array}{llll}
\text{First Family:} & (\mathbf{10})\mathbf{495_1} \to u_{\rm L}, u^c_{\rm L}, d_{\rm L}, e^c_{\rm L}; & (\overline{\mathbf{5}})\overline{\mathbf{220}}_{\mathbf{1}} \to d^c_{\rm L}, e_{\rm L}, \nu_{e{\rm L}}; & (\mathbf{1})\mathbf{1_1} \to N^c_{1{\rm L}} \\
\text{Second Family:} & (\mathbf{10})\overline{\mathbf{220}}_{\mathbf{2}} \to c_{\rm L}, c^c_{\rm L}, s_{\rm L}, \mu^c_{\rm L}; & (\overline{\mathbf{5}})\overline{\mathbf{12}}_{\mathbf{2}} \to s^c_{\rm L}, \mu_{\rm L}, \nu_{\mu{\rm L}}; & (\mathbf{1})\mathbf{1_2} \to N^c_{2{\rm L}} \\
\text{Third Family:} & (\mathbf{10})\mathbf{66_3} \to t_{\rm L}, t^c_{\rm L}, b_{\rm L}, \tau^c_{\rm L}; & (\overline{\mathbf{5}})\overline{\mathbf{12}}_{\mathbf{3}} \to b^c_{\rm L}, \tau_{\rm L}, \nu_{\tau{\rm L}}; & (\mathbf{1})\mathbf{1_3} \to N^c_{3,{\rm L}}
\end{array} \tag{23}$$

$$\begin{array}{ll}
\text{Massive fermions:} & \mathbf{12}\times\overline{\mathbf{12}}, \quad \mathbf{66}\times\overline{\mathbf{66}}, \quad \mathbf{220}\times\overline{\mathbf{220}}, \quad \mathbf{495}\times\overline{\mathbf{495}}, \quad \mathbf{792}\times\overline{\mathbf{792}} \\
\text{Higgs bosons:} & (\mathbf{5})\overline{\mathbf{495}}_{\rm H}, (\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H}, (\mathbf{24})\mathbf{5148}_{\rm H}, (\mathbf{24})\overline{\mathbf{5148}}_{\rm H}, (\mathbf{1})\mathbf{1}_{\rm H}, \\
 & (\mathbf{1})\mathbf{66}_{\rm H}, (\mathbf{1})\overline{\mathbf{66}}_{\rm H}, (\mathbf{1})\mathbf{792}_{\rm H}, (\mathbf{1})\overline{\mathbf{792}}_{\rm H}
\end{array} \tag{24}$$

From the above irreps appearing in the model, we can construct the leading-order contributions to each Yukawa matrix element. The complete list for this model is presented in Appendix B. for the **U**, **D**, **L**, **DN**, and **MN** matrix elements. For the **U** Yukawa matrix, dimensional contributions of order 4, 5, and 6 are found to appear, which are scaled according to the ratios $1 : \varepsilon : \varepsilon^2$, where ε is related to the ratio of the SU(5) scale to the SU(12) scale. More precisely, ε is set equal to the ratio of a singlet VEV, times its fermion coupling, divided by the SU(12) unification scale where the massive fermions obtain their masses. The $\mathbf{5}$ and $\bar{\mathbf{5}}$ EW VEVs are labeled v_u and v_d, respectively, while the $(\mathbf{1})\mathbf{1_H}$ $\Delta L = 2$ VEV is set equal to Λ_R. The VEVs for $(\mathbf{24})\mathbf{5148}_{\rm H}$ and $(\mathbf{24})\overline{\mathbf{5148}}_{\rm H}$ involve κ and κ^*, respectively, as noted earlier in Eq. (11).

The five mass matrices for the model in question then are found to have the following textures:

$$
\begin{aligned}
M_{\rm U} &= \begin{pmatrix} h^{\rm u}_{11}\varepsilon^2 & h^{\rm u}_{12}(\varepsilon^2 - \frac{2\kappa\varepsilon}{3}) & h^{\rm u}_{13}\varepsilon \\ h^{\rm u}_{21}(\varepsilon^2 + \frac{2\kappa\varepsilon}{3}) & h^{\rm u}_{22}(\varepsilon^2 - \frac{4\kappa^2}{9}) & h^{\rm u}_{23}(\varepsilon + \frac{2\kappa}{3}) \\ h^{\rm u}_{31}\varepsilon & h^{\rm u}_{32}(\varepsilon - \frac{2\kappa}{3}) & h^{\rm u}_{33} \end{pmatrix} v_u, \\
M_{\rm D} &= \begin{pmatrix} 2h^{\rm d}_{11}\varepsilon^2 & h^{\rm d}_{12}\varepsilon & h^{\rm d}_{13}\varepsilon \\ h^{\rm d}_{21}\varepsilon & 2h^{\rm d}_{22}\varepsilon & 2h^{\rm d}_{23}\varepsilon \\ h^{\rm d}_{31}\varepsilon & h^{\rm d}_{32} & h^{\rm d}_{33} \end{pmatrix} v_d, \\
M_{\rm L} &= \begin{pmatrix} 2h^{\ell}_{11}\varepsilon^2 & h^{\ell}_{12}\varepsilon & h^{\ell}_{13}\varepsilon \\ h^{\ell}_{21}\varepsilon & 2h^{\ell}_{22}\varepsilon & h^{\ell}_{23} \\ h^{\ell}_{31}\varepsilon & 2h^{\ell}_{32}\varepsilon & h^{\ell}_{33} \end{pmatrix} v_d, \\
M_{\rm DN} &= \begin{pmatrix} 2h^{\rm dn}_{11}\varepsilon & 2h^{\rm dn}_{12}\varepsilon & 2h^{\rm dn}_{13}\varepsilon \\ h^{\rm dn}_{21}(2\varepsilon - \kappa) & h^{\rm dn}_{22}(2\varepsilon - \kappa) & h^{\rm dn}_{23}(2\varepsilon - \kappa) \\ h^{\rm dn}_{31}(2\varepsilon - \kappa) & h^{\rm dn}_{32}(2\varepsilon - \kappa) & h^{\rm dn}_{33}(2\varepsilon - \kappa) \end{pmatrix} v_u, \\
M_{\rm MN} &= \begin{pmatrix} h^{\rm mn}_{11} & h^{\rm mn}_{12} & h^{\rm mn}_{13} \\ h^{\rm mn}_{21} & h^{\rm mn}_{22} & h^{\rm mn}_{23} \\ h^{\rm mn}_{31} & h^{\rm mn}_{32} & h^{\rm mn}_{33} \end{pmatrix} \Lambda_R.
\end{aligned}
\tag{25}
$$

The corresponding h's are the prefactors to be determined numerically and are all required to lie in the range $\pm[0.1, 10]$ to achieve a satisfactory model that avoids fine tuning. Note that $M_{\rm U}$ exhibits a hierarchical structure, $M_{\rm DN}$ and $M_{\rm MN}$ do not, while $M_{\rm D}$ and $M_{\rm L}$ have

no simple hierarchical structure.

V. MODEL SCAN AND FITTING PROCEDURE

In this section we explain the aforementioned computerized model scan in more detail. The scan determines anomaly-free sets of family assignments for SU(N) irreps and scans possible unification models by adding EW Higgs fields and SU(5) Higgs singlets, as well as sets of massive fermions in a systematic way. The scan is built on top of LieART for the determination of tensor products extended to handle products of embeddings as described in Sect. III. Potential models are fit to phenomenological particle data, such as masses, mixing angles and phases, to analyze their viability. The scan is not restricted to SU(12) or a specific anomaly-free set of family assignments as discussed in this article, but we found SU(12) to be the lowest rank yielding realistic models not requiring discrete group extensions of the symmetry, and its lowest anomaly-free set of irreps is maximally economical as it assigns all SU(12) irreps to SU(5) family irreps.

A pure brute-force scan of all possible family assignments and sets of Higgs and massive fermions has proven impractical due to the enormous number of possible combinations. Instead, we break up the full number of combinations into independent parts that are organized in enclosed loops: (1) Fermions embedded in **10**'s of SU(5), which include all up-type quarks, are first assigned to suitable chiral irreps of the SU(12) anomaly-free set and prove sufficient to construct the M_{U} mass matrix, once the sets of Higgs and massive fermion irreps are defined. (2) Likewise, assignment of fermions embedded in $\overline{\mathbf{5}}$ of SU(5) complete the quark and charged lepton sectors and allows one to compute the M_{D} and M_{L} mass matrices and thus the CKM matrix. We fit the M_{U}, M_{D} and M_{L} mass matrix prefactors and four of the model parameters to the known quark masses and mixing angles, as well as charged lepton masses, at the GUT scale according to [21]. (3) Only for viable quark models do we loop over assignments of SU(12) irreps embedding SU(5) singlets as Majorana neutrinos. These assignments allow the construction of the M_{DN} and M_{MN} mass matrices and thus a fit to the lepton sector phenomenology. To this end we fit the M_{DN} and M_{MN} prefactors, as well as the righthanded scale Λ_R, to the known neutrino mass squared differences and two PMNS mixing angles. The M_{U}, M_{D} and M_{L} prefactors and all other parameters from the quark sector remain fixed as determined by the first fit to avoid the variation of too many

fit parameters at once. The lepton sector fit is performed twice: one favoring normal and the other inverted hierarchy of the light neutrino masses. Further details follow below.

A. Scan of Assignments

First, a list of anomaly-free sets of totally antisymmetric SU(N) irreps that yield three families on the SU(5) level is constructed, where $N > 5$. The list is ordered by the total number of SU(N) irreps in the sets and, since there is an infinite number of anomaly-free sets, is cut off at some chosen maximum. For SU(12) a list of the simpler anomaly-free sets has been given in (3). In looping over this list, the scan performs family assignments only for irreps from one set at a time to ensure freedom from anomalies.

For each anomaly-free set the scan loops over the SU(12) irreps containing **10**'s of SU(5) for the assignment of the three up-type quarks to construct the M_{U} mass matrix. In terms of Young tableaux the **10**'s are embedded in the upper part of the column for the SU(12) irreps, i.e., the regular embedding. Similarly, the scan loops over the SU(12) irreps containing $\overline{\mathbf{5}}$'s of SU(5) for the assignment of the three down-type quarks and leptons in a later step.

In a third loop the scan constructs subsets of possible assignments of EW Higgs doublets, SU(5) Higgs singlets, and massive fermion pairs. Both, the SU(12) Higgs irreps and the massive fermion pairs are selected from all totally antisymmetric complex irreps with the SU(5) EW Higgs and Higgs singlet irreps being regularly embedded. For our special SU(12) scenario at hand we add the $(\mathbf{24})\mathbf{5148}_{\mathrm{H}}$ and $(\mathbf{24})\overline{\mathbf{5148}}_{\mathrm{H}}$ to accommodate a CP phase and to abet the breaking of SU(5) to the SM. To reduce the number of Higgs sets from the beginning, we keep only those EW Higgses that yield a dim-4 mass term for the **U33** element with the selected third-family fermion assignment, i.e., the largest contribution to the top-quark mass term at lowest order, as pointed out in Sect. IV B. For the simple anomaly-free set of SU(12) the only possible **U33** at dim 4 using the regular embedding is given in expression (17). The loop over Higgs and massive-fermion-pair subsets starts with the smallest set of Higgses and massive fermions increasing to larger ones. Limits on the subset size can be imposed to focus on economical models.

With the assignments of the SU(12) irreps containing the **10**'s of SU(5), the $\mathbf{5}_{\mathrm{H}}$'s associated with EW doublets, and SU(5) Higgs singlets, as well as massive fermions assigned to SU(12) irreps, the **U** matrix elements can be constructed. For a given set of fermion,

Higgs and massive-fermion assignments determined by the iteration of the enclosing loops, the scan tries to construct diagrams for each matrix element beginning with a minimum, dim-4 or higher. If none is found at some dimension, it tries a higher dimension up to an adjustable upper limit. If one or more diagrams for a given dimension is found, the scan will turn to the next matrix element. Thus, only the lowest order contribution is taken into account. The algorithm allows one to set a range of admissible dimensions for each matrix element, e.g., the **U11** element must not be of dimension 4 or 5, but may be of dimension 6 or 7. It is also possible to allow for no contribution up to a maximum dimension, i.e., there may be no contribution at all amounting to a texture zero or a contribution of an even higher dimension, which is not analyzed further. A mass-term diagram is constructed from Higgs and massive fermion insertions depending on its dimension. The validity of the constructed mass-term diagrams is ensured if all vertices are singlets on their own at both the SU(12) and SU(5) levels and under application of the Young-tableaux multiplication rules. A mass-term diagram can then be translated to powers of ε and κ according to the orders of singlet VEVs and VEVs of $(\mathbf{24})\mathbf{5148}_{\mathrm{H}}$ and $(\mathbf{24})\overline{\mathbf{5148}}_{\mathrm{H}}$, respectively.

With M_{U} mass matrices matching the desired texture set by the dimension requirements, the M_{D} mass matrix is constructed from subsets of three unassigned irreps of the anomaly-free set containing $\overline{\mathbf{5}}$'s of SU(5), looping over the regular embedding. The construction of the mass matrix elements is analogous to that for the M_{U} mass matrix. The M_{L} matrix can be constructed from the reverse of the **D** matrix element diagrams. With the M_{D} and M_{L} matrices constructed, all assignments of the quark and charged lepton sectors are fixed.

We fit the quark sector and the charged leptons to phenomenological data run to the GUT scale taken from [21]. The description of this fit and the lepton sector fit is deferred to the next section. Since the quark sector is fully determined without the assignment of lefthanded conjugate Majorana neutrinos, we detach quark and lepton sector fits, to avoid fitting seemingly complete models where the quark sector itself does not reproduce SM phenomenology.

For models with quark and charged lepton sectors determined to be viable by the fit, a last loop over subsets of irreps assigned to Majorana neutrinos is performed. This requires SU(12) irreps containing SU(5) singlets. They are taken from unassigned irreps of the anomaly-free set or from additional SU(12) singlets, since they do not need to be chiral. The M_{DN} and M_{MN} matrices are constructed in analogy with the M_{U} and M_{D} matrices.

Once they are known, the lepton sector can be fit as well using the fit results of the quark sector performed in the stage prior to the assignment of Majorana neutrinos.

B. Quark and Lepton Sector Fits

Now we return from a more general description of the scanning procedure to our specific model setup to describe the separate quark and lepton sector fits to phenomenological data using the simplest anomaly-free set of SU(12) and the addition of $(\mathbf{24})\mathbf{5148}_{\mathrm{H}}$ and $(\mathbf{24})\overline{\mathbf{5148}}_{\mathrm{H}}$ scalars with complex valued VEVs introducing a source of CP violation.

1. *Quark Sector Fit*

The M_{U}, M_{D}, and M_{L} matrices enter the quark and charged lepton fit in terms of their prefactors h^{u}_{ij}, h^{d}_{ij} and h^{ℓ}_{ij}, powers of ε related to the SU(5) singlet VEVs appearing, and the complex-valued VEVs of $(\mathbf{24})\mathbf{5148}_{\mathrm{H}}$ and $(\mathbf{24})\overline{\mathbf{5148}}_{\mathrm{H}}$ involving κ and κ^*, respectively. The two EW VEVs of the 2-Higgs-Doublet-Model are labeled v_u and v_d, with only one independent and chosen to be v_u since $v^2 = v_u^2 + v_d^2$ must give $v = 174\,\mathrm{GeV}$. Because $(\mathbf{24})\mathbf{5148}_{\mathrm{H}}$ and $(\mathbf{24})\overline{\mathbf{5148}}_{\mathrm{H}}$ give different contributions to the M_{D} and M_{L} mass matrices according to (11) and asymmetric contributions to the M_{U} matrix, we refrain from imposing any symmetries on the prefactors and allow them to remain independent parameters. Thus, we have 27 real prefactors (9 per mass matrix), one real ratio ε, one complex ratio κ, and the EW VEV v_u, yielding a total of 31 parameters for the quark sector fit.

As initial values of the fit parameters we choose $\varepsilon{=}|\kappa| = 1/6.5^2{=}0.0237$, motivated by [22], $\arg(\kappa) = 45^\circ$ and $v_u = \sqrt{v^2/(1+\varepsilon^2)}$, where $v = 174\,\mathrm{GeV}$. The choice of unity for all initial values of the prefactors leads to cancellations in the matrix diagonalizations, thus resulting in fine tuning. Hence, we choose to set initial prefactor values randomly in the intervals $[-1.3, -0.7]$ and $[0.7, 1.3]$. Models with any prefactor fit to absolute values lower than $|0.1|$ or higher than $|10|$ are discarded. Fits with other randomly assigned prefactor initial values for such a model are tried until we either find a successful fit, or after a certain number of trials have been performed without success, we discard the model.

We perform the fit against phenomenological data at the SU(5) unification scale using values for the six quark and three charged lepton masses from [21]. We use the measured

values of the three quark mixing angles and phase. The renormalization group flow of the CKM matrix is governed by the Yukawa couplings, which are small except for the top quark. According to [23] the running of the matrix elements of the first two families is negligible and small for the third family. Thus we have neglected the running of the quark mixing angles and phase. In total we use 13 phenomenological data points.

The phenomenological implications of the models are compared with data by diagonalizing the mass matrices to obtain the quark and charged lepton masses and determine the CKM matrix from the unitary transformations diagonalizing $M_{\rm U}$ and $M_{\rm D}$. By transforming the CKM matrix into the standard parametrization, the three mixing angles and the CKM phase are easily obtained, as we explain in the following.

Since the Dirac matrices $M_{\rm U}$, $M_{\rm D}$ and $M_{\rm L}$ are generally not Hermitian, we form their lefthanded Hermitian products and diagonalize them with lefthanded rotations to obtain positive real eigenvalues as squares of the corresponding masses:

$$\begin{aligned} U_{\rm U}^\dagger M_{\rm U} M_{\rm U}^\dagger U_{\rm U} &= {\rm diag}(m_u^2, m_c^2, m_t^2), \\ U_{\rm D}^\dagger M_{\rm D} M_{\rm D}^\dagger U_{\rm D} &= {\rm diag}(m_d^2, m_s^2, m_b^2), \\ U_{\rm L}^\dagger M_{\rm L} M_{\rm L}^\dagger U_{\rm L} &= {\rm diag}(m_e^2, m_\mu^2, m_\tau^2). \end{aligned} \tag{26}$$

The Cabibbo-Kobayashi-Maskawa (CKM) matrix $V_{\rm CKM}$ encodes the mismatch of the mass and flavor eigenstates of the up- and down-type quarks and is calculated from the unitary transformations $U_{\rm U}$ and $U_{\rm D}$:

$$V_{\rm CKM} = U_{\rm U}^\dagger U_{\rm D}. \tag{27}$$

The CKM matrix in standard parametrization of the Particle Data Group [24] with $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ is given by

$$V_{\rm CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \tag{28}$$

A CKM matrix obtained by (27) can be brought into standard form by redefining five relative phases of quark fields, that are unphysical, or by extracting the three angles and the phase directly. The angles can be obtained from

$$\theta_{12} = \arctan\left(\frac{|V_{12}|}{|V_{11}|}\right), \quad \theta_{23} = \arctan\left(\frac{|V_{23}|}{|V_{33}|}\right) \quad \text{and} \quad \theta_{13} = \arcsin\left(|V_{13}|\right). \tag{29}$$

To determine the phase we perform a phase rotation of columns one and two such that V'_{11} and V'_{12} become real, where the prime denotes the rotated columns. We equate the quotient of the V'_{22} and V'_{21} elements with the corresponding expression in the standard form

$$r = \frac{V'_{22}}{V'_{21}} = \frac{V_{22}}{V_{21}} \mathrm{e}^{i(\phi_{11}-\phi_{12})} = \frac{c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{i\delta}}{-s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{i\delta}}, \tag{30}$$

where ϕ_{11} and ϕ_{12} are the phases of V_{11} and V_{12}, respectively, and solve for the phase δ yielding

$$\delta = \arg\left(\frac{c_{12}c_{23} + r s_{12}c_{23}}{s_{12}s_{23}s_{13} - r c_{12}s_{23}s_{13}}\right). \tag{31}$$

2. Lepton Sector Fit

Only quark models with a reasonably good fit are extended to include assignments of the lefthanded conjugate Majorana neutrinos in a loop over all their possibilities. For the simplest anomaly-free SU(12) model of interest here, since all six non-trivial irreps have been assigned to the SU(5) **10** and $\overline{\mathbf{5}}$ family irreps, the three heavy neutrinos are all assigned to SU(12) singlets. The M_{DN} and M_{MN} matrices are then determined, and the complex symmetric light-neutrino mass matrix is obtained via the Type I seesaw mechanism,

$$M_\nu = -M_{\mathrm{DN}} M_{\mathrm{MN}}^{-1} M_{\mathrm{DN}}^T. \tag{32}$$

By convention, the complex symmetric M_ν matrix is to be diagonalized by the unitary transformation

$$U_\nu^T M_\nu U_\nu = \mathrm{diag}(m_1, m_2, m_3), \tag{33}$$

to yield positive real eigenvalues m_i. This requires a very special unitary U_ν transformation, for in general the eigenvalues will be complex. To acquire the desired result, we form the Hermitian product $M_\nu^\dagger M_\nu$ and perform the unitary transformation by using (33),

$$U_\nu^\dagger M_\nu^\dagger M_\nu U_\nu = \mathrm{diag}(m_1^2, m_2^2, m_3^2), \tag{34}$$

to obtain positive real eigenvalues, m_i^2, and the transformation matrix U_ν. Clearly, Eq. (34) is invariant to a phase transformation Φ from the right together with its conjugate phase transformation from the left. We now define $U'_\nu = U_\nu \Phi'$ to be the special unitary transformation, operating on M_ν as in Eq. (33), which makes the neutrino mass eigenvalues real for the appropriate diagonal phase matrix Φ'. The Pontecorvo-Maki-Nakagawa-Sakata (PMNS)

matrix [25], $V_{\rm PMNS}$, follows from U'_ν and the unitary transformation $U_{\rm L}$ diagonalizing the charged lepton mass matrix $M_{\rm L}$, according to

$$V_{\rm PMNS} = U_{\rm L}^\dagger U'_\nu. \tag{35}$$

The PDG phase convention [24] for the neutrino mixing matrix $U_{\rm PMNS}$ follows by phase transforming the left- and right-hand sides of $V_{\rm PMNS}$ and then writing

$$V_{\rm PMNS} \equiv U_{\rm PMNS}\Phi_{\rm Majorana}, \tag{36}$$

where $\Phi_{\rm Majorana} = {\rm diag}(e^{i\phi_1/2},\ e^{i\phi_2/2},\ 1)$ with Majorana phases ϕ_1 and ϕ_2 is the adjoint of the required righthanded phase transition matrix, so effectively U'_ν is left untransformed from the right. The neutrino mixing angles and Dirac phase are determined in analogy with the CKM matrix.

To accomplish the phase transformations of Eq. (35) in detail, we follow the procedure as for $V_{\rm CKM}$ in Eqs. (29) – (31) to obtain $U_{\rm PMNS}$ in the PDG convention. To restore the correct untransformed U'_ν as appears in (35), we then multiply by $\Phi^\dagger$ on the right to obtain Eq. (36) with $\Phi_{\rm Majorana} = \Phi^\dagger$.

The effective mass $|\langle m_{ee}\rangle|$ for neutrinoless double beta decay [24] follows from Eq. (36) according to

$$|\langle m_{ee}\rangle| = |\Sigma_i\ (V_{{\rm PMNS},ei})^2 m_i|, \qquad i = 1, 2, 3. \tag{37}$$

This assumes that the light neutrino masses are the major contributors to the corresponding loop diagrams for the effective mass contribution to neutrinoless double beta decay [26].

We fit the lepton sector with recent neutrino data [27] for the mass squared differences of the light neutrinos $|\Delta_{21}|$, $|\Delta_{31}|$ and $|\Delta_{32}|$ and the sines squared of the neutrino mixing angles, $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$. We do not fit to the Dirac CP phase or $\sin^2\theta_{23}$, but discard models that are not within the bounds of $0.34 \leq \sin^2\theta_{23} \leq 0.66$, since values from current global fits of neutrino data can only provide this range or smaller. In total we fit to five data points. Since the $M_{\rm MN}$ matrix is symmetric, and involves the righthanded scale Λ_R as an additional parameter and the $M_{\rm DN}$ matrix is not symmetric, the lepton sector fit encompasses 16 fit parameters. Initial values for the prefactors are chosen in analogy to the quark sector fit, and we set the initial value of the righthanded scale to $\Lambda_R = 4 \times 10^{14}\,{\rm GeV}$. Models with prefactors not within the range $\pm[0.1, 10]$ are discarded and refit with other randomly assigned prefactors, as was done for the fits of quark sector models. The leptonic

fits for satisfactory models are carried out favoring first normal hierarchy (NH) and then inverted hierarchy (IH). In some cases, satisfactory models for both hierarchies can be obtained with the same set of mass matrix textures, but with different sets of prefactors of course.

	Quark and Lepton Data	**Fitted Results**
m_u (MeV)	0.3963 ± 0.1395	0.3950
m_c (GeV)	0.1932 ± 0.0243	0.1932
m_t (GeV)	80.4472 ± 2.7643	80.45
m_d (MeV)	0.9284 ± 0.3796	0.9143
m_s (MeV)	17.6097 ± 4.7855	17.60
m_b (GeV)	1.2424 ± 0.0599	1.243
m_e (MeV)	0.3569 ± 0.0003	0.3509
m_μ (MeV)	75.3570 ± 0.0713	75.42
m_τ (GeV)	1.6459 ± 0.0160	1.646
θ^q_{12}	13.04 ± 0.05°	13.04°
θ^q_{23}	2.38 ± 0.06°	2.381°
θ^q_{13}	0.201 ± 0.011°	0.2037°
δ^q	68.75 ± 4.584°	68.76°
Δ_{21} ($10^{-5} eV^2$)	7.50 ± 0.18	7.4
$\lvert\Delta_{31}\rvert$ ($10^{-3} eV^2$)	2.45 ± 0.047	2.5 (2.4)
$\lvert\Delta_{32}\rvert$ ($10^{-3} eV^2$)	2.45 ± 0.047	2.4 (2.5)
$\sin^2\theta_{12}$	0.304 ± 0.012	0.304
$\sin^2\theta_{13}$	0.0218 ± 0.001	0.0218

Table I: Phenomenological data with masses at the GUT scale and fitted model results for the special case illustrated. The NH (IH) results are indicated without (with) parentheses for $|\Delta_{31}|$ and $|\Delta_{32}|$. The quark data are taken from Ref. [21] and the neutrino data from [27].

C. Fitting Results for Special Case Illustrated

We begin with the known data, evaluated at the SU(5) GUT scale, which will be fitted with the five model parameters and prefactors for the five mass matrices. For the quark and charged lepton sectors, this consists of the nine masses and three CKM mixing angles and one phase listed in Table I. For the lepton sector, we make use of the three neutrino mass squared differences and two of the three neutrino mixing angles which are also given in Table I. The unknown neutrino quantities then involve the mass hierarchy (MH), the righthanded Majorana scale Λ_R fit parameter, the light and heavy neutrino masses, the octant and values of $\sin^2\theta_{23}$ and δ, along with the Majorana phases, and the effective neutrinoless double beta decay mass.

Following the above scanning and fitting procedures for the special case illustrated in Sect. IV C., the following matrices have been obtained in terms of the parameters $\varepsilon,\ \kappa,\ \kappa^*,\ v_u,\ v_d,\ \Lambda_R$ with the prefactors indicated explicitly. For the quark and charged lepton mass matrices the results obtained are

$$
\begin{aligned}
M_{\rm U} &= \begin{pmatrix} -0.47\varepsilon^2 & 0.48(\varepsilon^2-\frac{2\kappa\varepsilon}{3}) & 0.28\varepsilon \\ -1.0(\varepsilon^2+\frac{2\kappa\varepsilon}{3}) & 1.9(\varepsilon^2-\frac{4\kappa^2}{9}) & 0.61(\varepsilon+\frac{2\kappa}{3}) \\ 0.67\varepsilon & 2.5(\varepsilon-\frac{2\kappa}{3}) & -0.46 \end{pmatrix} v_u, \\
M_{\rm D} &= \begin{pmatrix} -2.8\varepsilon^2 & 1.1\varepsilon & 0.15\varepsilon \\ 0.84\varepsilon & 5.3\varepsilon & 0.2\varepsilon \\ -1.3\varepsilon & -0.97 & -0.15 \end{pmatrix} v_d, \\
M_{\rm L} &= \begin{pmatrix} 1.8\varepsilon^2 & 0.85\varepsilon & -1.2\varepsilon \\ 0.55\varepsilon & 4.6\varepsilon & 0.37 \\ -0.91\varepsilon & -1.3\varepsilon & -1.3 \end{pmatrix} v_d,
\end{aligned}
\tag{38}
$$

with the parameters found to be $\varepsilon = 0.01453,\ \kappa = 0.02305\ e^{i27.53^\circ},\ v_u = 174$ GeV, $v_d = 1.262$ GeV.

It turns out for this special model, both NH and IH solutions can be found. With the

parameters determined as above, the two sets of neutrino mass matrices are given by

$$
\begin{aligned}
M_{\rm DN}^{\rm NH} &= \begin{pmatrix} -1.6\varepsilon & 1.5\varepsilon & 3.3\varepsilon \\ -1.5(2\varepsilon-\kappa) & 0.43(2\varepsilon-\kappa) & 0.76(2\varepsilon-\kappa) \\ -1.9(2\varepsilon-\kappa) & 0.7(2\varepsilon-\kappa) & -1.9(2\varepsilon-\kappa) \end{pmatrix} v_u, \\
M_{\rm MN}^{\rm NH} &= \begin{pmatrix} 0.94 & -1.1 & -0.16 \\ -1.1 & 1.5 & 1.3 \\ -0.16 & 1.3 & -0.38 \end{pmatrix} \Lambda_R^{\rm NH}.
\end{aligned}
\tag{39}
$$

for NH and by

$$
\begin{aligned}
M_{\rm DN}^{\rm IH} &= \begin{pmatrix} -2.8\varepsilon & 2.4\varepsilon & -1.9\varepsilon \\ 1.3(2\varepsilon-\kappa) & 1.3(2\varepsilon-\kappa) & 0.78(2\varepsilon-\kappa) \\ 1.7(2\varepsilon-\kappa) & 1.6(2\varepsilon-\kappa) & -1.6(2\varepsilon-\kappa) \end{pmatrix} v_u, \\
M_{\rm MN}^{\rm IH} &= \begin{pmatrix} -1.2 & 0.26 & -0.4 \\ 0.26 & 0.62 & 2.0 \\ -0.4 & 2.0 & 0.61 \end{pmatrix} \Lambda_R^{\rm IH}.
\end{aligned}
\tag{40}
$$

for IH, where $\Lambda_R^{\rm NH} = 1.5 \times 10^{12}$ GeV and $\Lambda_R^{\rm IH} = 1.3 \times 10^{12}$ GeV.

For the NH case the unknown neutrino mixing parameters and masses are determined to be

$$
\begin{gathered}
\sin^2\theta_{23} = 0.460,\ \delta = -121^\circ,\ \phi_1 = -219^\circ,\ \phi_2 = -67.4^\circ,\ |\langle m_{ee}\rangle| = 2.45 \text{ meV} \\
m_1 = 2.76 \text{ meV}, \qquad m_2 = 9.03 \text{ meV}, \qquad m_3 = 50.0 \text{ meV}, \\
M_1 = 5.86 \times 10^{11} \text{ GeV},\ M_2 = 1.76 \times 10^{12} \text{ GeV},\ M_3 = 4.35 \times 10^{12} \text{ GeV},
\end{gathered}
\tag{41}
$$

while for the IH case,

$$
\begin{gathered}
\sin^2\theta_{23} = 0.600,\ \delta = -50.2^\circ,\ \phi_1 = -319^\circ,\ \phi_2 = 12.8^\circ,\ |\langle m_{ee}\rangle| = 47.2 \text{ meV}, \\
m_1 = 49.3 \text{ meV}, \qquad m_2 = 50.0 \text{ meV}, \qquad m_3 = 3.39 \text{ meV}, \\
M_1 = 1.06 \times 10^{12} \text{ GeV},\ M_2 = 2.26 \times 10^{12} \text{ GeV},\ M_3 = 3.33 \times 10^{12} \text{ GeV}.
\end{gathered}
\tag{42}
$$

VI. RESULTS FOR ACCEPTABLE MODELS

We now present the results for acceptable SU(12) neutrino mixing models obtained with the scanning and fitting procedures outlined in Sect. V. While in general we noted that two sets of EW Higgs doublets are available for giving Dirac masses to the quarks and leptons,

$(\mathbf{5})\mathbf{12}_{\rm H}$ and $(\mathbf{5})\overline{\mathbf{495}}_{\rm H}$ for the up-type quarks and Dirac neutrinos, and $(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H}$ and $(\overline{\mathbf{5}})\mathbf{495}_{\rm H}$ for the down-type quarks and charged leptons, only the Higgs doublet in the $(\mathbf{5})\overline{\mathbf{495}}_{\rm H}$ could provide a dim-4 contribution to the top quark mass. Thus for simplicity we considered the $(\mathbf{5})\mathbf{12}_{\rm H}$ to contain an inert doublet and to be of no further interest. On the other hand, both irreps leading to $(\overline{\mathbf{5}})$ doublets seemed to be possible contributers to the EW VEVs of the down-type quarks and charged leptons. But the full scan results to be displayed below have shown that only the $(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H}$ appeared in successful models for the simplest anomaly-free set. Hence it suggests that we also consider the $(\overline{\mathbf{5}})\mathbf{495}_{\rm H}$ to contain an inert Higgs doublet.

Concerning the permissible family assignments for the three $(\mathbf{10})$'s and the three $(\overline{\mathbf{5}})$'s displayed in Eq. (18), along with their permutations, no satisfactory model appeared involving the second $(\mathbf{10})$ assignment, $(\mathbf{10})\overline{\mathbf{220}}_1$, $(\mathbf{10})\overline{\mathbf{220}}_2$, $(\mathbf{10})\mathbf{66_3}$. The family assignments leading to acceptable models are labeled I and II for the two remaining $(\mathbf{10})$ choices,

$$\begin{array}{llll} \text{I}: & (\mathbf{10})\mathbf{495_1}, & (\mathbf{10})\overline{\mathbf{220}}_{\mathbf{2}}, & (\mathbf{10})\mathbf{66_3} \\ \text{II}: & (\mathbf{10})\overline{\mathbf{220}}_{\mathbf{1}}, & (\mathbf{10})\mathbf{495_2}, & (\mathbf{10})\mathbf{66_3} \end{array} \tag{43}$$

and A, B, and C for the three $(\overline{\mathbf{5}})$ permutations,

$$\begin{array}{llll} \text{A}: & (\overline{\mathbf{5}})\overline{\mathbf{12}}_{\mathbf{1}}, & (\overline{\mathbf{5}})\overline{\mathbf{12}}_{\mathbf{2}}, & (\overline{\mathbf{5}})\overline{\mathbf{220}}_{\mathbf{3}} \\ \text{B}: & (\overline{\mathbf{5}})\overline{\mathbf{12}}_{\mathbf{1}}, & (\overline{\mathbf{5}})\overline{\mathbf{220}}_{\mathbf{2}}, & (\overline{\mathbf{5}})\overline{\mathbf{12}}_{\mathbf{3}} \\ \text{C}: & (\overline{\mathbf{5}})\overline{\mathbf{220}}_{\mathbf{1}}, & (\overline{\mathbf{5}})\overline{\mathbf{12}}_{\mathbf{2}}, & (\overline{\mathbf{5}})\overline{\mathbf{12}}_{\mathbf{3}} \end{array} \tag{44}$$

Table II gives a summary of the models found acceptable by the scanning and fitting proceedure where the types of models are numbered according to their massive fermion content labeled MF1, MF2, etc. and their Higgs structure. For simplicity only the irreps are given with their conjugate irreps understood to be included. The Higgs irreps include $(\mathbf{5})\overline{\mathbf{495}}_{\rm H}$, $(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H}$, the $\Delta L = 2$ Higgs singlet $(\mathbf{1})\mathbf{1}_{\rm H}$, $(\mathbf{24})\mathbf{5148}_{\rm H}$ and $(\mathbf{24})\overline{\mathbf{5148}}_{\rm H}$ in all models. In addition, it is found that proper models can be constructed either with the minimum number of Higgs singlets $(\mathbf{1})\mathbf{792}_{\rm H}$ and $(\mathbf{1})\overline{\mathbf{792}}_{\rm H}$ in a few cases, or with these singlets plus the $(\mathbf{1})\mathbf{66}_{\rm H}$ and $(\mathbf{1})\overline{\mathbf{66}}_{\rm H}$ pair in the majority of cases as indicated in the Table. In fact, while seven other sets of Higgs singlets can be added to each model, the results are unmodified, for diagrams including those additional Higgs irreps all occur with higher-dimensional contributions to the matrix elements which we choose to neglect. On the other hand, additional $\mathbf{792}$ and $\overline{\mathbf{792}}$ fermions will add extra contributions to the $(\mathbf{DNij})$ Yukawa matrix elements.

Family Assignments: **I. (10): 495_1, $\overline{220}_2$, 66_3** **II. (10): $\overline{220}_1$, 495_2, 66_3**

A. ($\overline{5}$): $\overline{12}_1$, $\overline{12}_2$, $\overline{220}_3$ **B. ($\overline{5}$): $\overline{12}_1$, $\overline{220}_2$, $\overline{12}_3$** **C. ($\overline{5}$): $\overline{220}_1$, $\overline{12}_2$, $\overline{12}_3$**

	Run	MF1	MF2	MF3	MF4	MF5	(1)Higgs	ε	κ	arg(κ)
IA1:	1	**66**	**220**	**495**			**66**, **792**	-0.0117	0.0436	35.8°
IA1:	2	**66**	**220**	**495**			**792**	-0.0470	0.0241	−119°
IA1:	5	**66**	**220**	**495**			**66**, **792**	-0.0062	0.0464	−164°
IA2:	1	**66**	**220**	**495**	**792**		**66**, **792**	-0.0117	0.0436	35.8°
IA2:	2	**66**	**220**	**495**	**792**		**792**	-0.0470	0.0241	−119°
IA2:	5	**66**	**220**	**495**	**792**		**66**, **792**	-0.0062	0.0464	−164°
IA3:	1	**12**	**66**	**495**			**66**, **792**	0.0042	0.0268	76.3°
IA3:	3	**12**	**66**	**495**			**66**, **792**	-0.0111	0.0070	43.1°
IA4:	3	**12**	**66**	**495**	**792**		**66**, **792**	-0.0111	0.0070	43.1°
IA4:	4	**12**	**66**	**495**	**792**		**66**, **792**	-0.0077	0.0383	−15.0°
IA5:	3	**12**	**66**	**220**	**495**		**66**, **792**	0.0177	0.0742	−171°
IA5:	5	**12**	**66**	**220**	**495**		**66**, **792**	0.0114	0.0170	−112°
IA6:	1	**12**	**66**	**220**	**495**	**792**	**66**, **792**	-0.0117	0.0061	40.1°
IA6:	3	**12**	**66**	**220**	**495**	**792**	**66**, **792**	0.0177	0.0742	−171°
IA6:	5	**12**	**66**	**220**	**495**	**792**	**66**, **792**	0.0114	0.0170	−112°
IB1:	4	**66**	**220**	**495**			**66**, **792**	-0.0051	0.0245	−128°
IB1:	5	**66**	**220**	**495**			**66**, **792**	0.0116	0.0155	56.6°
IB2:	4	**66**	**220**	**495**	**792**		**66**, **792**	-0.0051	0.0245	−128°
IB2:	5	**66**	**220**	**495**	**792**		**66**, **792**	0.0116	0.0155	56.6°
IB3:	4	**12**	**66**	**495**			**66**, **792**	-0.0087	0.0060	119°
IB4:	1	**12**	**66**	**495**	**792**		**66**, **792**	-0.0094	0.0230	121°
IB4:	4	**12**	**66**	**495**	**792**		**66**, **792**	-0.0087	0.0060	119°
IB5:	5	**12**	**66**	**220**	**495**		**66**, **792**	-0.0125	0.0124	121°
IB6:	3	**12**	**66**	**220**	**495**	**792**	**792**	-0.0244	0.1430	10.2°
IB6:	5	**12**	**66**	**220**	**495**	**792**	**792**	-0.0125	0.0124	121°

	Run	MF1	MF2	MF3	MF4	MF5	(1)Higgs	ε	κ	arg(κ)
IC1:	1	**66**	**220**	**495**			**66**, **792**	0.0174	0.0262	$-130°$
IC1:	4	**66**	**220**	**495**			**66**, **792**	0.0064	0.0272	$-97.1°$
IC1:	5	**66**	**220**	**495**			**66**, **792**	-0.0125	0.0080	52.8°
IC2:	1	**66**	**220**	**495**	**792**		**66**, **792**	0.0174	0.0263	$-130°$
IC2:	4	**66**	**220**	**495**	**792**		**66**, **792**	0.0064	0.0272	$-97.1°$
IC2:	5	**66**	**220**	**495**	**792**		**66**, **792**	-0.0125	0.0080	52.8°
IC3:	1	**12**	**66**	**495**			**66**, **792**	-0.0088	0.0381	13.4°
IC3:	2	**12**	**66**	**495**			**66**, **792**	-0.0066	0.0327	29.6°
IC3:	3	**12**	**66**	**495**			**66**, **792**	-0.0079	0.0317	18.8°
IC3:	5	**12**	**66**	**495**			**66**, **792**	-0.0105	0.0035	61.8°
IC4:	1	**12**	**66**	**495**	**792**		**66**, **792**	-0.0088	0.0381	13.4°
IC4:	2	**12**	**66**	**495**	**792**		**66**, **792**	-0.0066	0.0327	29.6°
IC4:	3	**12**	**66**	**495**	**792**		**66**, **792**	-0.0079	0.0317	18.9°
IC4:	5	**12**	**66**	**495**	**792**		**66**, **792**	-0.0105	0.0035	61.8°
IC5:	1	**12**	**66**	**220**	**495**		**66**, **792**	-0.0149	0.0357	16.7°
IC5:	4	**12**	**66**	**220**	**495**		**66**, **792**	0.0145	0.0231	27.5°
IC6:	1	**12**	**66**	**220**	**495**	**792**	**66**, **792**	-0.0149	0.0357	16.8°
IC6:	4	**12**	**66**	**220**	**495**	**792**	**66**, **792**	0.0145	0.0231	27.5°
IIA1:	2	**66**	**220**	**495**			**66**, **792**	-0.0026	0.0138	77.2°
IIA1:	4	**66**	**220**	**495**			**66**, **792**	-0.0103	0.0154	58.4°
IIA2:	2	**66**	**220**	**495**	**792**		**66**, **792**	-0.0026	0.0138	77.2°
IIA2:	4	**66**	**220**	**495**	**792**		**66**, **792**	-0.0103	0.0154	58.4°
IIA3:	1	**12**	**66**	**495**			**66**, **792**	-0.0150	0.0337	$-9.1°$
IIA3:	2	**12**	**66**	**495**			**66**, **792**	-0.0139	0.0087	60.8°
IIA3:	3	**12**	**66**	**495**			**66**, **792**	0.0119	0.0396	170°
IIA3:	5	**12**	**66**	**495**			**66**, **792**	0.0157	0.0312	$-132°$
IIA4:	1	**12**	**66**	**495**	**792**		**66**, **792**	-0.0150	0.0337	$-9.1°$
IIA4:	2	**12**	**66**	**495**	**792**		**66**, **792**	-0.0139	0.0087	60.8°
IIA4:	3	**12**	**66**	**495**	**792**		**66**, **792**	0.0158	0.0274	84.6°
IIA4:	5	**12**	**66**	**495**	**792**		**66**, **792**	0.0119	0.0396	170°
IIA5:	1	**12**	**66**	**220**	**495**		**66**, **792**	-0.0113	0.0334	9.3°
IIA5:	3	**12**	**66**	**220**	**495**		**66**, **792**	-0.0125	0.0080	79.7°
IIA6:	1	**12**	**66**	**220**	**495**	**792**	**66**, **792**	-0.0113	0.0334	9.3°
IIA6:	2	**12**	**66**	**220**	**495**	**792**	**66**, **792**	0.0079	0.0129	37.7°
IIA6:	3	**12**	**66**	**220**	**495**	**792**	**66**, **792**	-0.0125	0.0080	79.7°
IIA6:	4	**12**	**66**	**220**	**495**	**792**	**66**, **792**	0.0067	0.0225	$-124°$

	Run	MF1	MF2	MF3	MF4	MF5	(1)Higgs	ε	κ	arg(κ)
IIB1:	3	**66**	**220**	**495**			**66**, **792**	0.0074	0.0719	52.3°
IIB2:	3	**66**	**220**	**495**	**792**		**792**	0.0318	0.0282	65.7°
IIB2:	3	**66**	**220**	**495**	**792**		**66**, **792**	0.074	0.0719	52.3°
IIB2:	5	**66**	**220**	**495**	**792**		**66**, **792**	0.0071	0.0221	34.4°
IIB3:	1	**12**	**66**	**495**			**66**, **792**	0.0102	0.0202	36.1°
IIB3:	2	**12**	**66**	**495**			**66**, **792**	0.0219	0.0253	34.4°
IIB3:	3	**12**	**66**	**495**			**66**, **792**	-0.0131	0.0077	37.1°
IIB3:	4	**12**	**66**	**495**			**66**, **792**	-0.0100	0.0276	27.2°
IIB4:	1	**12**	**66**	**495**	**792**		**66**, **792**	0.0102	0.0202	36.1°
IIB4:	2	**12**	**66**	**495**	**792**		**66**, **792**	0.0219	0.0253	34.4°
IIB4:	3	**12**	**66**	**495**	**792**		**66**, **792**	-0.0100	0.0276	27.2°
IIB4:	4	**12**	**66**	**495**	**792**		**66**, **792**	-0.015	0.0387	14.4°
IIB5:	2	**12**	**66**	**220**	**495**		**66**, **792**	-0.0080	0.0254	−122°
IIB5:	3	**12**	**66**	**220**	**495**		**66**, **792**	0.0147	0.0115	−146°
IIB5:	4	**12**	**66**	**220**	**495**		**66**, **792**	-0.037	0.0232	87.4°
IIB6:	2	**12**	**66**	**220**	**495**	**792**	**66**, **792**	-0.0080	0.0254	−122°
IIB6:	3	**12**	**66**	**220**	**495**	**792**	**66**, **792**	0.0044	0.0359	91.0°
IIB6:	4	**12**	**66**	**220**	**495**	**792**	**66**, **792**	-0.0038	0.0232	87.5°
IIC1:	1	**66**	**220**	**495**			**66**, **792**	0.0111	0.0230	63.3°
IIC1:	2	**66**	**220**	**495**			**66**, **792**	-0.0037	0.0281	28.6°
IIC2:	2	**66**	**220**	**495**	**792**		**66**, **792**	-0.0037	0.0254	57.0°
IIC2:	5	**66**	**220**	**495**	**792**		**66**, **792**	0.0072	0.0160	−87.7°
IIC3:	2	**12**	**66**	**495**			**66**, **792**	0.0028	0.0408	80.3°
IIC3:	4	**12**	**66**	**495**			**66**, **792**	-0.0053	0.0138	13.4°
IIC4:	2	**12**	**66**	**495**	**792**		**66**, **792**	-0.0037	0.0281	28.6°
IIC4:	4	**12**	**66**	**495**	**792**		**66**, **792**	-0.0053	0.0138	13.4°
IIC5:	1	**12**	**66**	**220**	**495**		**792**	-0.0575	0.0266	20.9°
IIC5:	4	**12**	**66**	**220**	**495**		**66**, **792**	0.0086	0.0223	169°
IIC6:	1	**12**	**66**	**220**	**495**	**792**	**792**	-0.0575	0.0266	20.9°
IIC6:	4	**12**	**66**	**220**	**495**	**792**	**66**, **792**	0.0086	0.0222	169°

Table II: Summary of irreps and model parameters for successful SU(12) models found in the five search and fitting runs. The classes of models are labeled by their SU(5) (**10**) and ($\overline{\mathbf{5}}$) family assignments, while the righthanded Majorana neutrinos all belong to SU(5) and SU(12) singlets. It is to be understood that the massive fermions and Higgs singlets occur in both unbarred and barred irreps.

Also included in Table II are the fit parameters ε, $|\kappa|$, and $\arg(\kappa)$ which are adjusted to help give good fits to the charged lepton masses and to the quark mass and mixing data. The additional adjusted Higgs VEVs, v_u and $v_d = \sqrt{174^2 - v_u^2}$, are found to lie in the range $v_d = 1 - 5$ GeV and $v_u \simeq 174$ GeV and are not included in the Table. Recalling that the initial values for the fit parameters were chosen as $\varepsilon = |\kappa| = 1/6.5^2 = 0.0237$, and $\arg(\kappa) = 45°$, we see that the resulting fit parameters for ε and $|\kappa|$ are reasonably close to their starting values. In particular, $|\kappa| \sim (0.5 - 5)|\varepsilon|$ in most cases, so that the (**24**) Higgs contributions, which serve to split the down quark and charged lepton spectra, and the Higgs singlet contributions are comparable. To obtain a satisfactory model, we have required that all prefactors lie in the range $\pm[0.1, 10]$, i.e., within a factor of 10 of unity. In most cases, the range is considerably tighter. Not surprisingly, in all cases the quark mass and mixing parameters at the GUT scale can be fit accurately with the above model parameters and the overwhelming number of matrix element prefactors far exceeding the number of data points.

To give a more complete picture of the results obtainable for successful models, we have made five separate complete runs of the scanning and fitting procedure outlined above and labeled them by their run number in Table II. Due to the Monte Carlo nature of the prefactor fitting, it is apparent from the table that no successful model assignments were obtained for all five runs, and in many cases for only two or three of the runs. Nevertheless, the results are instructive.

For succesful models found in run 4, we present in Table III the predictions for the neutrino mass and mixing parameters that were obtained with fits of Λ_R and the known neutrino mass and mixing parameters, namely, the three Δm_{ij}^2's and the two sine squares of θ_{12} and θ_{13}. In particular, we list for each model the neutrino mass hierarchy MH, Λ_R, the unknown heavy righthanded Majorana neutrino masses $M_1,\ M_2,\ M_3$, the light neutrino masses $m_1,\ m_2,\ m_3$, $\sin^2\theta_{23}$, the Dirac leptonic CP phase δ, the Majorana phases ϕ_1 and ϕ_2, and the effective mass parameter $|\langle m_{ee}\rangle|$ for neutrinoless double beta decay. The latter prediction assumes the light neutrino masses are the major contributors to the corresponding loop diagrams. Of course there are large spreads in the resulting predictions due to the Monte Carlo adjusted fit parameters and large number of matrix element prefactors. In a number of family symmetry cases listed, both normal and inverted hierarchy models are acceptable with quite different sets of matrix element prefactors and Λ_R. Of the 31 models found in

	MH	Λ_R	M_1	M_2	M_3	m_1	m_2	m_3	$\sin^2\theta_{23}$	δ	ϕ_1	ϕ_2	$\lvert\langle m_{ee}\rangle\rvert$
		——	(10^{14} GeV)	——		——	(meV)	——					(meV)
IA4:	NH	0.174	0.0569	0.189	0.388	0.116	8.6	49.9	0.387	$-1.3°$	$5.6°$	$185°$	1.39
	IH	0.0358	0.0496	0.0583	0.112	49.3	50.0	3.67	0.390	$178°$	$-174°$	$6.4°$	18.6
IB1:	NH	0.0499	0.0055	0.0371	0.101	1.85	8.79	49.9	0.510	$-10.6°$	$-132°$	$40.6°$	2.45
IB2:	IH	0.0102	0.0093	0.0182	0.0217	49.2	49.9	1.61	0.545	$-12.1°$	$235°$	$-133°$	48.2
IB3:	NH	0.0011	0.0002	0.0062	0.0187	14.0	16.5	51.9	0.469	$134°$	$167°$	$-345°$	6.48
	IH	0.0048	0.0021	0.0094	0.0097	49.2	49.9	1.06	0.647	$-121°$	$-44.0°$	$-64.8°$	47.7
IB4:	NH	0.0152	0.0223	0.0484	0.0496	0.121	8.60	49.9	0.443	$17.3°$	$-79.8°$	$114°$	1.65
	IH	0.0088	0.0086	0.0128	0.0196	49.2	49.9	0.642	0.426	$-158°$	$279°$	$85.6°$	19.4
IC1:	NH	3.38	0.0142	0.0243	10.1	0.0022	8.6	49.9	0.543	$82.7°$	$185°$	$219°$	3.58
	IH	0.0009	0.0009	0.0013	0.0059	49.2	49.9	0.394	0.481	$-62.6°$	$-263°$	$241°$	23.2
IC2:	NH	0.0520	0.0542	0.111	0.145	1.73	8.77	49.9	0.630	$97.0°$	$-227°$	$341°$	0.65
	IH	0.0187	0.0228	0.0336	0.0544	49.2	49.9	0.269	0.658	$81.0°$	$-163°$	$-316°$	21.2
IC5:	IH	0.013	0.0074	0.242	0.0319	49.2	49.9	0.321	0.560	$-90.2°$	$200°$	$172°$	47.1
IC6:	NH	0.0155	0.0057	0.0176	0.0435	2.76	9.03	50.0	0.460	$-121°$	$-219°$	$-67.4°$	2.45
	IH	0.0127	0.0105	0.0226	0.0332	49.3	50.0	3.39	0.600	$-50.2°$	$-319°$	$12.8°$	47.2
IIA1:	NH	0.0320	0.0069	0.0515	0.0543	0.469	8.61	49.9	0.360	$22.8°$	$70.8°$	$-86.0°$	3.2
	IH	0.0078	0.0123	0.0159	0.0190	49.2	50.0	2.91	0.578	$140°$	$-32.5°$	$166°$	20.0
IIA2:	NH	0.313	0.0501	0.250	0.750	0.00027	8.6	49.9	0.590	$-152°$	$-103°$	$64.0°$	2.23
IIA6:	NH	0.127	0.0307	0.140	0.295	0.0143	8.6	49.9	0.348	$11.1°$	$308°$	$317°$	3.61
	IH	0.0252	0.0149	0.0383	0.0507	49.2	49.9	0.317	0.657	$-161°$	$-251°$	$102°$	48.2
IIB3:	NH	0.0458	0.0635	0.0956	0.107	1.91	8.81	49.9	0.447	$-12.4°$	$-121°$	$-313°$	2.44
IIB4:	NH	0.225	0.0416	0.400	0.438	0.0104	8.6	49.9	0.419	$-9.3°$	$-135°$	$-325°$	3.61
	IH	0.741	0.0532	0.138	1.24	49.2	49.9	0.0041	0.413	$-167°$	$134°$	$304°$	19.0
IIB5:	NH	0.0055	0.0051	0.0130	0.0157	13.3	15.8	51.7	0.449	$-7.4°$	$-324°$	$207°$	5.46
	IH	0.0072	0.0066	0.0182	0.0243	49.3	50.0	3.7	0.504	$11.3°$	$310°$	$-224°$	18.9
IIB6:	NH	0.0337	0.0261	0.0634	0.0671	2.72	9.02	50.0	0.415	$170°$	$-146°$	$42.3°$	1.86
IIC3:	NH	0.0071	0.0127	0.0158	0.0215	1.92	8.80	49.9	0.639	$177°$	$-168°$	$13.2°$	2.40
IIC4:	NH	0.0200	0.0327.	0.0580	0.0717	0.121	8.6	49.9	0.519	$-178°$	$353°$	$172°$	1.39
IIC5:	IH	0.0085	0.0161	0.0323	0.0439	50.0	50.7	9.07	0.469	$179°$	$-177°$	$-176°$	48.9
IIC6:	NH	0.191	0.0419	0.216	0.508	0.286	8.6	49.9	0.598	$179°$	$2.7°$	$-177°$	1.28
	IH	0.196	0.0352	0.0473	0.0492	49.4	50.2	5.01	0.551	$-179°$	$-182°$	$177°$	48.5

Table III: Summary of successful SU(12) neutrino models for the 4th run with Λ_R as an additional fit parameter. The models are grouped according to the family assignments listed in Table II. In many cases, both normal hierarchy NH and inverted hierarchy IH models have been generated for the same class of family assignment and matrix textures with different sets of matrix element prefactors and Λ_R.

run 4, it is interesting to note that 17 of them correspond to normal hierarchy, while 14 have inverted hierarchy. For all five runs with an average of 28 successful models each, 76 have normal hierarchy while 64 have inverted hierarchy.

In order to better grasp the distributions of results obtained in all five runs, we present several scatterplots. In Fig. 1, δ is plotted vs. $\sin^2\theta_{23}$ for normal hierarchy in (a) and for inverted hierarchy in (b). The circles and squares for NH (upright and inverted triangles for IH) refer to the I and II family ($\mathbf{10}$) assignments, respectively, while the shadings distinguish the family ($\overline{\mathbf{5}}$) assignments. With regard to the $\sin^2\theta_{23}$ distributions, the NH one slightly prefers the second octant, while the IH one is equally split between the first and second octants. It is apparent that most of the models favor small leptonic CP violation, for δ tends to lie near 0° or 180°. The IC6 run 5 model we have illustrated earlier with both NH and IH variations is among the exceptions.

In Fig. 2 is displayed the effective mass parameter for neutrinoless double beta decay vs. the lightest neutrino mass, $m_0 = m_1$ for NH and $m_0 = m_3$ for IH. As expected the IH points lie higher than the NH ones. The IH $|\langle m_{ee}\rangle|$ values cluster around 20 and 50 meV, while the NH ones generally fall below 10 meV with the smallest value occurring for $m_1 \sim 2$ meV. The two clusterings occur because the difference of the two phases ϕ_1 and ϕ_2 for many of the IH models tends to be near 0° or 180°. From this figure we can conclude that the neutrinoless double beta decay experiments by themselves must be able to reach down to $m_{ee} = 10$ meV in order to rule out the inverse mass hierarchy in the framework of three righthanded neutrinos. These results are well within the ranges found by the PDG in [24], based on a 2σ variation of the best fit values as of 2014.

VII. SUMMARY

To explain quark and lepton masses and mixing angles, one has to extend the standard model by putting the quarks and leptons into irreducible representations of a discrete group. We argue that discrete flavor symmetries can be avoided, if we extend the gauge group to the point where the discrete symmetry is no longer needed. By consolidating flavor and family symmetries into a single gauged Lie group we eliminate the problems associated with discrete symmetry, e.g., violation by gravity, domain walls, etc. We have given explicit examples of models having varying degrees of predictability obtained by scanning over groups and representations and identifying cases with operators contributing to mass and mixing matrices

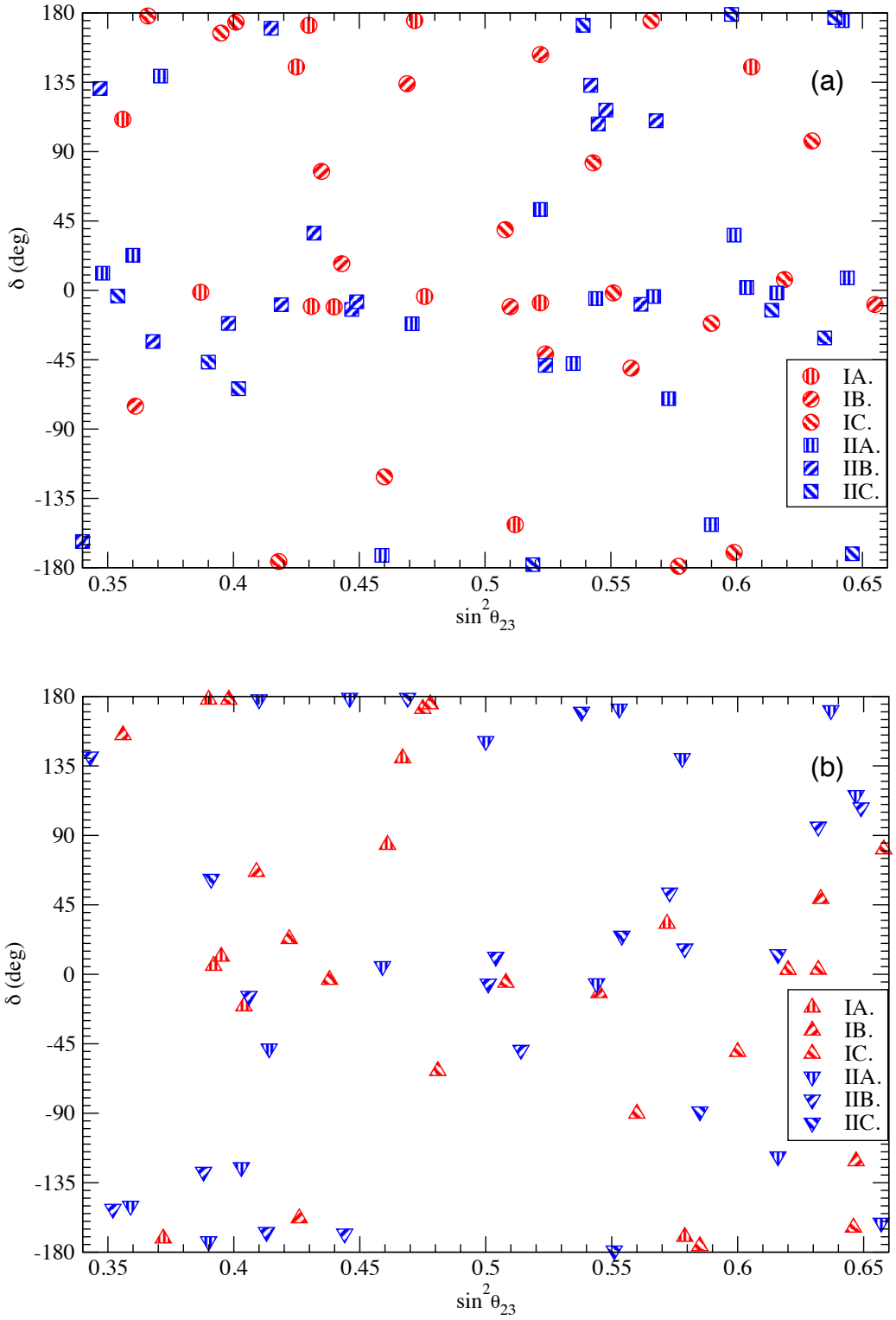

Figure 1: Scatterplots of the CP δ vs. $\sin^2\theta_{23}$ for the NH models in (a) and for the IH models in (b). The symbols label the sets of models listed in Table II. for both normal and inverted hierarchy.

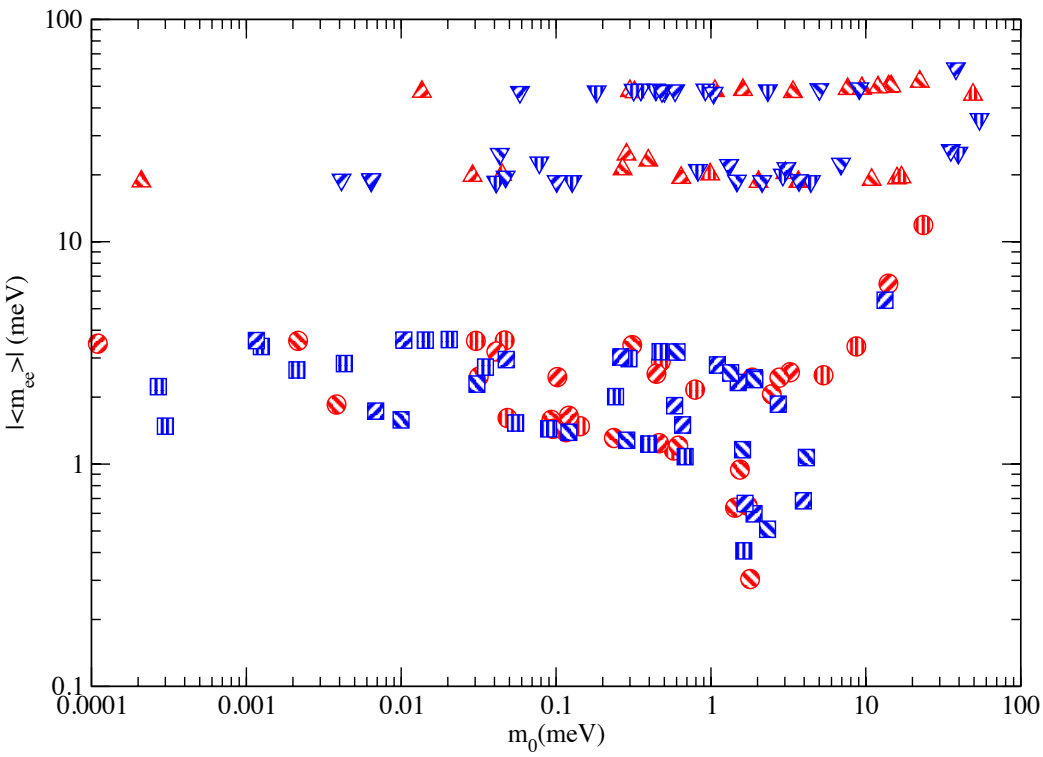

Figure 2: Effective mass for neutrinoless double beta decay vs. the lightest neutrino mass for both NH and IH models. The symbols for the models are the same as in Fig. 1.

that need little fine-tuning of prefactors. Models in SU(12) are particularly interesting.

We have been guided by simplicity. Starting with SM$\times\mathbf{G}_{\text{flavor}}$, we let SM$\rightarrow$ SU(N) and increase N until we can eliminate $\mathbf{G}_{\text{flavor}}$ and still fit known mass and mixing data. This process is rather involved. First we place the SM particles in SU(5) irreps. Beginning with anomaly-free sets of irreps containing three families of fermions, we then assign the family $\bar{\mathbf{5}}$ and $\mathbf{10}$ irreps to SU(N) irreps in a way that is consistent with known data. This requires scanning over fermion assignments and Higgs irreps to allow the necessary Yukawa coupling terms in the Largangian to generate successful models. The Higgs irreps are also required to be capable of breaking the symmetry directly from SU(N) to the SM without breaking SUSY. Since there is an infinity of possible models, the scan is directed and limited in various ways toward finding the simplest class of examples.

We begin to find satisfactory models that require no discrete $\mathbf{G}_{\text{flavor}}$ symmetry at $N = 12$. Smaller N is insufficient to fit the data without keeping at least a small discrete flavor group. Larger N typically gives too many parameters, hence we have focused on $N = 12$, which

seems to be the "sweet spot" for model building. In particular, the smallest anomaly-free set in SU(12) which is $\mathbf{66} + \mathbf{495} + 2(\overline{\mathbf{220}}) + 2(\overline{\mathbf{12}})$ stands out for its simplicity. It contains six irreps, and it turns out that we can assign a single one of the six to each of the $\bar{\mathbf{5}}$'s and $\mathbf{10}$'s in the three families of SU(5). All other three-family sets in SU(12) have more than six irreps, hence some of the irreps in the anomaly free sets cannot contain light fermions. We have limited our focus to this simplest anomaly-free set; however, there are still numerous issues to consider, e.g., which $\bar{\mathbf{5}}$ or $\mathbf{10}$ to assign to each of the SU(12) irreps, which Higgs fields to include, how to include righthanded fermion singlet neutrinos, etc. To handle these issues we rely on scans over assignments. As described in the text, the scans systematically consider models, generate mass and mixing matrices, compare them with data, and those that do not drop by the wayside are kept, while those that fail tests along the way are eliminated from consideration. The result is approximately 30 models for each complete scan (labeled I and II for their $\mathbf{10}$ assignments, and A, B, and C for the assignment of their $\bar{\mathbf{5}}$'s) that satisfy our criteria of providing a fit to all known mass and mixing data which is little fine-tuned, while providing predictive power that can distinguish amongst our models and also discriminate between them and other models in the literature. Once the fits of masses and mixings are complete for our models, they allow us to make further predictions for the neutrino masses and hierarchy, the octant of the atmospheric mixing angle, leptonic CP violation, Majorana phases, and the effective mass observed in neutrinoless double beta decay.

Our purpose has been to unify family and flavor symmetries into a single gauge group. What we have achieved is a demonstration that mass and mixing data can be fit within a class of models where the only symmetry is a gauged SU(12). Furthermore, these models can be predictive and distinguished from discrete flavor symmetry models. In addition, $N = 12$ is small enough that it is conceivable that a model of this type can be contained within a compactification of the superstring.

Among interesting features that have arisen in finding apparent satisfactory models are the following. While we have emphasized the simplicity in assigning the three families of quarks and leptons to irreps of the smallest anomaly-free set of SU(12), the massive lefthanded conjugate (or righthanded) neutrinos must be placed in singlets of both SU(5) and SU(12). We have made the conventional choice of three such neutrinos, but it is clear that one could also have considered only two, or included several additional singlet sterile neutrinos in the model.

Two pairs of Higgs doublets appear in the models, $(\mathbf{5})\overline{\mathbf{495}}_{\rm H}$ and its conjugate along with $(\overline{\mathbf{5}})\overline{\mathbf{12}}_{\rm H}$ and its conjugate, but only the first of each pair listed here are required to get EW VEVs in the successful models. The other two Higgs doublets, $(\overline{\mathbf{5}})\mathbf{495}_{\rm H}$ and $(\mathbf{5})\mathbf{12}_{\rm H}$, play no role in these models and can be considered inert. Note the "mismatch" nature of the pair that develops VEVs and the other pair which does not.

In order to break the SU(5) GUT symmetry, an adjunct **24** must be present in the model, but it can not be present in the SU(12) adjunct **143** which would break SUSY at the SU(12) scale. Instead, two **24**'s emerge with the inclusion of $\mathbf{5148}_{\rm H}$ and $\overline{\mathbf{5148}}_{\rm H}$ and their breakings at the SU(12) scale. Since they originate from an SU(12) complex pair, a ready means arises of introducing CP phases in the models. At the same time, their VEVs serve to split the spectra of the down quarks and charged leptons.

As for the model predictions for the unknown masses and mixing in the neutrino sector, the successful models favor NH over IH by a ratio of 76 to 64 for the five runs considered.. The NH models slightly favor the second octant for $\sin^2\theta_{23}$, while the IH models are impartial to the first and second octants. Many models favor small leptonic CP violation, while a few favor larger violations. Many pairs of successful models with identical Yukawa matrix textures have both NH and IH solutions due to different prefactors and Λ_R scales emerging in the fitting procedure.

Finally, we have shown for the successful models that the neutrinoless double beta decay experiments may need to reach down to an effective mass $|\langle m_{ee}\rangle| \sim 10$ meV, in order to eliminate an inverted light neutrino mass hierarchy. We also note that the present cosmological constraints on the sum of the light neutrino masses [28], $\Sigma m_{\nu,\rm IH} < 0.20$ eV, are insufficient to eliminate any of the apparently successful IH models.

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APPENDIX A. Direct Breaking of SU(12) → SU(5)

Complex irreps of SU(N) all have non vanishing charges in the Cartan subalgebra. If we give them VEVs, they then break a portion of that subalgebra. This in turn lowers the overall rank of the remaining symmetry group. Our interest here is in giving VEVs to antisymmetric irreps of SU(12) to break the gauge symmetry directly to SU(5). These irreps are all complex except for the antisymmetric tensor with 6 indices which is real. A scheme for giving VEVs to antisymmetric tensor irreps of SU(N) to reduce rank was devised in [18, 19] where vanishing total Dynkin weights for the VEVs provides gauge spontaneous symmetry breaking (SSB) without SUSY breaking.

Here we demonstrate that the direct gauge SSB of SU(12) → SU(5) is possible without breaking SUSY by providing two solutions, one for the fourth anomaly-free set of Eq. (3), and the other for the simplest anomaly-free set of Eq. (3).

The first example uses just the supermultiplets of the model where the chiral families live. There the irreps we have to work with are $\mathbf{66} + 2(\mathbf{495}) + \overline{\mathbf{792}} + 2(\overline{\mathbf{220}}) + 8(\overline{\mathbf{12}})$. Let us write VEVs with upper indices for the unbarred irreps, e.g., $v^{a,b}$ for the $\mathbf{66}$, and with lower indices for the barred irreps, e.g., $v_{a,b,c,d,e}$ for the $\overline{\mathbf{792}}$. (We could instead use an epsilon symbol with 12 indices to write the $\overline{\mathbf{792}}$ with 7 upper indices and likewise for other barred irreps, but it will not be necessary here.)

One set of chiral superpartner VEVs that breaks SU(12) directly to SU(5) is

$$v_{12,11,10,9,8}, \quad v^{12,11,10,9}, \quad v^{9,8,7,6}, \quad v_7, \quad v_6, \quad v_9, \tag{45}$$

where the VEVs can be in three different $\overline{\mathbf{12}}$s and in the two different $\mathbf{495}$s. We take all VEVs to be of equal magnitude. The corresponding Dynkin weights given in the same order as the above VEVs are:

$$\begin{array}{cccccccccccc}
[& 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 &] \\
[& -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 &] \\
[& 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 &] \\
[& 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 &] \\
[& 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 &]
\end{array} \tag{46}$$

Summing the weights as in vector addition, we get zero total weight so SUSY remains unbroken. Note that these VEVs give a minimum for the Higgs potential at zero, as required by SUSY. There could be some flat directions at this minimum, but since there are many terms in the superpotential this is not the generic situation.

The second example involves the simplest anomaly-free set which is of most interest in this paper, $\mathbf{66} + \mathbf{495} + 2(\overline{\mathbf{220}}) + 2(\overline{\mathbf{12}})$, with its scalar superpartners which are assumed to get VEVs, aside from the two $\overline{\mathbf{12}}$'s, along with a pair of Higgs singlets $\mathbf{12}_{\rm H}$ and $\overline{\mathbf{12}}_{\rm H}$. With the same tensor notation as above, we can form the following tensor contraction of the VEVS,

$$v^{12,11,10,9}\ v_{12,11,10}\ v_{9,8,7}\ v^{8}\ v^{7,6}\ v_{6}. \tag{47}$$

Again with all VEVs equal in magnitude, the ordered Dynkin weights are:

$$\begin{array}{cccccccccccc}
[& 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 &] \\
[& -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &] \\
[& 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 &] \\
[& 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 &] \\
[& 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 &] \\
[& 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 &]
\end{array} \tag{48}$$

The sum of the Dynkin weights vanishes, so SU(12) $\rightarrow$ SU(5) and SUSY remains unbroken.

APPENDIX B. Matrix Element Contributions to the Selected Model

Here we present in Table IV the leading diagrams contributing to the Yukawa matrix elements for the quark and lepton mass matrices of the special model considered in Sect. IV C. Several diagrams of the same dimension contribute to a given matrix element in many cases as listed. These diagrams apply for the **IC**6 class of models listed in Table II.

Matrix Element Contributions for Model IC6

Fermions: $(10)495_1$, $(10)\overline{220}_2$, $(10)66_3$, $(\overline{5})\overline{220}_1$, $(\overline{5})\overline{12}_2$, $(\overline{5})\overline{12}_3$
Massive Fermions: 12, $\overline{12}$, 66, $\overline{66}$, 220, $\overline{220}$, 495, $\overline{495}$, 792, $\overline{792}$
Higgs: $(5)\overline{495}_{\rm H}$, $(\overline{5})\ \overline{12}_{\rm H}$, $(24)5148_{\rm H}$, $(24)\overline{5148}_{\rm H}$, $(1)1_{\rm H}$, $(1)66_{\rm H}$, $(1)\overline{66}_{\rm H}$, $(1)792_{\rm H}$, $(1)\overline{792}_{\rm H}$

Leading Up-Type Diagrams:

Dim 4:

U33: $(10)66_3.(5)\overline{495}_{\rm H}.(10)66_3$

Dim 5:

U13: $(10)495_1.(1)\overline{66}_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66_3$
U31: $(10)66_3.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(1)\overline{66}_{\rm H}.(10)495_1$
U23: $(10)\overline{220}_2.(1)792_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66_3$
U23: $(10)\overline{220}_2.(24)5148_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66_3$
U32: $(10)66_3.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(1)792_{\rm H}.(10)\overline{220}_2$
U32: $(10)66_3.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(24)5148_{\rm H}.(10)\overline{220}_2$

Dim 6:

U11: $(10)495_1.(1)\overline{66}_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(1)\overline{66}_{\rm H}.(10)495_1$
U12: $(10)495_1.(1)\overline{66}_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(1)792_{\rm H}.(10)\overline{220}_2$
U12: $(10)495_1.(1)\overline{66}_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(24)5148_{\rm H}.(10)\overline{220}_2$
U21: $(10)\overline{220}_2.(1)792_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(1)\overline{66}_{\rm H}.(10)495_1$
U21: $(10)\overline{220}_2.(24)5148_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(1)\overline{66}_{\rm H}.(10)495_1$
U22: $(10)\overline{220}_2.(1)792_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(1)792_{\rm H}.(10)\overline{220}_2$
U22: $(10)\overline{220}_2.(1)792_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(24)5148_{\rm H}.(10)\overline{220}_2$
U22: $(10)\overline{220}_2.(24)5148_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(1)792_{\rm H}.(10)\overline{220}_2$
U22: $(10)\overline{220}_2.(24)5148_{\rm H}.(\overline{10})\overline{66}\times(10)66.(5)\overline{495}_{\rm H}.(10)66\times(\overline{10})\overline{66}.(24)5148_{\rm H}.(10)\overline{220}_2$

Leading Down-Type Diagrams:

Dim 4:

D32: $(10)66_3.(\overline{5})\overline{12}_{\rm H}.(\overline{5})\overline{12_2}$
D33: $(10)66_3.(\overline{5})\overline{12}_{\rm H}.(\overline{5})\overline{12_3}$

Dim 5:

D12: $(10)495_1.(1)\overline{66}_{\rm H}.(\overline{10})\overline{66}\times(10)66.(\overline{5})\overline{12}_{\rm H}.(\overline{5})\overline{12}_2$
D21: $(10)\overline{220}_2.(\overline{5})\overline{12}_{\rm H}.(\overline{5})495\times(5)\overline{495}.(1)\overline{792}_{\rm H}.(\overline{5})\overline{220}_1$
D13: $(10)495_1.(1)\overline{66}_{\rm H}.(\overline{10})\overline{66}\times(10)66.(\overline{5})\overline{12}_{\rm H}.(\overline{5})\overline{12}_3$
D31: $(10)66_3.(\overline{5})\overline{12}_{\rm H}.(\overline{5})\overline{12}\times(5)12.(1)66_{\rm H}.(\overline{5})\overline{220}_1$
D22: $(10)\overline{220}_2.(\overline{5})\overline{12}_{\rm H}.(\overline{5})495\times(5)\overline{495}.(1)792_{\rm H}.(\overline{5})\overline{12}_2$
D22: $(10)\overline{220}_2.(1)792_{\rm H}.(\overline{10})\overline{66}\times(10)66.(\overline{5})\overline{12}_{\rm H}.(\overline{5})\overline{12}_2$
D22: $(10)\overline{220}_2.(\overline{5})\overline{12}_{\rm H}.(\overline{5})495\times(5)\overline{495}.(24)5148_{\rm H}.(\overline{5})\overline{12}_2$
D22: $(10)\overline{220}_2.(24)5148_{\rm H}.(\overline{10})\overline{66}\times(10)66.(\overline{5})\overline{12}_{\rm H}.(\overline{5})\overline{12}_2$
D23: $(10)\overline{220}_2.(\overline{5})\overline{12}_{\rm H}.(\overline{5})495\times(5)\overline{495}.(1)792_{\rm H}.(\overline{5})\overline{12}_3$
D23: $(10)\overline{220}_2.(1)792_{\rm H}.(\overline{10})\overline{66}\times(10)66.(\overline{5})\overline{12}_{\rm H}.(\overline{5})\overline{12}_3$
D23: $(10)\overline{220}_2.(\overline{5})\overline{12}_{\rm H}.(\overline{5})495\times(5)\overline{495}.(24)5148_{\rm H}.(\overline{5})\overline{12}_3$
D23: $(10)\overline{220}_2.(24)5148_{\rm H}.(\overline{10})\overline{66}\times(10)66.(\overline{5})\overline{12}_{\rm H}.(\overline{5})\overline{12}_3$

Dim 6:

D11: $(10)495_1.(1)\overline{66}_{\rm H}.(\overline{10})\overline{66}\times(10)66.(\overline{5})\overline{12}_{\rm H}.(\overline{5})\overline{12}\times(5)12.(1)66_{\rm H}.(\overline{5})\overline{220}_1$
D11: $(10)495_{.}(1)792_{\rm H}.(\overline{10})220\times(10)\overline{220}.(\overline{5})\overline{12}_{\rm H}.(\overline{5})495\times(5)\overline{495}.(1)\overline{792}_{\rm H}.(\overline{5})\overline{220}_1$

Leading Dirac Neutrino Diagrams:
Dim 5:

DN11: $(\overline{5})\overline{220}_1.(5)\overline{495}_{\rm H}.(1)\overline{792}\times(1)792.(1)\overline{792}_{\rm H}.(1)1_1$
DN11: $(\overline{5})\overline{220}_1.(1)\overline{792}_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_1$
DN12: $(\overline{5})\overline{220}_1.(5)\overline{495}_{\rm H}.(1)\overline{792}\times(1)792.(1)\overline{792}_{\rm H}.(1)1_2$
DN12: $(\overline{5})\overline{220}_1.(1)\overline{792}_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_2$
DN21: $(\overline{5})\overline{12}_2.(5)\overline{495}_{\rm H}.(1)792\times(1)\overline{792}.(1)792_{\rm H}.(1)1_1$
DN21: $(\overline{5})\overline{12}_2.(1)792_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_1$
DN21: $(\overline{5})\overline{12}_2.(24)5148_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_1$
DN13: $(\overline{5})\overline{220}_1.(5)\overline{495}_{\rm H}.(1)\overline{792}\times(1)792.(1)\overline{792}_{\rm H}.(1)1_3$
DN13: $(\overline{5})\overline{220}_1.(1)\overline{792}_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_3$
DN31: $(\overline{5})\overline{12}_3.(5)\overline{495}_{\rm H}.(1)792\times(1)\overline{792}.(1)792_{\rm H}.(1)1_1$
DN31: $(\overline{5})\overline{12}_3.(1)792_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_1$
DN31: $(\overline{5})\overline{12}_3.(24)5148_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_1$
DN22: $(\overline{5})\overline{12}_2.(5)\overline{495}_{\rm H}.(1)792\times(1)\overline{792}.(1)792_{\rm H}.(1)1_2$
DN22: $(\overline{5})\overline{12}_2.(1)792_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_2$
DN22: $(\overline{5})\overline{12}_2.(24)5148_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_2$
DN23: $(\overline{5})\overline{12}_2.(5)\overline{495}_{\rm H}.(1)792\times(1)\overline{792}.(1)792_{\rm H}.(1)1_3$
DN23: $(\overline{5})\overline{12}_2.(1)792_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_3$
DN23: $(\overline{5})\overline{12}_2.(24)5148_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_3$
DN32: $(\overline{5})\overline{12}_3.(5)\overline{495}_{\rm H}.(1)792\times(1)\overline{792}.(1)792_{\rm H}.(1)1_2$
DN32: $(\overline{5})\overline{12}_3.(1)792_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_2$
DN32: $(\overline{5})\overline{12}_3.(24)5148_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_2$
DN33: $(\overline{5})\overline{12}_3.(5)\overline{495}_{\rm H}.(1)792\times(1)\overline{792}.(1)792_{\rm H}.(1)1_3$
DN33: $(\overline{5})\overline{12}_3.(1)792_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_3$
DN33: $(\overline{5})\overline{12}_3.(24)5148_{\rm H}.(5)\overline{495}\times(\overline{5})495.(5)\overline{495}_{\rm H}.(1)1_3$

Leading Majorana Neutrino Diagrams:
Dim 4:

MN11: $(1)1_1.(1)1_{\rm H}.(1)1_1$
MN12: $(1)1_1.(1)1_{\rm H}.(1)1_2$
MN21: $(1)1_2.(1)1_{\rm H}.(1)1_1$
MN22: $(1)1_2.(1)1_{\rm H}.(1)1_2$
MN13: $(1)1_1.(1)1_{\rm H}.(1)1_3$
MN31: $(1)1_3.(1)1_{\rm H}.(1)1_1$
MN23: $(1)1_2.(1)1_{\rm H}.(1)1_3$
MN32: $(1)1_3.(1)1_{\rm H}.(1)1_2$
MN33: $(1)1_3.(1)1_{\rm H}.(1)1_3$

Table IV: Yukawa diagrams for the up and down quark and Dirac neutrino matrices, as well as the righthanded Majorana neutrino diagrams, for the special model singled out in Sect. IV C. which belongs to the **IC**6 class of models of Table II. The diagrams for the charged leptons are the transpose of the down quark diagrams.

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